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Absolute cyclicity, Lyapunov quantities and center conditions

M. Caubergh *, A. Gasull

Universitat Autònoma de Barcelona, Facultat de Ciencies (Ed. C), Departament de Matemàtiques, Cerdanyola del Vallès, Barcelona, Spain

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ABSTRACT

In this paper we consider analytic vector fields X_0 having a non-degenerate center point e. We estimate the maximum number of small amplitude limit cycles, i.e., limit cycles that arise after small perturbations of X_0 from e. When the perturbation (X_λ) is fixed, this number is referred to as the cyclicity of X_λ at e for λ near 0. In this paper, we study the so-called absolute cyclicity; i.e., an upper bound for the cyclicity of any perturbation X_λ for which the set defined by the center conditions is a fixed linear variety. It is known that the zero-set of the Lyapunov quantities correspond to the center conditions (Caubergh and Dumortier (2004) [6]). If the ideal generated by the Lyapunov quantities is regular, then the absolute cyclicity is the dimension of this so-called Lyapunov ideal minus 1. Here we study the absolute cyclicity in case that the Lyapunov ideal is not regular.

1. Introduction

1.1. Cyclicity problem and center conditions

The existential part of Hilbert's 16th problem asks whether there exists a uniform upper bound for the number of limit cycles that appear in a planar polynomial vector field, only depending on its degree n. By the so-called Roussarie reduction this global problem is reduced to the investigation of local 'cyclicity problems'; in this reduction one looks for 'limit periodic sets', from which limit cycles can arise when slightly perturbing the vector field (cf. [15]). Let $(X_\lambda)_\lambda$ be an analytic family of vector fields, such that Γ is a limit periodic set of X_{λ^0}; then, the cyclicity of X_λ at (Γ, λ^0) is defined by

$$Cycl\big(X_\lambda, (\Gamma, \lambda^0)\big) = \lim_{\lambda \to \lambda^0} \sup_{\gamma \to \Gamma} \{\# \text{ limit cycles } \gamma \text{ of } X_\lambda\},$$

where the limit $\gamma \to \Gamma$ is taken in the sense of the Haussdorf distance. If for every given limit periodic set of an analytic family of vector fields, the cyclicity is finite, then there exists a uniform upper bound for the number of limit cycles of (X_λ).

There exist several (equivalent) techniques to study this number. Poincaré reduced the study of limit cycles to the study of zeroes of maps $(\delta_\lambda)_\lambda$, associated to the family of vector fields $(X_\lambda)_\lambda$ near the limit periodic set Γ. These maps are called displacement maps. In this paper we only consider analytic families of vector fields and isolated singularities; then, by Poincaré–Bendixson's theorem, a limit periodic set is one of the following compact invariant sets: a singularity, a periodic orbit or a graphic. The cyclicity in the first two cases corresponds to the local study of zeroes of an analytic family of maps; it is theoretically well understood. For instance, the cyclicity is finite; knowing a non-identically zero jet of finite order of the maps δ_{λ^0} at the limit periodic set, an explicit upper bound for the cyclicity is known and given in terms of the order

* Corresponding author.
E-mail address: leen@mat.uab.cat (M. Caubergh).

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of the first non-zero jet of δ_λ. This result is often referred to as the theorem of Melnikov–Pontryagin and is obtained by a division–derivation algorithm. In fact, in this case, the bifurcation diagram can completely be described by use of the Weierstrass Preparation Theorem.

When all jets of the map δ_{λ^0} at $r = 0$ vanish identically, then the vector field is called to be of center type near Γ. This means that the vector field near Γ consists of a disc or annulus of non-isolated periodic orbits. In this case, the division–derivation algorithm cannot be applied in a straight forward way. One first has to remove the degeneracy caused by the center type; this is done by dividing the displacement maps δ_λ in the Bautin ideal, i.e. the ideal generated by the (analytic) coefficients in the asymptotic expansion of $(\delta_\lambda)_\lambda$. By Hilbert's base theorem, we know that this ideal is finitely generated, and the division of $(\delta_\lambda)_\lambda$ in a so-called minimal set of generators provides the upper bound for the cyclicity in the center case [15]. The parameter values λ at which the generators of the Bautin ideal vanish, correspond to vector fields X_λ of center type, and give the center conditions.

In general, it is a difficult problem to calculate the asymptotic expansion of the maps δ_λ of infinite order; often only a finite number of coefficients in this expansion can be calculated. In practice, only the first non-zero coefficient can be calculated, and this is sufficient to draw conclusions. Therefore, one restricts the calculations of these coefficients to parameter values for which the previous coefficients vanish. If at some order the coefficient is not identically zero for all parameter values, one can give an upper bound for the cyclicity.

In [6] it is proven that the Bautin ideal coincides with the so-called Lyapunov ideal; furthermore, they coincide at each order of asymptotic expansion. There exist algorithms in computer-algebra packages to calculate the Lyapunov quantities (cf. [11]).

The definition and properties of Lyapunov quantities can for instance be found in [6,7,11]. Among specialists it is well known that for classical Liénard equations, the Lyapunov ideal corresponding to the singularity at the origin, are given by the 'odd' coefficients (Cherkas). Using the theory developed in [6], an asymptotic expansion of the maps $(\delta_\lambda)_\lambda$ is provided in [7], and the cyclicity is thus calculated, see also [10,16]. However, no such explicit center conditions can be given for the generalized Liénard equation in terms of its coefficients.

In general, there does not exist any theory to determine the order of non-vanishing coefficient, nor of stabilizing of the Bautin ideal in terms of the coefficients of the vector field $(X_\lambda)_\lambda$. Knowing this order enables us to bound the cyclicity of the family [15].

When the center condition is generated by merely a 1-dimensional parameter, say $\varepsilon = 0$, then the technique based on the Bautin ideal corresponds to the technique of computing Melnikov functions (Abelian integrals).

By the difficulty of calculating all Lyapunov quantities that make the Bautin ideal stabilize, the question arised to apply this 1-parameter technique to estimate the cyclicity in the multi-parameter family. When the center conditions are generated by a multi-dimensional parameter, say $\varphi_1, \ldots, \varphi_l$, then we know from [3] that the cyclicity of the multi-dimensional family can be studied by means of a 1-dimensional parameter subfamily. More precisely, there always exists an analytic curve in parameter space on which the cyclicity is attained, a so-called curve of maximal cyclicity (mcc). As a consequence, the 1-dimensional technique can be applied as soon as we know an mcc. Under certain generic conditions the existence of an algebraic mcc is guaranteed (cf. [5]). In general, there does not exist a linear mcc and we only know the existence of an mcc. If the Bautin ideal is regular, then there exists a linear curve of maximal multiplicity (mmc); this is the case of the classical Liénard equations. If the Bautin ideal is principal, there always exists a linear curve of maximal index (mic). As a consequence, if the Bautin ideal is regular or principal, an upperbound for the cyclicity can be found by calculating Melnikov functions in 1-parameter subfamilies induced by a straight line through λ^0 (cf. [5]).

To verify the conditions for existence of linear mcc, mmc or mic, one has to compute the Bautin ideal; hence, their existence cannot always be ensured. Now the question arises how to estimate the cyclicity at a center by the knowledge of only a few number of Lyapunov quantities. This is the subject of this paper.

1.2. Results

In this paper, we suppose that for a given analytic family $(X_\lambda)_\lambda$, the center conditions can be found by a geometric argument; suppose that the vector field is of center type for parameter values that satisfy $\{f(\lambda) = 0\}$, where $f : (\mathbb{R}^m, \lambda^0) \to (\mathbb{R}^n, 0)$ is an analytic function, that is not identically zero. Without loss of generality, we can assume that $\lambda^0 = 0$.

If $m = 1$, then the Bautin ideal is principal, hence there exists a linear mic (cf. [5]). So the 1-parameter technique can be applied. If, e.g., the first Lyapunov quantity is given by $(f(\lambda))^5$, then, using standard techniques, one finds that the cyclicity is at most 4. Furthermore, there exist examples for which the cyclicity is exactly 4 (cf. [9,15]). In [9], a precise description of the bifurcation diagram of limit cycles is given in case that the Bautin ideal is an arbitrary ideal of dimension 1; there the approach is based on Lyapunov quantities (which is an equivalent approach, cf. e.g., [6]).

Here, we investigate the case that the Bautin ideal is not principal; i.e., the case when the dimension of the Bautin ideal is at least 2. Throughout this paper, we will often deal with the 2-dimensional case in order to simplify the reading; however, the results can be generalized in a natural way to any dimension. When the dimension of the Bautin ideal is greater than 2, the bifurcation diagram becomes more complicated: besides Hopf bifurcations also boundary bifurcations can occur (cf. [6]). This extra complexity is also reflected in the analysis of the bifurcation diagram of the 2-dimensional case study in Section 3.2.

Suppose that the mapping

$$\big(\mathbb{R}^m, 0\big) \to \mathbb{R}^2 : \lambda \mapsto \big(f_1(\lambda), f_2(\lambda)\big)$$

is a local submersion at $\lambda = 0$.

Then, the absolute cyclicity for a class of analytic families for which the first Lyapunov quantity takes the form $f_1(\lambda) f_2(\lambda)$ is infinite (Theorem 3). Recall that by *absolute cyclicity* we mean the maximal cyclicity for an analytic family of vector fields, satisfying a given property; here the absolute cyclicity concerns the maximal cyclicity taken over all analytic families having a center in the origin with centers generated by either $f_1(\lambda) = 0$ or $f_2(\lambda) = 0$, and first Lyapunov quantity given by $f_1(\lambda)^{k_1} f_2(\lambda)^{k_2}$.

More generally, if we suppose that the first Lyapunov quantity is given by $\lambda_1^{k_1}\lambda_2^{k_2} \cdot \cdots \cdot \lambda_m^{k_m}$, where $k_i \in \mathbb{N}$, $1 \leqslant i \leqslant m$, $k_1 + \cdots + k_m \geqslant 1$. Then, the study of the maximal possible cyclicity is reduced to a problem of estimating the maximum number of small positive zeroes in analytic families of functions $(\delta_\lambda)_\lambda$, $\lambda = (\lambda_1, \ldots, \lambda_m)$, satisfying

$$\delta_\lambda|_{\lambda=0} \equiv 0 \quad \text{and} \quad \delta_\lambda(r) = r^n\big(\lambda_1^{k_1}\lambda_2^{k_2} \cdot \cdots \cdot \lambda_m^{k_m} + O(r)\big), \tag{1}$$

for $r \to 0$, $\|\lambda\| \to 0$, and for certain $n, k_i \in \mathbb{N}$, $1 \leqslant i \leqslant m$. For $m = 1$, the answer is contained in the Weierstrass Preparation Theorem. As far as we know, for $m \geqslant 2$, there does not exist an analogue of the Weierstrass Preparation Theorem, where the standard family $(\bar{\delta}_\lambda)_\lambda$ is a family of multivariate polynomials $\bar{\delta}_\lambda$. Furthermore, we point out that the property (1) is too wild to find a uniform upperbound for the maximal number of small positive zeroes of δ_λ, $\|\lambda\| \downarrow 0$; in other words, the absolute cyclicity is infinite (Theorem 3).

However, if, instead of (1), the derivatives of $\delta_\lambda(r)$ with respect to r satisfy

$$\delta_\lambda|_{\lambda=0} \equiv 0 \quad \text{and} \quad \delta_\lambda^{(2j-1)}(0) = \lambda_j^{k_j}, \quad 1 \leqslant j \leqslant m, \tag{2}$$

for $\|\lambda\| \to 0$, $\forall 1 \leqslant j \leqslant m$, for certain $n_j, k_j \in \mathbb{N}$, $1 \leqslant j \leqslant m$, with $n_1 < n_2 < \cdots < n_m$, then by a division–derivation algorithm, the absolute cyclicity $\mathcal{C}_m^{\mathrm{abs}}(k_1, \ldots, k_m)$ is shown to be finite (Theorem 4):

$$\mathcal{C}_m^{\mathrm{abs}}(k_1, \ldots, k_m) \leqslant k_1 \cdot \cdots \cdot k_m + m - 2, \tag{3}$$

where

$$\mathcal{C}_m^{\mathrm{abs}}(k_1, \ldots, k_m) = \sup\big\{\mathcal{C}_m\big(\delta_\lambda, \big(0^+, 0\big)\big)\colon (\delta_\lambda)_\lambda \text{ satisfies (2)}\big\},$$

and

$$\mathcal{C}_m\big(\delta_\lambda, \big(0^+, 0\big)\big) = \lim_{\lambda \to 0} \sup_{r \downarrow 0}\{\#\text{ positive zeroes } r \text{ of } \delta_\lambda\}.$$

Notice that the result in (3) can be generalized in a trivial way for families δ_λ satisfying

$$\delta_\lambda|_{\lambda=0} \equiv 0, \qquad \delta_\lambda^{(p)}(0) = 0, \quad \forall 0 \leqslant p \leqslant n_m - 1,\ n \neq n_j \quad \text{and} \quad \delta_\lambda^{(n_j)}(0) = \lambda_j^{k_j}.$$

In particular, from [9], we obtain Theorem 2:

$$\mathcal{C}_1^{\mathrm{abs}}(k) = k - 1,$$

and for $m = 2$, we find the following finer bounds (Theorem 6):

$$\mathcal{C}_2^{\mathrm{abs}}(k_1, 1) = k_1 \quad \text{and for } k_1 \geqslant 2, \qquad \left[\frac{3k_1 + 1}{2}\right] \leqslant \mathcal{C}_2^{\mathrm{abs}}(k_1, 2) \leqslant 2k_1,$$

where $[s]$ denotes the integer part of s. Furthermore, in Section 3.2, we investigate the germ of the bifurcation diagram for $\|\lambda\| \to 0$, $r \downarrow 0$ for $m = k_1 = k_2 = 2$ in more detail; in general it corresponds to the case where the first (respectively second) Lyapunov quantity is given by

$$f_1(\lambda)^2 \quad \big(\text{respectively } f_2(\lambda)^2\big).$$

Our results seem to indicate that in this case the absolute cyclicity would be 3.

The paper is organized as follows. In Section 2, we investigate the existence of upper bounds for the absolute cyclicity with respect to (1) as well as (2); if it exists, we provide an upper bound. Next, in Section 3, we concentrate on the case that the Bautin ideal is 2-dimensional; as such, we obtain finer estimates.

In the analysis of the bifurcation diagram of the zeroes of the analytic function δ, we use standard tools, such as the Bautin ideal (cf. [15]), Newton's diagram (cf. [1]), discriminant (cf. [13]) and Descartes' Rule.

Let us finally remark that for a given analytic family of functions $\delta(r, \lambda)$ with $\delta(r, \lambda) = \delta(-r, \lambda)$, we can construct the analytic family of vector fields

$$X_\lambda = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \delta_\lambda(r)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right),$$

where $r = \sqrt{x^2 + y^2}$ and $\delta(r,\lambda) = \delta_\lambda(r)$. Then, the function δ_λ is a displacement map for X_λ, up to a non-zero analytic factor. However, in the study of limit cycles, these functions play the same role and have the same properties as the traditional displacement map. In particular, by symmetry with respect to the center in the origin, if we have locally the following asymptotics:

$$\delta(r,\lambda) = \sum_{i=1}^{t} \alpha_i(\lambda) r^i + O\left(r^{t+1}\right), \quad r \downarrow 0,$$

then it is well known that there exist analytic functions $A_{ij}(\lambda)$ such that locally

$$\alpha_{2i}(\lambda) = \sum_{j=1}^{i} A_{ij}(\lambda)\alpha_{2j-1}(\lambda)$$

(cf. [4]). This is the reason why we provide with examples that are even with respect to r.

2. Upper bounds for the absolute cyclicity

We first study analytic families of functions satisfying (1). For $m = 1$, the absolute cyclicity can be calculated exactly. This result has been proven in [9]. For sake of completeness, we here include the precise result – rephrased in terms of zeroes of analytic functions – and its proof. In particular the proof provides insight in the multi-dimensional case ($m > 1$). Its proof relies on the curve selection lemma for subanalytic sets, which we state below:

Lemma 1. *(See [2,8,14].) Suppose that V is an open subanalytic set in $\mathbb{R}^p$, and λ^0 is an accumulation point of V, then there exists an analytic curve $\gamma : [0,1] \to \mathbb{R}^p$ such that $\gamma(]0,1[) \subset V$ and $\gamma(0) = \lambda^0$.*

Theorem 2. *(See [9].) Consider any analytic family $(\delta_\lambda)_\lambda$ of functions with $\lambda \in \mathbb{R}$, such that*

$$\delta_\lambda(r) = r^p\left(\lambda^k + \lambda g(r,\lambda)\right), \tag{4}$$

for $p \in \mathbb{N}$ with

$$g(r,\lambda) = O(r), \quad r \downarrow 0. \tag{5}$$

Then, $\mathcal{C}_1^{\mathrm{abs}}(k) = k - 1$.

Proof. Let $(\delta_\lambda)_\lambda$ be a fixed analytic family of maps satisfying (4) and (5); then there exists $1 \leqslant i \leqslant k$, such that

$$\delta_\lambda(r) = \delta(r,\lambda) = \lambda^i r^p \widehat{\delta}(r,\lambda),$$

for an analytic map $\widehat{\delta}$ with

$$\widehat{\delta}(r,\lambda) = \lambda^{k-i} + O(r), \quad r \to 0.$$

First we show that

$$\mathcal{C}_m\left(\delta_\lambda, \left(0^+, 0\right)\right) \leqslant k - i \leqslant k - 1;$$

then, by providing an example in which $\mathcal{C}_m(\delta_\lambda, (0^+, 0)) = k - 1$, the result follows. Suppose that the cyclicity $\mathcal{C}_m(\delta_\lambda, (0^+, 0)) > k - i$. As a consequence of the curve selection lemma (Lemma 1), there exist continuous functions $\xi_j : [0, A] \to \mathbb{R}$, $1 \leqslant j \leqslant k - i + 1$ (that are even analytic outside $\lambda = 0$) such that for $0 < \lambda < A$:

$$0 < \xi_1(\lambda) < \xi_2(\lambda) < \cdots < \xi_{k+1}(\lambda) \tag{6}$$

with $\forall 1 \leqslant j \leqslant k + 1$,

$$\widehat{\delta}\left(\xi_j(\lambda), \lambda\right) \equiv 0 \quad \text{and} \quad \xi_j(0) = 0$$

(cf. [4]). From the Intermediate Value Theorem for continuous functions, it follows that for any r small enough and any $0 < A_0 < A$ small enough, i.e., $r \in \bigcap_{j=1}^{k-i+1} \xi_j(]0, A_0[)$, we find $(k - i + 1)$ values λ, say $\lambda_1, \ldots, \lambda_{k-i+1}$, such that $\xi_j(\lambda_j) = r$. By (6), these $\lambda_1, \ldots, \lambda_{k-i+1}$ are disjoint zeroes of $\widehat{\delta}(r, \cdot)$ in $[0, A_0]$. However, by Rolle's theorem, for r and A_0 small enough, the map $\widehat{\delta}(r, \cdot)$ has at most $k - i$ zeroes in $[0, A_0]$. Contradiction.

Consider now the analytic family of functions defined by

$$\bar{\delta}_\lambda(r) = \bar{\delta}(r,\lambda) = r\lambda\big(\lambda^{k-1} + \nu_1\lambda^{k-2}r + \nu_2\lambda^{k-3}r^2 + \cdots + \nu_{k-2}\lambda r^{k-2} + \nu_{k-1}r^{k-1}\big),$$

where $\nu = (\nu_1,\ldots,\nu_k) \in \mathbb{R}^k$, $\lambda \in \mathbb{R}$, $r > 0$. For an appropriate choice of $\nu = (\nu_1,\ldots,\nu_k)$, we have

$$\mathcal{C}_m\big(\bar{\delta}_\lambda, \big(0^+, 0\big)\big) = k-1. \qquad \square$$

In particular, from this result it follows that the absolute cyclicity of an analytic family of functions δ_λ cannot be bounded for the class of functions defined by (1), as soon as $m \geqslant 2$.

Theorem 3. *Let $m \geqslant 2$ arbitrary but fixed. Consider analytic families of functions satisfying* (1) *such that $\sum_{1\leqslant i_1<i_2\leqslant m}(k_{i_1}k_{i_2})^2 \neq 0$ (i.e., at least two indices are non-zero). Then, the absolute cyclicity is infinite. In particular, let $1 \leqslant i_1 < i_2 \leqslant m$ be integers such that $k_{i_1}k_{i_2} \neq 0$. Then, $\forall M \in \mathbb{N}$, there exists an analytic family $(\widehat{\delta_\lambda})_\lambda$ of the above form with*

$$\mathcal{C}_m\big(\widehat{\delta_\lambda}, \big(0^+, 0\big)\big) = M.$$

Proof. Write $K = \sum_{i=1, i\neq i_2}^m k_i - 1$, then we can choose $q \in \mathbb{N}$ such that $N = K + q(k_{i_2} - 1) - 1 \geqslant M$. Choose real constants $\alpha_1,\ldots,\alpha_N$ such that

$$1 + \alpha_1\rho + \alpha_2\rho^2 + \alpha_3\rho^3 + \cdots + \alpha_K\rho^K + \cdots + \alpha_N\rho^N$$

has exactly N positive zeroes. Let $1 \leqslant i_1 < i_2 \leqslant m$ be the smallest integers for which $k_{i_1}k_{i_2} \neq 0$. Then, define $(\widehat{\delta_\lambda})_\lambda$ by

$$\widehat{\delta_\lambda}(r) = \lambda_1^{k_1}\lambda_2^{k_2}\ldots\lambda_m^{k_m} + \alpha_1\lambda_{i_1}^{N+1}\lambda_{i_2}r + \alpha_2\lambda_{i_1}^{N}\lambda_{i_2}r^2 + \cdots + \alpha_{K-1}\lambda_{i_1}^{N+3-K}\lambda_{i_2}r^{K-1}$$
$$+ \alpha_K\lambda_{i_1}\lambda_{i_2}^{k_{i_2}}r^K + \alpha_{K+1}\lambda_{i_1}^{N+1-K}\lambda_{i_2}r^{K+1} + \cdots + \alpha_N\lambda_{i_1}^2\lambda_{i_2}r^N.$$

Consider the 1-parameter subfamily defined by the curve $\zeta(C) = (\zeta_1(C),\ldots,\zeta_m(C))$, $C > 0$ with

$$\zeta_i(C) = C, \quad \forall i \neq i_2 \quad \text{and} \quad \zeta_{i_2}(C) = C^q;$$

this yields to the 1-parameter family

$$\widehat{\delta_{\zeta(C)}}(r) = C^{q+1}\big[C^{N+1} + \alpha_1C^Nr + \alpha_2C^{N-1}r^2 + \cdots + \alpha_{K-1}C^{N+2-K}r^{K-1}\alpha_KC^{q(k_{i_2}-1)}r^K$$
$$+ \alpha_{K+1}C^{N-K}r^{K+1} + \cdots + \alpha_{N-1}Cr^N\big].$$

Next we perform the rescaling $r = C\rho$, and we can factorize $\widehat{\delta_\lambda}$ as follows:

$$\widehat{\delta_{\zeta(C)}}(\lambda_1\rho) = C^{N+q+2}\big(1 + \alpha_1\rho + \alpha_2\rho^2 + \cdots + \alpha_{N-1}\rho^{N-1} + \alpha_N\rho^N\big).$$

This map has N positive zeroes ρ. As a consequence, for parameter values λ that belong to the curve ζ, the map $\widehat{\delta_\lambda}$ has N positive zeroes r, that tend to zero when $\|\lambda\| \to 0$. The result follows. $\square$

If $\sum_{i=1}^m k_i \neq 0$, but for every $1 \leqslant i_1 < i_2 \leqslant m$, $k_{i_1}k_{i_2} = 0$, then without refining the class of analytic families of functions $(\delta_\lambda)_\lambda$ in (1), the absolute cyclicity also is infinite. This fact is illustrated by the following family of analytic functions $(\delta_\lambda)_\lambda$, in case $m = 2$:

$$\delta_\lambda(r) = \lambda_1 + \alpha_1\lambda_2^l r + \alpha_2\lambda_2^{l-1}r^2 + \cdots + \alpha_l\lambda_2 r^l.$$

For an appropriate choice of the constants α_i, $1 \leqslant i \leqslant l$, the cyclicity of this family, $\mathcal{N}_2(\delta_\lambda, (0^+, 0))$, is l. If we now refine this class of analytic families of functions $(\delta_\lambda)_\lambda$ to the class determined by (2), then we have the following absolute finiteness result:

Theorem 4. *Suppose that $(\delta_\lambda)_\lambda$ is an analytic family of functions with asymptotic expansion for $r \downarrow 0$,*

$$\delta_\lambda(r) = \sum_{i=1}^m \bar{A}_i\lambda_i^{k_i}r^{n_i} + O\big(r^{n_m+1}\big) \tag{7}$$

for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m$, λ near 0, $k_i \in \mathbb{N} \setminus \{0\}$, $\bar{A}_i \neq 0$, $\forall 1 \leqslant i \leqslant m$, and $n_i \in \mathbb{N}$ with $n_1 < n_2 < \cdots < n_m$. Then, its cyclicity at $(0^+, 0)$ is bounded by $k_1k_2\ldots k_m + m - 2$. As a consequence,

$$\mathcal{C}_m^{\text{abs}}(k_1, k_2, \ldots, k_m) \leqslant k_1k_2\ldots k_m + m - 2.$$

In particular, $\mathcal{C}_2^{\text{abs}}(k, l) \leqslant kl$.

Proof. The Taylor expansion of $\delta(\cdot,\lambda)$ at $r=0$, defines analytic functions $\alpha_j(\lambda)$, $j \geqslant n_m+1$, in a neighbourhood of $\lambda=0$, where $\alpha_j(\lambda)$ is the coefficient corresponding to the power r^j. Then, in the local ring of germs of analytic functions at $\lambda=0$, we consider the ideal generated by

$$\left\{\lambda_1^{k_1}, \lambda_2^{k_2}, \dots, \lambda_m^{k_m}, \alpha_j(\lambda)\colon\ j \geqslant n_m+1\right\};$$

this ideal is called the Bautin ideal associated to the analytic family of functions $(\delta_\lambda)_\lambda$. By Hilbert's base theorem, the Bautin ideal is finitely generated; in particular, we can choose a set of generators of the form:

$$\left\{\lambda_1^{k_1}, \lambda_2^{k_2}, \dots, \lambda_m^{k_m}, \alpha_{n_j}(\lambda)\colon\ m+1 \leqslant j \leqslant \mathcal{L}\right\},$$

such that $n_1 < n_2 < \cdots < n_{\mathcal{L}}$ and $\forall n_m+1 \leqslant j < n_t$:

$$\alpha_j \in \left\{\lambda_1^{k_1}, \lambda_2^{k_2}, \dots, \lambda_m^{k_m}, \alpha_j(\lambda)\colon\ n_{m+1} \leqslant j < n_t\right\}.$$

As a consequence, by a regrouping of the terms, we can write:

$$\delta(r,\lambda) = \sum_{i=1}^{m} \lambda_i^{k_i} \bar{h}_i(r,\lambda) + \sum_{i=m+1}^{\mathcal{L}} \alpha_{n_i}(\lambda) \bar{h}_i(r,\lambda), \tag{8}$$

such that the factor functions $\bar{h}_i$ have the following asymptotics for $\lambda \to 0$, $r \downarrow 0$:

$$\bar{h}_i(r,\lambda) = A_i r^{n_i} + o\left(r^{n_i}\right), \quad 1 \leqslant i \leqslant \mathcal{L},$$

for non-zero constants A_i, $1 \leqslant i \leqslant \mathcal{L}$. Next, by the multi-variate Taylor's theorem at $\lambda=0$, we find for each α_{n_i}, $m+1 \leqslant i \leqslant \mathcal{L}$ a polynomial $\varphi_i(\lambda)$ in the parameter variable $\lambda_1, \lambda_2, \dots, \lambda_m$ such that

$$\alpha_{n_i} - \varphi_i \in \mathcal{I}\left(\lambda_1^{k_1}, \dots, \lambda_m^{k_m}\right); \tag{9}$$

i.e., the analytic function $\alpha_{n_i} - \varphi_i$ can be divided in the ideal generated by $\lambda_1^{k_1}, \dots, \lambda_m^{k_m}$ and φ is a polynomial of degree $k_j - 1$ with respect to λ_j, $1 \leqslant j \leqslant m$. By again regrouping the terms in (8), we find the local division:

$$\delta(r,\lambda) = \sum_{i=1}^{m} \lambda_i^{k_i} h_i(r,\lambda) + \sum_{i=m+1}^{\mathcal{L}} \varphi_i(\lambda) h_i(r,\lambda), \tag{10}$$

with $\forall 1 \leqslant i \leqslant \mathcal{L}$:

$$h_i(r,\lambda) = A_i r^{n_i} + o\left(r^{n_i}\right), \quad 1 \leqslant i \leqslant \mathcal{L}. \tag{11}$$

By the division–derivation algorithm, we find a compact neighborhood $\mathcal{W}$ of $\lambda=0$ in $\mathbb{R}^2 \times \mathbb{R}^n$ and a neighborhood $\mathcal{V}$ of $r=0$ in $\mathbb{R}^+$ such that the function $\delta(\cdot,\lambda)$ has at most $\mathcal{L}-1$ zeroes in $\mathcal{V}$, $\forall \lambda \in \mathcal{W}$.

In other words,

$$C_m^{\text{abs}}(k_1, \dots, k_m) \leqslant \mathcal{L} - 1.$$

By Newton's diagram in $\mathbb{N}^m$, (9) and the fact that the family $(\delta_\lambda)_\lambda$ is not identically 0, it follows that

$$\mathcal{L} - m \leqslant k_1 k_2 \cdot \cdots \cdot k_m - 1,$$

and the result follows. □

Remark 5. Using a division–derivation argument, Theorem 4 can be generalized to e.g., a displacement map δ_λ having an asymptotic expansion given by

$$\delta_{(\lambda)}(r) = \sum_{i=1}^{m} \bar{A}_i(\lambda) \lambda_i^{k_i} r^{n_i} + \lambda_1^l f(r,\lambda) + O\left(r^{n_m+1}\right), \quad r \downarrow 0,$$

for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{R}^m$, $\|\lambda\| \to 0$, $n_i \in \mathbb{N}$ with $n_1 + 1 < n_2 < \cdots < n_m$ and for analytic functions $\bar{A}_i : (\mathbb{R}^m, 0) \to \mathbb{R}$ with $\bar{A}_i(0) \neq 0$, $1 \leqslant i \leqslant m$, and $l \in \mathbb{N}_1 = \mathbb{N} \setminus \{0\}$, f an analytic function such that

$$f(r,\lambda) = A(\lambda) r^{n_1+1} + o\left(r^{n_1+1}\right), \quad r \downarrow 0,$$

and $A(0) \neq 0$. Then,

$$C_m^{\text{abs}} \leqslant k_1 k_2 \cdot \cdots \cdot k_m + m - 1.$$

3. Bounds for the absolute cyclicity for $\lambda \in \mathbb{R}^2$

3.1. Lower bounds

We now investigate the absolute cyclicity in the case that the Bautin ideal is 2-dimensional, and we look for lower bounds for the absolute cyclicity. In the 2-dimensional case the parameter $\lambda \in \mathbb{R}^m$ can be expressed by analytic coordinates (a, b, ν), where $(a, b) \in \mathbb{R}^2$ and $\nu \in \mathbb{R}^{m-2}$. In what follows we will forget about the parameter variable ν, and we will simply write $\lambda = (a, b)$. However, all the results can evenly be stated for $\lambda = (a, b, \nu)$.

Theorem 6. *Suppose that $(\delta_\lambda)_\lambda$ is the analytic family of functions, $\lambda = (a, b)$, that satisfy*

$$\delta_\lambda(r) = \bar{A}_1 a^k r^{\bar{n}_1} + \bar{A}_2 b^l r^{\bar{n}_2} + O\left(r^{\bar{n}_2+1}\right), \quad r \downarrow 0,$$

for $\lambda \to 0$, for certain positive integers $\bar{n}_1 < \bar{n}_2$ and certain analytic functions $\bar{A}_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$ with $\bar{A}_1(0)\bar{A}_2(0) < 0$. Then,

$$\frac{(k+1)(l+1) - \gcd(k, l) - 1}{2} \leqslant \mathcal{C}_2^{\mathrm{abs}}(k, l) \leqslant kl.$$

In particular $\mathcal{C}_2^{\mathrm{abs}}(k, 1) = k$ and $[\frac{3k+1}{2}] \leqslant \mathcal{C}_2^{\mathrm{abs}}(k, 2) = 2k$.

Proof. What needs to be proved is the lower bound. As for any lower bound, it suffices to construct an example realizing it. Therefore, we look for a 'standard polynomial' in $\lambda = (a, b)$ with given highest order terms and with coefficients that are powers of r^2,

$$\delta(r, a, b) = a^k + b^l r^2 + g(r, a, b),$$

where $g(r, a, b) = O(r^3)$, $r \to 0$, and the function vanishes when $a = b = 0$. Generalizing the 1-parameter case, where the proof is based on the Preparation Theorem, we construct a polynomial g in powers $a^i b^j$, $0 \leqslant i \leqslant k$, $0 \leqslant j \leqslant l$ and $i + j \neq 0$. To this end we use Newton's diagram exhibiting two leading monomials, which define some natural quasi-homogeneous degree for the problem. Joining them with a line yields to a finite number of monomials below that line, with lower quasi-homogeneous degree. These correspond to the degrees of freedom available to produce the limit cycles. Then one needs to find a natural ordering of these monomials with respect to quasi-homogeneous degree so that each monomial can be used to create a limit cycle. Further in the article – after Remark 7 – we illustrate the ideas and notations in this proof in a concrete example (cf. also Fig. 1).

First we select a maximal number of powers $a^i b^j$, that are independent in the following sense; a set $\{a^{i_s} b^{j_s},\ s = 1, 2, \ldots, \mathcal{L}\}$ is independent if there exist a vector (K, L) such that the straight lines perpendicular to (K, L) and through the points (i_s, j_s), $s = 1, 2, \ldots, \mathcal{L}$, all are different. The set is ordered by the following ordering with respect to (K, L):

$$a^{i_s} b^{j_s} \prec a^{i_r} b^{j_r} \quad \Longleftrightarrow \quad K i_s + L j_s > K i_r + L j_r;$$

geometrically, the powers $a^{i_s} b^{j_s}$ are identified with the vectors (i_s, j_s); if the vector perpendicular to (K, L) through the point (i_r, j_r) lies below the one through the point (i_s, j_s), then the corresponding powers receive the reverse ordering.

We choose (K, L) such that the cardinality of the corresponding independent set $\mathcal{S}$ is maximal, $K, L \geqslant 1$, and such that $a^k, b^l \in \mathcal{S}$ with

$$Kk > Ll > Ki + Lj, \quad \text{for all } a^i b^j \in \mathcal{S}.$$

Now we construct the set S by considering the corresponding set of vectors, the so-called 'admissible' exponents. Let us denote by $\mathcal{K}$ and $\mathcal{H}$ the following sets: $\mathcal{K} = \{(i, j):\ 1 \leqslant li + kj \leqslant kl\}$ and

$$\mathcal{H} = \left\{ \frac{j_2 - j_1}{i_2 - i_1}:\ (i_1, j_1), (i_2, j_2) \in \mathcal{K},\ i_2 \geqslant i_1 \text{ and } j_2 \leqslant j_1 \right\}.$$

Geometrically, $\mathcal{H}$ corresponds to the set of all rational numbers that appear as slope of the line between two points in $\mathcal{K}$. We consider the straight line through $(0, k)$ and $(l, 0)$; then, we rotate this line clockwise through the point $(0, l)$ slightly, in such a way that we do not pass through another point of $\mathcal{K}$. Notice that the slope, say $-\mu$, of the rotated line can be chosen to be any rational number different from l/k such that

$$\frac{l}{k} < \frac{K}{L} = \mu \quad \text{and} \quad \frac{K}{L} \notin \mathcal{H}. \tag{12}$$

Furthermore, we can suppose that

$$Kk > Ll + 2. \tag{13}$$

Condition (13) is not restrictive but a sufficient condition to end up with an analytic map δ with respect to r^2, at $r = 0$; indeed, this condition can be obtained after replacing K resp. L by nK resp. nL, where $n \in \mathbb{N}_1$. In particular, we can suppose that K and L are even integers.

Set $\mathcal{K}_1 = \mathcal{K} \setminus \{(i, j)\colon\ li + kj = kl\}$, $N = \#\mathcal{K}_1$ and denote the elements of $\mathcal{K}_1$ by (i_s, j_s), $1 \leqslant s \leqslant N$, using the order introduced above with respect to (K, L):

$$(i_N, j_N) \prec (i_{N-1}, j_{N-1}) \prec \cdots \prec (i_1, j_1). \tag{14}$$

Now we define the corresponding powers n_s, $1 \leqslant s \leqslant N$. Denote $\sigma(i_s, j_s) = \sigma_s := Ki_s + Lj_s$; by construction, by (14), (12) and (13), we have

$$Ll > \sigma_1 > \sigma_2 > \cdots > \sigma_N.$$

Next, define $n_s = lL - \sigma_s + 2$, then

$$2 < n_1 < n_2 < \cdots < n_N.$$

Notice that the integers n_s, $s = 1 \ldots N$, can be supposed to be even (because K and L can be taken to be even).

Then we define the 'standard polynomial' with respect to (k, l) by

$$\delta_\pm(r, a, b) = a^k \pm b^l r^2 + \sum_{s=1}^{N} \alpha_s r^{n_s} a^{i_s} b^{j_s}. \tag{15}$$

For a good choice of the coefficients α_s, $1 \leqslant s \leqslant N$, this polynomial δ has cyclicity $N + 1$ for $(a, b) \to (0, 0)$. More concretely, this cyclicity will be attained along an algebraic curve (an mcc) of the form

$$a = C^K, \qquad b = \alpha_0 C^L, \qquad r = C\rho,$$

where C is the regular parameter, and α_0 a real constant to be determined now.

Choose real constants α_s, $0 \leqslant s \leqslant N$, such that the polynomial map

$$\rho \mapsto \alpha_0^l + \sum_{s=1}^{N} \alpha_0^{i_s} \alpha_s \rho^{n_s - 2}$$

has exactly N disjoint, strictly positive simple zeroes $\rho_1^* < \rho_2^* < \cdots < \rho_N^*$. Then, by the implicit function theorem,

$$C^{kK - lL - 2} + \rho^2 \left(\alpha_0^l + \sum_{s=1}^{N} \alpha_0^{i_s} \alpha_s \rho^{n_s - 2} \right)$$

has $N + 1$ disjoint zeroes $\rho_0(C) < \rho_1(C) < \cdots < \rho_N(C)$ that depend smoothly on C, for C sufficiently small, with

$$\rho_0(0) = 0, \qquad \rho_s(C) = \rho_s^*, \quad \forall 1 \leqslant s \leqslant N.$$

To end we prove that

$$N = \frac{(k+1)(l+1) - \gcd(k, l) - 3}{2}, \tag{16}$$

and the theorem follows.

Recall that $N = \#\mathcal{K}_1$. If $d(k, l)$ represents the number of integer couples on the segment joining $(0, k)$ and $(l, 0)$, then N can be expressed as

$$N = \frac{\#\{\text{integer couples in } [0, k] \times [0, l]\} - d(k, l)}{2} - 1 = \frac{(k+1)(l+1) - d(k, l)}{2} - 1, \tag{17}$$

where the 'minus one' corresponds with the point $(0, 0)$ that is not in the set $\mathcal{K}_1$. Now we are left with finding the number $d(k, l)$; by definition,

$$d(k, l) = \bigl\{(x, y) \in \mathbb{N}^2\colon\ lx + ky = kl,\ 0 \leqslant x \leqslant k,\ 0 \leqslant y \leqslant l\bigr\}.$$

Therefore we look for points (x, y) with integer coordinates satisfying the diophantine equation

$$lx + ky = kl, \quad \text{with } 0 \leqslant x \leqslant k.$$

Its solutions are

$$x = 0 + \frac{k}{\gcd(k, l)} t, \qquad y = l - \frac{l}{\gcd(k, l)} t, \quad \text{with } t = 0, 1, \ldots, \gcd(k, l).$$

Hence $d(k, l) = \gcd(k, l) + 1$ and formula (16) follows. $\square$

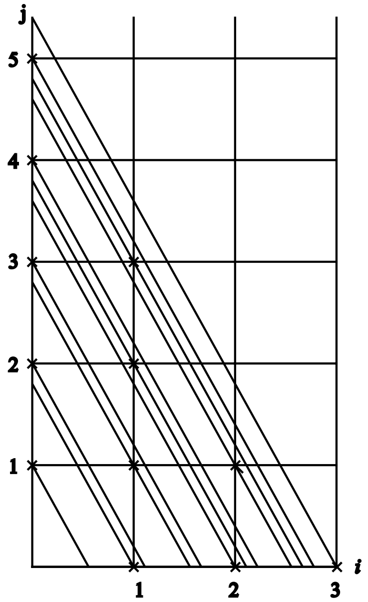

Fig. 1. Illustration for the proof of Theorem 6 in case $k = 3$ and $l = 5$.

Remark 7. The number N can also be obtained using the celebrated Pick's formula. Consider a simple polygon constructed on a grid of the plane whose coordinates are integers and such that all its vertices are points of the grid. Then

$$A = i + \frac{b}{2} - 1,$$

where A is the area of the polygon, i is the number of points of the grid located in its interior and b is the number of points of the grid on the polygon's perimeter, see [12]. By applying it to the triangle A with vertices at $(0,0)$, $(0,k)$ and $(l,0)$, the expression in (16) also follows.

To illustrate the ideas and notations of the above proof we develop a concrete example (cf. Fig. 1). Consider the case $l = 5$ and $k = 3$. Then if we take $K = 9$ and $L = 5$, conditions (12) are satisfied, i.e., $\frac{5}{3} < \frac{K}{L} < 2$, but condition (13) is not satisfied because $Ll = 25$ and $Kk = 27$. So we can consider $K = 18$ and $L = 10$. With these values the set $\mathcal{K} \setminus \{(i, j)\colon li + kj = kl\}$ consists of the points

$$(0,1),\ (0,2),\ (0,3),\ (0,4),\ (1,0),\ (1,1),\ (1,2),\ (1,3),\ (2,0),\ (2,1).$$

Note that there are precisely $N = \frac{6\times 4-4}{2} = 10$ points. These ten points, together with $(0,5)$ and $(3,0)$, give the twelve points which we use in the construction of the map $\delta_\pm$. On each of these points the weight $\sigma(i,j) = 18i + 10j$ gives a different value, and they range between $\sigma(0,1) = 10$ and $\sigma(3,0) = 54$, giving rise to twelve different parallel lines $18x + 10y = 18i + 10j$. By ordering the twelve points according to $\sigma(i_s, j_s)$ and by defining the corresponding n_s we get that the polynomial (15) is:

$$\begin{aligned}\delta_\pm(r,a,b) = {} & a^3 \pm b^5 r^2 + \alpha_1 ab^3 r^4 + \alpha_2 a^2 b r^6 + \alpha_3 b^4 r^8 + \alpha_4 ab^2 r^{14} + \alpha_5 a^2 r^{16} + \alpha_6 b^3 r^{22} \\ & + \alpha_7 abr^{24} + \alpha_8 b^2 r^{32} + \alpha_9 ar^{34} + \alpha_{10} br^{42}.\end{aligned}$$

This polynomial on the algebraic curve

$$a = C^{18}, \qquad b = \alpha_0 C^{10},$$

writes as

$$\begin{aligned}\delta_\pm(r,a,b) = {} & C^{54} \pm \alpha_0^5 C^{50} r^2 + \alpha_1 \alpha_0^3 C^{48} r^4 + \alpha_2 \alpha_0 C^{46} r^6 + \alpha_3 \alpha_0^4 C^{40} r^{12} + \alpha_4 \alpha_0^2 C^{38} r^{14} + \alpha_5 C^{36} r^{16} \\ & + \alpha_6 \alpha_0^3 C^{30} r^{22} + \alpha_7 \alpha_0 C^{28} r^{24} + \alpha_8 \alpha_0^2 C^{20} r^{32} + \alpha_9 C^{18} r^{34} + \alpha_{10} \alpha_0 C^{10} r^{42}.\end{aligned}$$

By substituting $r = C\rho$ in the expression of $\delta_\pm(r,a,b)$ it is not difficult to see that for suitable α_s it has $N + 1 = 11$ positive roots that go to zero when $C \downarrow 0$.

Remark 8. The method used to prove Theorem 6 can also be used to give a lower bound for the cyclicity $\mathcal{C}_m^{\mathrm{abs}}(k_1, k_2, \ldots, k_m)$ of the family of functions (7) studied in Theorem 4. This lower bound is

$$\frac{\prod_{j=1}^{m}(k_j+1) - d(k_1, k_2, \ldots, k_m)}{2} + m - 2,$$

where $d(k_1, k_2, \ldots, k_m)$ counts the number of points with non-negative integer coordinates that belong to the hyperplane

$$\sum_{j=1}^{m} k_1 k_2 \cdots k_{j-1} \widehat{k}_j k_{j+1} \cdots k_m x_j = \prod_{j=1}^{m} k_j.$$

Note that this formula comes from the natural extension of (17) to $\mathbb{R}^m$. The above linear diophantine equation can be studied to get more explicit expressions of $d(k_1, k_2, \ldots, k_m)$.

3.2. Detailed analysis of $\mathcal{C}_2^{\mathrm{abs}}(2, 2)$

As a corollary of Theorem 6, we get:

Corollary 9. *Suppose that δ is an analytic map with asymptotics*

$$\delta(r, \lambda) = \delta_\lambda(r) = r\big(a^2 + K_1 b^2 r^2 + O\big(r^4\big)\big), \quad r \to 0, \tag{18}$$

where $\lambda = (a, b)$ and K_1 is a non-zero real constant and $\delta(r, 0) \equiv 0$. Then,

$$3 \leqslant \mathcal{C}_2^{\mathrm{abs}}(2, 2) \leqslant 4.$$

This section gives several results that seem to indicate that the absolute cyclicity is 3.

Using Taylor's theorem with respect to (a, b, r) at $(0, 0, 0)$, we can distinguish the study of families $(\delta(\cdot, \lambda))_\lambda$, satisfying (18), in between the following 4 types: for $r \to 0$,

$$\delta(r, a, b) = r\big(a^2 + K_1 b^2 h_1(r, a, b) + K_2 ab h_2(r, a, b) + K_3 a h_3(r, a, b) + K_4 b h_4(r, a, b)\big), \tag{19}$$

$$\delta(r, a, b) = r\big(a^2 + K_1 b^2 h_1(r, a, b) + K_2 ab h_2(r, a, b) + K_3 b h_3(r, a, b) + K_4 a h_4(r, a, b)\big), \tag{20}$$

$$\delta(r, a, b) = r\big(a^2 + K_1 b^2 h_1(r, a, b) + K_2 a h_2(r, a, b) + K_3 b h_3(r, a, b)\big), \tag{21}$$

$$\delta(r, a, b) = r\big(a^2 + K_1 b^2 h_1(r, a, b) + K_2 b h_2(r, a, b) + K_3 a h_3(r, a, b)\big), \tag{22}$$

where K_2, K_3, K_4 are real constants and h_1, h_2, h_3, h_4 are analytic functions with the following asymptotics for $r \to 0$:

$$h_1(r, a, b) = r^2 + O\big(r^3\big), \qquad h_2(r, a, b) = r^{n_2} + O\big(r^{n_2+1}\big),$$
$$h_3(r, a, b) = r^{n_3} + O\big(r^{n_3+1}\big), \qquad h_4(r, a, b) = r^{n_4} + O\big(r^{n_4+1}\big),$$

for some integers $2 < n_2 < n_3 < n_4$.

By the theory based on Bautin ideal, the map δ of either type (21) or (22) can have at most 3 positive zeroes r shrinking to zero with the parameter (a, b). From the ideas of the proof of Theorem 6, we can easily construct examples of type (21) or (22) having 3 positive zeroes r shrinking to zero with the parameter. Clearly, a map of type (19) or (20) has at most 4 small positive zeroes shrinking to zero with the parameter (a, b).

In the rest of this section we study a particular case of the subcases (19) respectively (20) in which the functions h_i are monomials r^{2i}, $1 \leqslant i \leqslant 4$, and call the maps δ_λ by (F_λ) and (G_λ) respectively; if we write $S = r^2$ and $\lambda = (a, b)$, then

$$F_\lambda(S) = F(S, \lambda) = a^2 + K_1 b^2 S + K_2 ab S^2 + K_3 a S^3 + K_4 b S^4, \tag{23}$$

respectively

$$G_\lambda(S) = G(S, \lambda) = a^2 + L_1 b^2 S + L_2 ab S^2 + L_3 b S^3 + L_4 a S^4. \tag{24}$$

In Section 3.2.1 (respectively 3.2.2), we investigate what are the regions adhering at $(0, 0)$, existing of parameter values $\lambda = (a, b)$, for which the map F_λ (respectively G_λ) has a fixed number of positive zeroes. Next, we prove that for any sequence of parameters $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n \to (0, 0)$, $n \to \infty$, the map F_{λ_n} (respectively G_{λ_n}) have at most 2 positive zeroes, $\forall n$ sufficiently large. If one of the constants K_i, $1 \leqslant i \leqslant 4$ (respectively L_i, $1 \leqslant i \leqslant 4$) vanishes, then the maximal number of zeroes of F_λ and G_λ is strictly smaller than 4.

In what follows we show that for any choice of the constants K_i, $1 \leqslant i \leqslant 4$, there are at most 3 positive zeroes shrinking to 0 with the parameter (a, b). By use of the Newton polygon, we describe the bifurcation diagram of F and G near $\lambda = (0, 0)$. In this way, the study of the 2-parameter family $(F_\lambda)_\lambda$ (respectively $(G_\lambda)_\lambda$) can be reduced to the study in a 1-parameter family $(F_{\zeta(\varepsilon)})_\varepsilon$ (respectively $(G_{\zeta(\varepsilon)})_\varepsilon$); using again Newton polygons on these 1-parameter families, we find the following result:

Theorem 10. *For any fixed choice of the real constants K_i, L_i, $1 \leqslant i \leqslant 4$, the maximal number of positive zeroes $S = \xi(a,b)$ of the polynomial $F_{(a,b)}$ (resp. $G_{(a,b)}$) defined in (23) (resp. (24)) with $\xi(a,b) \downarrow 0$ when $(a,b) \to (0,0)$, is strictly smaller than 4.*

Applying Descartes' Rule, we notice that the map F_λ (respectively G_λ) can only have 4 positive zeroes for parameter values $\lambda = (a,b)$ that satisfy

$$K_1 < 0, \qquad K_2ab > 0, \qquad K_3a < 0, \qquad K_4b > 0,$$

respectively

$$L_1 < 0, \qquad L_2ab > 0, \qquad L_3b < 0, \qquad L_4a > 0. \tag{25}$$

In particular, it is necessary that

$$\operatorname{sgn}(K_1K_3) = \operatorname{sgn}(a), \tag{26}$$

respectively

$$\operatorname{sgn}(L_1L_3) = \operatorname{sgn}(b) \quad \text{and} \quad \operatorname{sgn}(L_4) = \operatorname{sgn}(a). \tag{27}$$

Hence, in the search for 4 positive zeroes we can assume that the constants K_i, $1 \leqslant i \leqslant 4$ (respectively L_i, $1 \leqslant i \leqslant 4$) are non-zero and that $K_1 < 0$ (respectively $L_1 < 0$).

3.2.1. Type (23)

The bifurcation diagram of the number of positive zeroes of F_λ with respect to the parameter $\lambda = (a,b)$ is determined by the following three curves, $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$:

$$\mathcal{L}_1 \leftrightarrow b = 0, \qquad \mathcal{L}_2 \leftrightarrow a = 0, \qquad \mathcal{L}_3 \leftrightarrow \mathcal{D}(a,b) = 0,$$

where $\mathcal{D}(a,b)$ is the discriminant of the polynomial $F_{(a,b)}$ (cf. [13]):

$$\begin{aligned}\mathcal{D}(a,b) = {} & -27K_3^4a^8 - 4K_3^3K_1^3a^3b^6 - 6K_4K_3^2K_1^2a^4b^5 - 192K_4^2K_3K_1a^5b^4 + \left(256K_4^3 + 18K_3^3K_1K_2\right)a^6b^3 \\ & + 144K_4K_2K_3^2a^7b^2 + 18K_4K_3K_1^3K_2a^2b^8 + 144K_4^2K_1^2K_2a^3b^7 + K_2^2K_3^2K_1^2a^4b^6 - 80K_4K_3K_1K_2^2a^5b^5 \\ & - 128K_4^2K_2^2a^6b^4 - 4K_2^3K_3^2a^7b^3 - 27K_4^2K_1^4b^{10} + 16K_2^4K_4a^6b^5 - 4K_2^3K_4K_1^2a^3b^8.\end{aligned}$$

For parameter values (a,b) belonging to $\mathcal{L}_1$, the polynomial $F_{(a,b)}$, looses at least one degree; therefore, when we let the parameter value cross $\mathcal{L}_1$ a zero can disappear (or appear). For parameter values (a,b) belonging to $\mathcal{L}_2$, the polynomial $F_{(a,b)}$ has a zero located in the origin; as such a positive zero can disappear, when we let the parameter value pass through $\mathcal{L}_2$. The zero-set of the discriminant, $\mathcal{L}_3$, determines the parameter values (a,b) for which $F_{(a,b)}$ has multiple zeroes. Since multiple zeroes are unstable, the bifurcation of zeroes is possible when crossing $\mathcal{L}_3$.

Therefore we study the behaviour of the graph $\mathcal{D}(a,b) = 0$ near $(a,b) = (0,0)$; in particular, we determine the asymptotics of its branches, using Newton's polygon (see [1]). The Newton polygon P is constructed from the set of points:

$$P = \big\{(8,0), (7,2), (7,3), (6,3), (6,4), (6,5), (5,4), (5,5), (4,5), (4,6), (3,6), (3,7), (3,8), (2,8), (0,10)\big\}.$$

Hence, there are two 'feasible lines' (cf. [1]), that bound the Newton polygon from below: the line through the points $(8,0)$ and $(3,6)$, and the line through the points $(3,6)$ and $(0,10)$. The slopes of the feasible lines are respectively $-6/5$ and $-4/3$. Therefore, the graph of $\mathcal{D}(a,b) = 0$ has two branches adhering at the origin, say γ_1 and γ_2; their asymptotic behaviour near the origin is given by

$$\begin{aligned}&\gamma_1 \leftrightarrow a = AC^6, \qquad b = BC^5 + O\left(C^6\right), \quad C \to 0, \\ &\gamma_2 \leftrightarrow a = EC^4, \qquad b = FC^3 + O\left(C^4\right), \quad C \to 0,\end{aligned}$$

for some non-zero constants A, B, E, F. In fact, these constants are determined by $D(\gamma_1(C)) = 0$ and $D(\gamma_2(C)) = 0$; as a consequence,

$$\begin{aligned}&-K_3^3A^3\left(27K_3A^5 + 4K_1^3B^6\right)C^{48} + O\left(C^{49}\right) = 0, \\ &-K_1^3F^6\left(27K_4^2K_1F^4 + 4K_3^3E^3\right)C^{30} + O\left(C^{31}\right) = 0\end{aligned}$$

or equivalently,

$$A^5 = -\frac{4K_1^3B^6}{27K_3} \quad \text{and} \quad E^3 = -\frac{27K_4^2K_1F^4}{4K_3^3}.$$

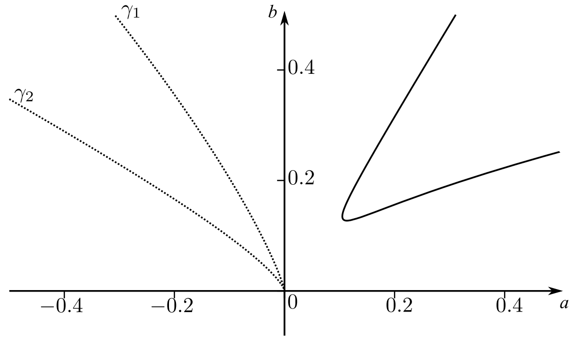

Fig. 2. Graph of $\mathcal{D}(a,b)=0$, near $(0,0)$, taking $K_1=K_3=-1$ and $K_2=K_4=1$.

From these expressions, it follows that the curves γ_1 and γ_2 lay in the half plane $\{K_1K_3a<0\}$, see Fig. 2. Now, by (26), the map $F(\cdot,a,b)$ has strictly less than 4 positive zeroes for (a,b) in this half plane. As a consequence, the half plane giving rise to possibly 4 positive zeroes, does not intersect $\mathcal{L}_3$ in a sufficiently small neighborhood of $(0,0)$. Furthermore, it follows that the distribution of the zeroes (positive, negative, imaginary) in a sufficiently small neighborhood of the origin, is stable at each of the quadrants in the half plane $\{K_1K_3a>0\}$.

Hence, to find out whether a region in parameter space realizes 4 positive zeroes, it suffices to investigate the maximum number of zeroes of $F(\cdot,a,b)$ induced by an arbitrary linear curve in each of the two quadrants $\{K_1K_3a>0,\ b>0\}$, and $\{K_1K_3a>0,\ b<0\}$.

For $a=\varepsilon\bar{a}$, $b=\varepsilon\bar{b}$, $\varepsilon\downarrow 0$, $\bar{a}\neq 0$, $\bar{b}\neq 0$, the map $F(\cdot,a,b)$ writes as

$$F(S,a,b)=\varepsilon\big(\varepsilon\bar{a}^2+\varepsilon K_1\bar{b}^2S+\varepsilon K_2\bar{a}\bar{b}S^2+K_3\bar{a}S^3+K_4\bar{b}S^4\big).$$

Using next lemma we find that $F(\cdot,\varepsilon\bar{a},\varepsilon\bar{b})$ has 2 positive zeroes, for sufficiently small $\varepsilon>0$.

Lemma 11. *Let $p_1>0$, $p_2<0$, $p_3>0$ and $p_4<0$ be fixed real constants. Then, for each sufficiently small $\varepsilon>0$, the polynomial P_ε defined by*

$$P_\varepsilon(S)=\varepsilon\big(p_1+p_2S+p_3S^2\big)+p_4S^3+S^4,$$

has exactly 2 *real zeroes which are simple and positive.*

Proof. By Descartes' Rule, the map P_ε has no negative zeroes; as a consequence, all real zeroes are positive. When ε is zero the polynomial has a triple root at 0 and a simple positive one at $S=-p_4$. When $\varepsilon>0$ is small we can find the number of positive zeroes of the polynomial P_ε by studying it as an algebraic curve in two variables (ε,S) in a neighborhood of $(0,0)$. By using again the Newton polygon we get that this curve has only one branch passing through the origin and it is given by

$$\varepsilon=At^3+O\big(t^4\big),\qquad S=Bt,\quad t\to 0,$$

where $t\in\mathbb{R}$ is a parameter and A and B satisfy $Ap_1+B^3p_4=0$. So, for $\varepsilon>0$ small, P_ε has only two real roots, which are positive and tend to 0 and $-p_4$ when $\varepsilon\downarrow 0$, as we wanted to prove. □

3.2.2. Type (24)

The bifurcation diagram of the number of positive zeroes of G_λ with respect to the parameter $\lambda=(a,b)$ is determined by the following two curves, $\mathcal{L}_1$ and $\mathcal{L}_2$:

$$\mathcal{L}_1\leftrightarrow a=0,\qquad \mathcal{L}_2\leftrightarrow \mathcal{D}'(a,b)=0,$$

where $\mathcal{D}'(a,b)$ is the discriminant of the polynomial $G(\cdot,a,b)$. For parameter values (a,b) belonging to $\mathcal{L}_1$, the polynomial $G(\cdot,a,b)$, looses at least one degree and has a zero fixed at the origin; for parameter values (a,b) belonging to $\mathcal{L}_2$, the polynomial $G(\cdot,a,b)$ has multiple zeroes. Therefore, when the parameter value crosses the set $\mathcal{L}_1\cup\mathcal{L}_2$, the number of positive zeroes can change. An analogous study as in Section 3.2.1 based on the Newton polygon, shows that the graph $\mathcal{L}_2$ in a sufficiently small neighborhood of $(0,0)$ is formed by two curves γ_1 and γ_2. Their asymptotics are given by

$$\gamma_1\leftrightarrow a=AC^4,\qquad b=BC^5+O\big(C^6\big),\quad C\to 0,$$
$$\gamma_2\leftrightarrow a=EC^5,\qquad b=FC^4+O\big(C^5\big),\quad C\to 0,$$

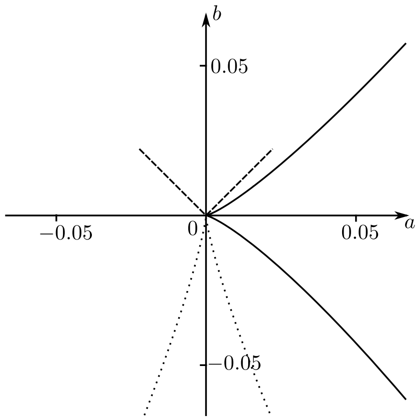

Fig. 3. Graph of $\mathcal{D}'(a,b)=0$, near $(0,0)$, where $L_1=L_3=-1$ and $L_2=L_4=1$. The continuous curve is γ_1 and the dotted one γ_2. The dashed lines are not in $\mathcal{D}'(a,b)=0$, and are used to get the number of real roots in the corresponding connected components.

for some non-zero constants A, B, E, F such that

$$A^5=\frac{27}{256}\frac{L_3^4}{L_4^3}B^4 \quad \text{and} \quad E^4=\frac{-4F^5L_1^3}{27L_3}. \tag{28}$$

From (28), it follows that γ_1 lies in the half plane $\{L_4a>0\}$ and γ_2 lies in the half plane $\{L_1L_3b<0\}$. Now, by (27), the map $G(\cdot,a,b)$ has strictly less than 4 positive zeroes for (a,b) in the half plane $\{L_1L_3b<0\}$. As a consequence, the half plane giving rise to possibly 4 positive zeroes, i.e., $\{L_1L_3b>0\}$, does not contain γ_2 in a sufficiently small neighborhood of $(0,0)$. Furthermore, it follows that the distribution of the zeroes (positive, negative, imaginary) in the half plane $\{L_1L_3b>0\}$, in a sufficiently small neighborhood of the origin, is stable in the regions bounded by γ_1 and $\mathcal{L}_1=\{a=0\}$. As in the previous case it suffices to study the number of real zeroes on a line on each of the three connected components of the half-plane $\{L_1L_3b>0\}$, minus the sets $\{a=0\}$ and γ_1, see Fig. 3. Indeed, in one of the three zones, viz. the smallest one between the two branches of γ_1, there are strictly less than four real zeroes, because it lies in the same connected component as the points which are in $\{L_1L_3b<0\}$. In short it suffices to find the number of positive real zeroes of $G(S,a,b)$ given in (24) moving along the two lines:

$$(a,b)=(-1,-1)\operatorname{sgn}(L_3)\varepsilon \quad \text{and} \quad (a,b)=(1,-1)\operatorname{sgn}(L_3)\varepsilon \tag{29}$$

for $\varepsilon>0$, small enough.

On these lines, $G(S,a,b)$ writes as

$$G(S,a,b)=\varepsilon\big[\varepsilon\big(1+L_1S\pm L_2S^2\big)-|L_3|S^3\mp\operatorname{sgn}(L_3)L_4S^4\big].$$

By using similar reasonings as in the proof of Lemma 11, we can conclude that G, restricted to these lines, has at most two positive real roots for $\varepsilon>0$ small enough, as we wanted to prove.

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