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Fourier analysis on domains in compact groups

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Abstract

Let Ω be a measurable subset of a compact group G of positive Haar measure. Let $\mu:\pi \mapsto \mu_\pi$ be a non-negative function defined on the dual space $\widehat{G}$ and let $L^2(\mu)$ be the corresponding Hilbert space which consists of elements $(\xi_\pi)_{\pi\in\operatorname{supp}\mu}$ satisfying $\sum \mu_\pi \operatorname{Tr}(\xi_\pi \xi_\pi^*) < \infty$, where ξ_π is a linear operator on the representation space of π, and is equipped with the inner product: $((\xi_\pi),(\eta_\pi)) = \sum \mu_\pi \operatorname{Tr}(\xi_\pi \eta_\pi^*)$. We show that the Fourier transform gives an isometric isomorphism from $L^2(\Omega)$ onto $L^2(\mu)$ if and only if the restrictions to Ω of all matrix coordinate functions $\sqrt{\mu_\pi}\pi_{ij}$, $\pi \in \operatorname{supp}\mu$, constitute an orthonormal basis for $L^2(\Omega)$. Finally compact connected Lie groups case is studied.

Keywords: Spectral pairs; Fourier transform; Compact groups; C^∞ vectors

1. Introduction

Let Ω be an open connected domain of finite volume in $\mathbb{R}^n$. Let $\frac{1}{i}\frac{\partial}{\partial x_j}$ be the partial derivatives with domain $C_c^\infty(\Omega)$, the space of smooth functions on Ω with compact support contained in Ω. With some regularity conditions on Ω, Fuglede showed that the partial derivatives $\frac{1}{i}\frac{\partial}{\partial x_1}, \ldots, \frac{1}{i}\frac{\partial}{\partial x_n}$ can be extended to n commuting self-adjoint operators $H_1, \ldots, H_n$ on $L^2(\Omega)$ is equivalent to the existence of a subset Λ of $\mathbb{R}^n$ such that the restrictions of the exponentials $e^{i\lambda x}$, $\lambda \in \Lambda$, to Ω form an orthogonal base for $L^2(\Omega)$ [2, Theorem I]. And this result was further studied by several authors in the cases of certain domains in $\mathbb{R}^n$ or in abelian groups [2,5,6]. More

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generally, when Ω is a measurable subset of a locally compact abelian group M and Ω has finite and positive measure, in [6, Corollary 1.11] Pedersen showed that if μ is a positive regular Borel measure on the dual group $\widehat{M}$ of M and put $\Lambda = \operatorname{supp}\mu$, then the following are equivalent:

(i) Fourier transform yields an isometric isomorphism from $L^2(\Omega)$ onto $L^2(\widehat{M}, \mu)$;
(ii) $\mu(\{\lambda\}) = 1$ for all $\lambda \in \Lambda$ and the restrictions $\lambda|_\Omega$'s, $\lambda \in \Lambda$, form an orthonormal basis for $L^2(\Omega)$.

This paper is motivated by their works. We are going to study in the case of compact groups. Throughout the paper G will denote a compact group with normalized Haar measure dx and Ω will denote a measurable subset of G of positive Haar measure. For a Borel subset E of G, write $|E|$ for its measure. The inner product on $L^2(\Omega)$ is given by

$$(f, g)_{L^2(\Omega)} = \int_\Omega f\bar{g}\,dx,$$

where $f, g \in L^2(\Omega)$. By regarding $L^2(\Omega)$ as a subspace of $L^2(G)$, i.e. put $f \equiv 0$ outside Ω for $f \in L^2(\Omega)$, we note that $\|f\|_{L^2(\Omega)} = \|f\|_{L^2(G)}$ for $f \in L^2(\Omega)$. Let $\widehat{G}$ be the dual space of G. It is well known that $\widehat{G}$ is a discrete space under the hull-kernel topology. For $f \in L^1(G)$, we recall that the Fourier transform of f is defined by

$$\widehat{f}(\pi) = \int_G f(x)\pi\left(x^{-1}\right)dx, \quad \pi \in \widehat{G}.$$

For $\pi \in \widehat{G}$, put d_π the multiplicity of π and if $\{e_i\}$ is an orthonormal basis for H_π, the matrix coordinate functions of π are given by $\pi_{ij}(x) = (\pi(x)e_j, e_i)$, $x \in G$, $i, j = 1, \ldots, d_\pi$. From now on, $\widetilde{\pi}_{ij}$ always denotes the restriction to Ω of π_{ij}. Notice that for $f \in L^2(\Omega)$ and $\pi \in \widehat{G}$, the (ij)th entry of $\widehat{f}(\pi)$ is equal to

$$\left[\widehat{f}(\pi)\right]_{ij} = (f, \widetilde{\pi}_{ji})_{L^2(\Omega)}. \tag{1.1}$$

For a non-negative function μ on $\widehat{G}$, we associate a Hilbert space $L^2(\mu)$ as follows (see [3, 28.24]): for $\pi \in \widehat{G}$, write $\mu_\pi = \mu(\pi)$ and let Λ be the support of μ, i.e. $\Lambda = \{\pi \mid \mu_\pi \neq 0\}$. Let $\mathcal{B}(H_\pi)$ be the space of bounded operators on the (finite-dimensional) Hilbert representation space H_π. We define

$$L^2(\mu) = \left\{(\xi_\pi) \in \prod_{\pi\in\Lambda} \mathcal{B}(H_\pi) \,\middle|\, \sum_{\pi\in\Lambda} \mu_\pi \operatorname{Tr}\left(\xi_\pi\xi_\pi^*\right) < \infty\right\}.$$

And the inner product on $L^2(\mu)$ is given by

$$\left((\xi_\pi), (\eta_\pi)\right)_{L^2(\mu)} = \sum_{\pi\in\Lambda} \mu_\pi \operatorname{Tr}\left(\xi_\pi\eta_\pi^*\right).$$

Then $L^2(\mu)$ becomes a Hilbert space. The following is an analogue of the Pedersen's definition [6, Definition 1.3] for non-abelian compact groups.

Definition 1.1. A measurable subset Ω of G of positive measure is called a spectral set if there exists a non-negative function μ on $\widehat{G}$ satisfying the following two conditions:

(α) for each $f \in L^2(\Omega)$, its Fourier transform $\widehat{f} \in L^2(\mu)$, namely,

$$\sum_{\pi \in \Lambda} \mu_\pi \operatorname{Tr}\big(\widehat{f}(\pi)\widehat{f}(\pi)^*\big) < \infty;$$

(β) Fourier transform "$\widehat{\cdot}$" yields an isometric isomorphism from $L^2(\Omega)$ onto $L^2(\mu)$.

In this case, (Ω, μ) is called a spectral pair and μ is called an exponent for Ω.

One of the main results in this paper, Theorem 3.3 gives a characterization of a spectral set.

On the other hand, the notion of "integrability property" for open subsets of locally compact groups was described by Fuglede [2], Jorgensen [4] and Pedersen [6, Definition 1.1]. Recall that an open subset Ω of a locally compact group G is said to have "integrability property" if there exits a unitary representation π of G on $L^2(\Omega)$ satisfying the following condition: for every x in Ω, there exist an open neighborhood E of x and an open neighborhood F of the identity of G such that $yt \in \Omega$ for all $(y,t) \in E \times F$, in addition, for any $t \in F$ and $f \in L^2(\Omega)$ we have

$$\big(\pi(t)f\big)(y) = f(yt) \quad \text{a.e. for } y \text{ in } E.$$

For convenience, we shall call such representation $(\pi, L^2(\Omega))$ locally right regular. Suppose further that G is a second countable connected Lie group and Ω is an open subset of G. In [6, Proposition 1.2], Pedersen claimed that Ω has the integrability property if and only if there exits a unitary representation π of G on $L^2(\Omega)$ satisfying $X\varphi(z) = \frac{d}{dh}\varphi(z \exp hX)|_{h=0} = \frac{d}{dh}\pi(\exp hX)\varphi|_{h=0}(z)$, for $\varphi \in C_c^\infty(\Omega)$, $z \in \Omega$ and X in the Lie algebra of G. But there is no detailed proof in his paper. Indeed, if $(\pi, L^2(\Omega))$ is a locally right regular representation, then it is easy to see that for X in the Lie algebra of G,

$$X\varphi(z) = \lim_{h\to 0} \frac{1}{h}\big(\pi(\exp hX)\varphi(z) - \varphi(z)\big) \quad \text{a.e. for } z \in \Omega$$

because G is second countable. But it seems that it is not clear whether $\varphi \in \operatorname{Dom} d\pi(X)$ for $\varphi \in C_c^\infty(\Omega)$, where the domain of $d\pi(X)$ is given by

$$\left\{\xi \in L^2(\Omega) \;\middle|\; d\pi(X)\xi = \lim_{h\to 0}\frac{1}{h}\big(\pi(\exp hX)\xi - \xi\big) \text{ exists in } L^2(\Omega)\right\}.$$

We shall show that it is in this case for a certain representation U (see (2.1)) which is associated with a spectral pair in a compact Lie group. Indeed we obtain that φ is a C^∞ vector for this associated representation U, for any φ in $C_c^\infty(\Omega)$. Furthermore, we show that if (Ω, μ) is a spectral pair in a compact Lie group, then the restrictions $\widetilde{\pi}_{ab}$'s of all matrix coordinate functions for $\pi \in \operatorname{supp} \mu$ are C^∞ vectors for U.

Remark 1.2. In view of the present results in this paper, it is plausible to extend them to a larger class of groups. Now let Υ be a measurable subset of a locally compact group K with finite non-zero Haar measure and let μ be a positive measure on the dual space $\widehat{K}$. We call (Υ, μ) a spectral pair if the Fourier transform maps from $L^1(\Upsilon) \cap L^2(\Upsilon)$ into $\int_{\operatorname{supp}\mu}^{\oplus} H_\pi \otimes H_{\overline{\pi}}\, d\mu(\pi)$ and it extends to an isometry from $L^2(\Upsilon)$ onto $\int_{\operatorname{supp}\mu}^{\oplus} H_\pi \otimes H_{\overline{\pi}}\, d\mu(\pi)$, roughly speaking, $\widehat{f}(\sigma)$ is a Hilbert–Schmidt operator on H_σ and $\int_\Upsilon f(x)\overline{g(x)}\, dx = \int_{\operatorname{supp}\mu} \operatorname{Tr}[\widehat{f}(\pi)\widehat{g}(\pi)^*]\, d\mu(\pi)$ for $f, g \in L^1(\Upsilon) \cap L^2(\Upsilon)$; $\sigma \in \operatorname{supp}\mu$ (see [1, Section 7.4] for the strict notion). For the further investigation, it may be interesting to study how to characterize spectral pairs in this general setting, for example, in the case of nilpotent Lie groups.

2. Spectral pairs

For a spectral pair (Ω, μ), we associate a pair of representations U and V of G on $L^2(\Omega)$ as follows:

$$\big(U(t)f\big)\widehat{\ }(\pi) = \pi(t)\widehat{f}(\pi), \quad \text{and} \tag{2.1}$$

$$\big(V(t)f\big)\widehat{\ }(\pi) = \widehat{f}(\pi)\pi(t)^*, \tag{2.2}$$

where $\pi \in \Lambda := \operatorname{supp}\mu$, $t \in G$ and $f \in L^2(\Omega)$.

It is easy to see that U and V both are algebraic unitary representations of G. In fact we have the following.

Proposition 2.1. *For a spectral pair (Ω, μ), with the above notation, both U and V are strongly continuous unitary representations. Moreover, we have*

$$U \cong \bigoplus_{\pi \in \Lambda} d_\pi \pi \quad \text{and} \quad V \cong \bigoplus_{\pi \in \Lambda} d_\pi \overline{\pi},$$

where $\overline{\pi}$ denotes the contragredient representation of π. Therefore V is the contragredient representation of U.

Proof. For $\pi \in \Lambda$, let $\mathcal{E}_\pi = \mathcal{B}(H_\pi)$. And the inner product on $\mathcal{E}_\pi$ is given by $(A, B)_{\mathcal{E}_\pi} = \mu_\pi \operatorname{Tr}(AB^*)$, where $A, B \in \mathcal{E}_\pi$. Define the representation u^π of G on $\mathcal{E}_\pi$ by $u^\pi(t)A = \pi(t)A$, $A \in \mathcal{E}_\pi$. Then $U \cong \bigoplus_{\pi\in\Lambda} u^\pi$. From this we see that U is strongly continuous. On the other hand, one can directly check that the character χ_{u^π} (i.e. $\chi_{u^\pi}(t) = \operatorname{Tr} u^\pi(t)$) of u^π is equal to $d_\pi \chi_\pi$. This implies that $u^\pi \cong d_\pi \pi$. Hence we have $U \cong \bigoplus_{\pi\in\Lambda} d_\pi \pi$. Similarly, if we define the representation v^π of G on $\mathcal{E}_\pi$ by $v^\pi(t) = A\pi(t)^*$, $A \in \mathcal{E}_\pi$, then we have $V \cong \bigoplus_{\pi\in\Lambda} v^\pi$ and $\chi_{v^\pi} = d_\pi \overline{\chi_\pi}$. Hence V is strongly continuous and $v^\pi \cong d_\pi \overline{\pi}$. The proof is finished. □

Lemma 2.2. *Let (Ω, μ) be a spectral pair. Let U be the associated representation of G as in* (2.1). *If E is a measurable subset of Ω and an element $t \in G$ satisfies $|Et \setminus \Omega| = 0$, namely, the set Et is almost contained in Ω, then for any $f \in L^2(\Omega)$, we have*

$$\big(U(t)f\big)(y) = f(yt) \quad \textit{a.e. for } y \textit{ in } E.$$

Furthermore, if Ω is open, then Ω has the integrability property and U is a locally right regular representation.

Proof. Let $t \in G$ be an element which satisfies $|Et \setminus \Omega| = 0$. Fix $f \in L^2(\Omega)$. Let $\varphi \in L^\infty(E)$. Since $\widehat{\varphi(\cdot\, t^{-1})}(\sigma) = \sigma(t^{-1})\widehat{\varphi}(\sigma)$ for all $\sigma \in \widehat{G}$, we see that

$$\int_E \big(U(t)f\big)(x)\overline{\varphi(x)}\,dx = \big(U(t)f, \varphi\big)_{L^2(\Omega)} = \big(f, U\big(t^{-1}\big)\varphi\big)_{L^2(\Omega)} = \big(\widehat{f}, \big(U\big(t^{-1}\big)\varphi\big)\widehat{\ }\big)_{L^2(\mu)}$$

$$= \big(\widehat{f}, \widehat{\varphi\big(\cdot t^{-1}\big)}\big)_{L^2(\mu)} = \big(f, \varphi\big(\cdot t^{-1}\big)\big)_{L^2(\Omega)}.$$

If we put $F(x) = f(x)$ for $x \in \Omega$, otherwise, set $F(x) \equiv 0$, then the above last equality becomes

$$\big(f, \varphi\big(\cdot\, t^{-1}\big)\big)_{L^2(\Omega)} = \int_G F(x)\overline{\varphi(xt^{-1})}\,dx = \int_G F(xt)\overline{\varphi(x)}\,dx = \int_E f(xt)\overline{\varphi(x)}\,dx.$$

The last equality follows from $|Et \setminus \Omega| = 0$. Therefore $(U(t)f)(y) = f(yt)$ a.e. for y in E. And the last assertion is obtained immediately. □

Remark 2.3. Following the similar arguments as in Lemma 2.2, if an element t in G satisfies $|t^{-1}E \setminus \Omega| = 0$ and V is the representation of G as in (2.2), then for any $f \in L^2(\Omega)$, we have $(V(t)f)(y) = f(t^{-1}y)$ a.e. for y in E.

3. Orthogonality relations on domains

Throughout this section, (Ω, μ) denotes a spectral pair and δ_{ij} denotes the Kronecker symbol, i.e. $\delta_{ij} = 1$ for $i = j$, otherwise, $\delta_{ij} = 0$. Let U, V be the associated representations as in (2.1) and (2.2), respectively. Before going to show the main theorem, we need the following calculations later. For $\pi, \sigma \in \operatorname{supp}\mu$ and $t \in G$, (1.1) gives

$$\big(U(t)\widetilde{\pi}_{ab}, \widetilde{\sigma}_{mn}\big)_{L^2(\Omega)} = \big[\big(U(t)\widetilde{\pi}_{ab}\big)\widehat{\ }(\sigma)\big]_{nm} = \big[\sigma(t)\widehat{\widetilde{\pi}}_{ab}(\sigma)\big]_{nm}$$
$$= \sum_r \sigma_{nr}(t)\big[\widehat{\widetilde{\pi}}_{ab}(\sigma)\big]_{rm} = \sum_r \sigma_{nr}(t)(\widetilde{\pi}_{ab}, \widetilde{\sigma}_{mr})_{L^2(\Omega)}. \tag{3.1}$$

Similarly, we have

$$\big(V(t)\widetilde{\pi}_{ab}, \widetilde{\sigma}_{mn}\big)_{L^2(\Omega)} = \sum_r \sigma_{rm}\big(t^{-1}\big)(\widetilde{\pi}_{ab}, \widetilde{\sigma}_{rn}.)_{L^2(\Omega)}. \tag{3.2}$$

Proposition 3.1. *Let (Ω, μ) be a spectral pair in a compact group G. Let Λ be the support of μ. Then for π and $\sigma \in \Lambda$, we have*

$$(\widetilde{\pi}_{ab}, \widetilde{\sigma}_{mn})_{L^2(\Omega)} = \begin{cases} \delta_{am}\delta_{bn}\mu_\pi^{-1} & \text{if } \pi \cong \sigma, \\ 0 & \text{otherwise}, \end{cases}$$

where $a, b = 1, \dots, d_\pi$ and $m, n = 1, \dots, d_\sigma$.

Proof. For $\pi, \sigma \in \Lambda$, (3.1) gives

$$\sum_r \sigma_{nr}(t)(\widetilde{\pi}_{ab}, \widetilde{\sigma}_{mr})_{L^2(\Omega)} = \big(U(t)\widetilde{\pi}_{ab}, \widetilde{\sigma}_{mn}\big)_{L^2(\Omega)} = \overline{\big(U\big(t^{-1}\big)\widetilde{\sigma}_{mn}, \widetilde{\pi}_{ab}\big)}_{L^2(\Omega)}$$
$$= \sum_s \pi_{sb}(t)(\widetilde{\pi}_{as}, \widetilde{\sigma}_{mn})_{L^2(\Omega)} \tag{3.3}$$

for all $t \in G$.

Then the orthogonality relations on compact groups [3, Theorem 27.19] implies that

$$(\widetilde{\pi}_{ab}, \widetilde{\sigma}_{mn})_{L^2(\Omega)} = 0 \tag{3.4}$$

for any $\pi, \sigma \in \Lambda$ with $\pi \ncong \sigma$.

As $\pi \cong \sigma$, fix $a, b, m, n = 1, \dots, d_\pi$. (3.3) infers that

$$\sum_r \pi_{nr}(t)(\widetilde{\pi}_{ab}, \widetilde{\pi}_{mr})_{L^2(\Omega)} = \sum_r \pi_{rb}(t)(\widetilde{\pi}_{ar}, \widetilde{\pi}_{mn})_{L^2(\Omega)} \tag{3.5}$$

for all $t \in G$.

By observing the coefficient of $\pi_{nn}(t)$ in (3.5), using the orthogonality relations on compact groups again, we see that

$$(\widetilde{\pi}_{ab}, \widetilde{\pi}_{mn})_{L^2(\Omega)} = 0 \tag{3.6}$$

for $\pi \in \Lambda$ and $n \neq b$.

By the same argument for the representation V, (3.2) implies that

$$\sum_r \pi_{rm}\big(t^{-1}\big)(\widetilde{\pi}_{ab}, \widetilde{\pi}_{rn})_{L^2(\Omega)} = \big(V(t)\widetilde{\pi}_{ab}, \widetilde{\pi}_{mn}\big)_{L^2(\Omega)} = \overline{\big(V\big(t^{-1}\big)\widetilde{\pi}_{mn}, \widetilde{\pi}_{ab}\big)}_{L^2(\Omega)}$$
$$= \sum_r \pi_{ar}\big(t^{-1}\big)(\widetilde{\pi}_{ra}, \widetilde{\pi}_{mn})_{L^2(\Omega)} \tag{3.7}$$

for all $t \in G$.

Hence by observing the coefficient of $\pi_{mm}(t^{-1})$ in (3.7), if $m \neq a$, then

$$(\widetilde{\pi}_{ab}, \widetilde{\pi}_{mn})_{L^2(\Omega)} = 0. \tag{3.8}$$

Combing (3.6) and (3.8), we assert that

$$(\widetilde{\pi}_{ab}, \widetilde{\pi}_{mn})_{L^2(\Omega)} = 0 \tag{3.9}$$

for $\pi \in \Lambda$ and $(ab) \neq (mn)$.

Finally it remains to show that $\|\widetilde{\pi}_{ab}\|^2_{L^2(\Omega)} = \mu_\pi^{-1}$ for $\pi \in \Lambda$ and $a, b = 1, \dots, d_\pi$. For $\pi \in \Lambda$ and $a, b = 1, \dots, d_\pi$, by the surjectivity of the Fourier transform $\widehat{\cdot} : L^2(\Omega) \to L^2(\mu)$, we can find a function $\psi \in L^2(\Omega)$ so that for $\tau \in \Lambda$, we have

$$\big[\widehat{\psi}(\tau)\big]_{mn} = \begin{cases} \delta_{bm}\delta_{an} & \text{if } \pi \cong \tau, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have $(\psi, \widetilde{\pi}_{ab})_{L^2(\Omega)} = \widehat{\psi}(\pi)_{ba} = 1$. This implies that $\widetilde{\pi}_{ab} \neq 0$ in $L^2(\Omega)$. On the other hand, the isometric property of the Fourier transform, (1.1), (3.4), and (3.9) imply:

$$\|\widetilde{\pi}_{ab}\|^2_{L^2(\Omega)} = \|\widehat{\widetilde{\pi}}_{ab}\|^2_{L^2(\mu)} = \sum_{\sigma \in \Lambda} \mu_\sigma \operatorname{Tr}\big(\widehat{\widetilde{\pi}}_{ab}(\sigma)\widehat{\widetilde{\pi}}_{ab}(\sigma)^*\big) = \mu_\pi \operatorname{Tr}\big(\widehat{\widetilde{\pi}}_{ab}(\pi)\widehat{\widetilde{\pi}}_{ab}(\pi)^*\big)$$
$$= \mu_\pi \sum_{r,s} \big|(\widetilde{\pi}_{ab}, \widetilde{\pi}_{sr})_{L^2(\Omega)}\big|^2 = \mu_\pi \|\widetilde{\pi}_{ab}\|^4_{L^2(\Omega)}.$$

Since $\widetilde{\pi}_{ab} \neq 0$, $\|\widetilde{\pi}_{ab}\|^2_{L^2(\Omega)} = \mu_\pi^{-1}$. This completes the proof. □

Corollary 3.2. *Let Ω be a spectral set. If μ is an exponent for Ω, then we have $d_\pi \leqq \mu_\pi$, for all π in the support of μ.*

Proof. For $\pi \in \operatorname{supp}\mu$, Proposition 3.1 and the orthogonality relations on compact groups assert that ${\mu_\pi}^{-1/2} = \|\widetilde{\pi}_{ab}\|_{L^2(\Omega)} \leqq \|\pi_{ab}\|_{L^2(G)} = {d_\pi}^{-1/2}$. Therefore we have $d_\pi \leqq \mu_\pi$. □

Theorem 3.3. *Let Ω be a measurable subset of a compact group G of positive measure. Let μ be a non-negative function on $\widehat{G}$. Then (Ω, μ) is a spectral pair if and only if the set $\{\sqrt{\mu_\pi}\widetilde{\pi}_{ab} \mid \pi \in \operatorname{supp}\mu;\ a, b = 1, \ldots, d_\pi\}$ forms an orthonormal base for $L^2(\Omega)$.*

Proof. Suppose that (Ω, μ) is a spectral pair. Put $\Lambda = \operatorname{supp}\mu$. Proposition 3.1 shows that the set $\{\sqrt{\mu_\pi}\widetilde{\pi}_{ab} \mid \pi \in \operatorname{supp}\mu;\ a, b = 1, \ldots, d_\pi\}$ forms an orthonormal subset of $L^2(\Omega)$. Hence we need to show that this set is total in $L^2(\Omega)$. In fact, if $f \in L^2(\Omega)$ and $(f, \widetilde{\pi}_{ab})_{L^2(\Omega)} = 0$, for all $\pi \in \Lambda$, and $a, b = 1, \ldots, d_\pi$, then (1.1) asserts that $\widehat{f}(\pi)_{ab} = 0$, for all $\pi \in \Lambda$ and $a, b = 1, \ldots, d_\pi$. Then $f = 0$ follows from the injectivity of the Fourier transform. Conversely, from the assumption, we can directly check that $(\widehat{\widetilde{\pi}}_{ab}, \widehat{\widetilde{\sigma}}_{jk})_{L^2(\mu)} = (\widetilde{\pi}_{ab}, \widetilde{\sigma}_{jk})_{L^2(\Omega)}$ for π, σ in Λ. Hence the Fourier transform is an isometry from $L^2(\Omega)$ into $L^2(\mu)$. And (1.1) implies that this transformation is surjective. □

Remark 3.4. Write $[\pi]$ for the unitary equivalence class of π and for a set of irreducible unitary representations $\breve{\Lambda}$ of G, put $[\breve{\Lambda}] = \{[\pi] \mid \pi \in \breve{\Lambda}\}$. Going back the construction of $L^2(\mu)$, strictly speaking, we select a fixed element in each $[\pi] \in \operatorname{supp}\mu$. Nevertheless, it is easy to see that up to unitary equivalence, $L^2(\mu)$ does not depend on such selections. Hence the definition of a spectral pair (Ω, μ) depends only on the unitary equivalence class of $[\pi]$ in $\operatorname{supp}\mu$. From this observation and Theorem 3.3, now for the sets $\breve{\Xi}$ and $\breve{\Lambda}$ of irreducible unitary representations of G with $[\breve{\Xi}] = [\breve{\Lambda}]$, we can assert that if for each element $\pi \in \breve{\Lambda}$, there is $c_\pi > 0$ such that $\{c_\pi \widetilde{\pi}_{ij} \mid \pi \in \breve{\Lambda};\ i, j = 1, \ldots, d_\pi\}$ forms an orthonormal basis for $L^2(\Omega)$, then Ξ shares the same property. Obviously, this assertion holds for the case of abelian groups, but it seems that this is not clear for the non-abelian case without employing Theorem 3.3. We give the following informative example of the non-abelian case.

Example 3.5. Let $G = SU(2)$, that is,

$$G = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \,\middle|\, |a|^2 + |b|^2 = 1 \right\}.$$

We first recall some basic properties of the representation theory of G. The details can be found in [1, Section 5.4]. For convenience, write

$$U_{a,b} = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in G.$$

We identify G with the unit sphere S^3 in $\mathbb{C}^2$ via $U_{a,b} \leftrightarrow (a,b)$. Then the normalized Haar measure on G is given by the normalized surface measure σ on S^3. Following the proof of [1, Lemma 5.32], one can directly check that for any Borel measurable subset E of S^3, if f is a homogeneous polynomial function in complex variables $z, \bar{z}; w, \bar{w}$ of degree $m \in \mathbb{N}$, i.e. $f = \sum c_{\alpha\beta\gamma\delta} z^\alpha \bar{z}^\beta w^\gamma \bar{w}^\delta$; $\alpha + \beta + \gamma + \delta = m$, then

$$\int_E f(Z')\,d\sigma(Z') = \frac{1}{\pi^2 \Gamma\left(\frac{1}{2}m + 2\right)} \int_{\widetilde{E}} f(Z) e^{-\|Z\|^2}\, d^4 Z, \tag{3.10}$$

where $\widetilde{E} = \{rZ' \mid Z' \in E,\ r \geqq 0\}$ and $d^4 Z$ denotes the Lebesgue measure on $\mathbb{C}^2$. For $m \in \mathbb{N}$, let H_m be the space of homogeneous polynomials in two complex variables of degree m and it is endowed with the inner product $(\xi, \eta) := \int_{S^3} \xi \bar{\eta}\, d\sigma$. Let π_m be the representation of G on H_m given by the natural action of G on $\mathbb{C}^2$. Then $\widehat{G} = \{[\pi_m] \mid m \in \mathbb{N}\}$. Under a suitable choice of an orthonormal basis for H_m, the (jk)th matrix coordinate function $\pi_m^{jk}(U_{a,b})$ of π_m is the linear combination of $a^{m-k-l} \bar{a}^h b^l \bar{b}^{k-h}$, where $0 \leqq h \leqq k$, $0 \leqq l \leqq m-k$ and $h + l = j$. In particular, $\pi_m^{00}(U_{a,b}) = a^m$. And we have $\|\pi_m^{jk}\|_{L^2(G)}^2 = \frac{1}{m+1}$ since $\dim H_m = m+1$.

Now let $\Omega = \{U_{a,b} \mid \operatorname{Im} a \geqq 0\}$. We claim that Ω is a spectral set. Note that we have $\int_{\mathbb{C}} z^r \bar{z}^s e^{-|z|^2}\, dz = 0$ if $r \neq s$ and $\int_{\mathbb{C}} z^p \bar{z}^q e^{-|z|^2}\, dz = 2\int_{\{\operatorname{Im} z \geqq 0\}} z^p \bar{z}^q e^{-|z|^2}\, dz$ if $p - q \in 2\mathbb{Z}$. From this and Eq. (3.10), a direct calculation shows that for $m, m_1 \in \mathbb{N}$ with $m - m_1 \in 2\mathbb{Z}$ and $0 \leqq j, k \leqq m$, $0 \leqq j_1, k_1 \leqq m_1$ then

$$\begin{aligned} &\left(a^{m-k-l} \bar{a}^h b^l \bar{b}^{k-h}, a^{m_1-k_1-l_1} \bar{a}^{h_1} b^{l_1} \bar{b}^{k_1-h_1}\right)_{L^2(G)} \\ &\quad = 2\left(a^{m-k-l} \bar{a}^h b^l \bar{b}^{k-h}, a^{m_1-k_1-l_1} \bar{a}^{h_1} b^{l_1} \bar{b}^{k_1-h_1}\right)_{L^2(\Omega)}, \end{aligned}$$

where $0 \leqq h \leqq k$, $0 \leqq l \leqq m-k$, $h + l = j$ and $0 \leqq h_1 \leqq k_1$, $0 \leqq l_1 \leqq m_1 - k_1$, $h_1 + l_1 = j_1$. This implies that $(\pi_m^{jk}, \pi_{m_1}^{j_1k_1})_{L^2(G)} = 2(\widetilde{\pi}_m^{jk}, \widetilde{\pi}_{m_1}^{j_1k_1})_{L^2(\Omega)}$ whenever $m - m_1 \in 2\mathbb{Z}$. Therefore $\{\widetilde{\pi}_{2m}^{jk} \mid m \geqq 0;\ j,k = 0,1,\ldots,2m\}$ and $\{\widetilde{\pi}_{2m+1}^{jk} \mid m \geqq 0;\ j,k = 0,1,\ldots,2m+1\}$ both are orthogonal subsets of $L^2(\Omega)$, moreover,

$$\|\widetilde{\pi}_m^{jk}\|_{L^2(\Omega)}^2 = \frac{1}{2(m+1)}.$$

Let

$$V_0 = \overline{\operatorname{span}}\left\{\widetilde{\pi}_{2m}^{jk} \mid m \geqq 0;\ j,k = 0,1,\ldots,2m\right\} \quad \text{and}$$
$$V_1 = \overline{\operatorname{span}}\left\{\widetilde{\pi}_{2m+1}^{jk} \mid m \geqq 0;\ j,k = 0,1,\ldots,2m+1\right\}.$$

Note that for $f \in V_0$, since $\{\widetilde{\pi}_{2m}^{jk} \mid m \geqq 0;\ j,k = 0,1,\dots,2m\}$ is an orthogonal set and

$$\|\widetilde{\pi}_{2m}^{jk}\|_{L^2(\Omega)}^2 = \frac{1}{2(2m+1)},$$

we have

$$\|f\|_{L^2(\Omega)}^2 = \sum_m \sum_{j,k} 2(2m+1)\left|\left(f, \widetilde{\pi}_{2m}^{jk}\right)_{L^2(\Omega)}\right|^2. \tag{3.11}$$

On the other hand, we may regard $L^2(\Omega)$ as a subspace of $L^2(G)$. Peter–Weyl theorem implies that

$$\begin{aligned}\|f\|_{L^2(\Omega)}^2 &= \|f\|_{L^2(G)}^2 \\ &= \sum_m \sum_{j,k} (2m+1)\left|\left(f, \pi_{2m}^{jk}\right)_{L^2(G)}\right|^2 + \sum_m \sum_{j,k} (2m+2)\left|\left(f, \pi_{2m+1}^{jk}\right)_{L^2(G)}\right|^2.\end{aligned}$$

Thus we have

$$\sum_m \sum_{j,k} (2m+1)\left|\left(f, \widetilde{\pi}_{2m}^{jk}\right)_{L^2(\Omega)}\right|^2 = \sum_m \sum_{j,k} (2m+2)\left|\left(f, \widetilde{\pi}_{2m+1}^{jk}\right)_{L^2(\Omega)}\right|^2 \tag{3.12}$$

because $(f, \pi_n^{pq})_{L^2(G)} = (f, \widetilde{\pi}_n^{pq})_{L^2(\Omega)}$. Equations (3.11) and (3.12) yield

$$\|f\|_{L^2(\Omega)}^2 = \sum_m \sum_{j,k} 2(2m+2)\left|\left(f, \widetilde{\pi}_{2m+1}^{jk}\right)_{L^2(\Omega)}\right|^2.$$

From this and $\|\widetilde{\pi}_{2m+1}^{jk}\|_{L^2(\Omega)}^2 = \frac{1}{2(2m+2)}$, we deduce that $f \in V_1$. Similarly, if $f \in V_1$, then Eq. (3.12) still holds. Consequently, $V_0 = V_1$. In particular, $\widetilde{\pi}_{2m+1}^{jk} \in V_0$ and $\widetilde{\pi}_{2m}^{jk} \in V_1$ for all m. It follows that $\{\widetilde{\pi}_{2m} \mid m \geqq 0\}$ and $\{\widetilde{\pi}_{2m+1} \mid m \geqq 0\}$ both are orthogonal bases for $L^2(\Omega)$. By Theorem 3.3 and Remark 3.4, we can now conclude that if we put $\mu^{\mathrm{ev}}(\pi_{2m}) = 2(2m+1)$ and $\mu^{\mathrm{ev}}(\pi_{2m+1}) = 0$, then $(\Omega, \mu^{\mathrm{ev}})$ is a spectral pair. Similarly, if we set $\mu^{\mathrm{odd}}(\pi_{2m+1}) = 2(2m+2)$ and $\mu^{\mathrm{odd}}(\pi_{2m}) = 0$, then $(\Omega, \mu^{\mathrm{odd}})$ is also a spectral pair. Besides μ^{ev} and μ^{odd}, there is no other exponent for Ω because $(\widetilde{\pi}_{2m}^{00}, \widetilde{\pi}_{2n+1}^{00})_{L^2(\Omega)} = (a^{2m}, a^{2n+1})_{L^2(\Omega)} \neq 0$.

Corollary 3.6. *Let (Ω, μ) be a spectral pair in a compact group G and let U be the associated representation of G on $L^2(\Omega)$ as in* (2.1). *Then for $\pi \in \operatorname{supp}\mu$; $a, b = 1, \dots, d_\pi$ and $x \in G$, we have*

$$U(x)\widetilde{\pi}_{ab} = \sum_{s=1}^{d_\pi} \pi_{sb}(x)\widetilde{\pi}_{as}.$$

Proof. Let $\pi \in \operatorname{supp}\mu$. Applying Proposition 3.1, for $\rho \in \operatorname{supp}\mu$, we have

$$\left[\widehat{\widetilde{\pi}}_{ab}(\rho)\right]_{mn} = \begin{cases} \delta_{an}\delta_{bm}\mu_\pi^{-1} & \text{if } \pi \cong \rho, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we have

$$\begin{aligned}\big(U(x)\widetilde{\pi}_{ab}, \widetilde{\sigma}_{rs}\big)_{L^2(\Omega)} &= \big(\big(U(x)\widetilde{\pi}_{ab}\big)\widehat{\ }, \widehat{\widetilde{\sigma}}_{rs}\big)_{L^2(\mu)} \\ &= \begin{cases} \delta_{ar}\mu_\pi^{-1}\pi(x)_{sb} & \text{if } \pi \cong \sigma, \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

for $x \in G$ and $\sigma \in \operatorname{supp}\mu$, $r, s = 1, \ldots, d_\sigma$. Then the result follows from Theorem 3.3. □

Proposition 3.7. *If Ω is an almost right G-invariant* (*respectively almost left G-invariant*) *spectral subset, that is, $|\Omega \vartriangle \Omega t| = 0$* (*respectively $|t\Omega \vartriangle \Omega| = 0$*) *for all $t \in G$, then there exists a unique exponent for Ω.*

Proof. Let μ, ν be the exponents for Ω. Suppose that Ω is almost right invariant. Let U^μ, U^ν be the corresponding associated representations of G on $L^2(\Omega)$ as in (2.1). Then Lemma 2.2 implies that for $f \in L^2(\Omega)$ and $t \in G$ we have $(U^\mu(t)f)(y) = (U^\nu(t)f)(y) = f(yt)$ for a.e. y in Ω. Thus we have $U^\mu = U^\nu$. Proposition 2.1 implies that $\operatorname{supp}\mu = \operatorname{supp}\nu$. Now for $\pi \in \operatorname{supp}\mu = \operatorname{supp}\nu$, by Proposition 3.1, we have $\mu_\pi^{-1} = \nu_\pi^{-1} = \|\widetilde{\pi}_{ab}\|^2_{L^2(\Omega)}$. Hence, $\mu = \nu$. Similarly if Ω is almost left invariant and V^μ, V^ν are the corresponding representations as in (2.2), then by Remark 2.3, we have $V^\mu = V^\nu$. Hence the result follows from the similar arguments as in the almost right invariance case. □

In general, the uniqueness of the exponents for a given spectral set does not hold. We have Example 3.5 and the following example to show this fact.

Example 3.8. Let G be the circle group and let $\Omega = \{e^{i2\pi\theta} \mid 0 \leqq \theta \leqq 1/2\}$. If we let $\chi_m(z) = z^m$ for $z \in G$ and $m \in \mathbb{Z}$, then we can directly check that $(\widetilde{\chi}_{2l+1}, \widetilde{\chi}_{2k+1})_{L^2(\Omega)} = (\widetilde{\chi}_{2l}, \widetilde{\chi}_{2k})_{L^2(\Omega)} = \frac{1}{2}\delta_{lk}$ and $(\widetilde{\chi}_{2l+1}, \widetilde{\chi}_{2k})_{L^2(\Omega)} = \frac{i}{\pi(2l-2k+1)}$ for $l, k \in \mathbb{Z}$. Put $\mu_{2m} = 2$ and $\mu_{2m+1} = 0$, for $m \in \mathbb{Z}$. We first claim that (Ω, μ) is a spectral pair. In fact, we have

$$\bigg\|\widetilde{\chi}_{2l+1} - \sum_{-m \leqq k \leqq m} 2(\widetilde{\chi}_{2l+1}, \widetilde{\chi}_{2k})_{L^2(\Omega)}\widetilde{\chi}_{2k}\bigg\|^2_{L^2(\Omega)} = \frac{1}{2} - \sum_{-m \leqq k \leqq m} \frac{2}{\pi^2(2l-2k+1)^2}$$

for all $l, m \in \mathbb{Z}$. From this we obtain that $\widetilde{\chi}_{2l+1}$ lies in the closed linear span of $\{\widetilde{\chi}_{2k} \mid k \in \mathbb{Z}\}$ because $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$. Hence if $f \in L^2(\Omega)$ satisfies $(f, \widetilde{\chi}_{2k})_{L^2(\Omega)} = 0$ for all $k \in \mathbb{Z}$, then we have $(f, \widetilde{\chi}_{2l+1})_{L^2(\Omega)} = 0$, for all $\ell \in \mathbb{Z}$. Therefore in this case, we have $(f, \chi_m)_{L^2(G)} = (f, \widetilde{\chi}_m)_{L^2(\Omega)} = 0$, for all $m \in \mathbb{Z}$. According to the Peter–Weyl theorem, f must be 0. Therefore the set $\{\sqrt{2}\widetilde{\chi}_{2k} \mid k \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\Omega)$. Hence (Ω, μ) is a spectral pair follows from Theorem 3.3. On the other hand, if we put $\nu_{2m+1} = 2$ and $\nu_{2m} = 0$ for $m \in \mathbb{Z}$, then by the same argument as above, (Ω, ν) is also a spectral pair.

4. Lie groups case

In this section, we are going to investigate the spectral pairs in compact Lie groups. We refer to [8, Section X.1], [9, Chapter 0] and [7] for the basic tools that will be used later. When L is a connected Lie group (not necessarily compact) with its Lie algebra $\mathfrak{l}$ and Ω is an open subset of L, for $X \in \mathfrak{l}$, then iX may be regarded as a symmetric operator in $L^2(\Omega)$ with domain $C_c^\infty(\Omega)$. It is known that for any strongly unitary representation (π, H_π) of L, the infinitesimal generator $d\pi(X)$ of the one parameter unitary group $h \mapsto \exp hX$, i.e. $d\pi(X) = \frac{d}{dh}\pi(\exp hX)|_{h=0}$, is a skew-adjoint operator [9, Chapter 0, Theorem 1.5]. Also recall that a vector $\xi \in H_\pi$ is said to be a C^∞ vector for π if the map $x \mapsto \pi(x)\xi$ is a C^∞ map from L to H_π. It is equivalent to say that the map $x \mapsto (\pi(x)\xi, \eta)$ is a C^∞ map on L, for any $\eta \in H_\pi$ [9, Chapter 0, Lemma 3.1]. It is known that the space of all C^∞ vectors for π is contained in the domain of $d\pi(X)$ for any $X \in \mathfrak{l}$ [9, Chapter 0, Proposition 2.6]. Moreover, the space of all C^∞ vectors for π is dense in H_π [9, Chapter 0, p. 11]. In particular, if H_π is of finite dimension, then every vector in H_π is a C^∞ vector. On the other hand, we shall use the following result of Jorgensen later. In [4, Theorem 2], Jorgensen showed that for an open connected subset Ω of L, if Ω has finite measure and ρ is a strongly continuous unitary representation of L on $L^2(\Omega)$ such that ρ is multiplicative, namely, $\rho(x)(fg) = (\rho(x)f)(\rho(x)g)$ for $x \in G$ and $f, g \in L^2(\Omega)$ with $fg \in L^2(\Omega)$, and $d\rho(X)$ is an extension of X for any $X \in \mathfrak{l}$, then Ω is a fundamental domain for a discrete subgroup of L.

Theorem 4.1. *Assume that G is a second countable compact connected Lie group and Ω is an open spectral subset of G with an exponent μ. Let U be the associated representation of G on $L^2(\Omega)$ as before. We have*

(i) *If $\varphi \in C_c^\infty(\Omega)$, then φ is a C^∞-vector for U.*
(ii) *For X in the Lie algebra $\mathfrak{g}$ of G, then the operator $dU(iX)$ is an self-adjoint extension of the vector field iX.*
(iii) *If Ω is connected and U is multiplicative, then Ω is a fundamental domain for a finite subgroup of G.*
(iv) *For any $\pi \in \operatorname{supp}\mu$ and $a, b = 1, \ldots, d_\pi$, then $\widetilde{\pi}_{ab}$ is a C^∞ vector for U. Consequently, $\widetilde{\pi}_{ab}$ lies in the domain of $dU(X)$ for any $X \in \mathfrak{g}$.*

Proof. (i) Let $\varphi \in C_c^\infty(\Omega)$ and let $X \in \mathfrak{g}$. Choose finitely many open subsets $\Omega_1, \ldots, \Omega_N$ of Ω which cover the support of φ, and open symmetric neighborhoods $\widetilde{W}_1, \ldots, \widetilde{W}_N$ of the identity of G, such that $\Omega_i \widetilde{W}_i \subseteq \Omega$, for all $i = 1, \ldots, N$. Using the partition of unity for the cover $\{\Omega_1, \ldots, \Omega_N\}$, we can decompose φ as $\sum_{i=1}^N \varphi_i$ with $\varphi_i \in C_c^\infty(\Omega_i)$, $i = 1, \ldots, N$. We now fix some Ω_i. Take a symmetric open neighborhood W_i of the identity such that $W_i \subseteq \widetilde{W}_i$ and $(\operatorname{supp}\varphi_i)W_i \subseteq \Omega_i$. We claim that the map $y \mapsto U(y)\varphi_i$ is a C^∞ map from W_i to $L^2(\Omega_i)$. In fact, recall that for $X \in \mathfrak{g}$, we have

$$X\varphi_i(z) = \lim_{h\to 0} \frac{1}{h}\big(\varphi_i(z\exp hX) - \varphi_i(z)\big) \quad \text{for } z \in \Omega_i.$$

Now fix $y \in W_i$. Regarding $C_c^\infty(\Omega_i)$ as a subspace of $C^\infty(G)$, the Taylor theorem implies that there exits $\varepsilon > 0$ and a bounded function $K(h, z)$ for $|h| < \varepsilon$ and $z \in G$ satisfy the conditions

$$y\exp hX \in W_i \quad \text{and} \quad \varphi_i(x\exp hX) = \varphi_i(x) + hX\varphi_i(x) + h^2K(h, x)$$

for all x in G. Therefore for $|h| < \varepsilon$, Lemma 2.2 implies that

$$U(y\exp hX)\varphi_i(z) = \varphi_i(zy\exp hX) \quad \text{a.e. for } z \in \Omega_i.$$

For $y \in W_i$, put $R(y)\varphi_i(z) = \varphi_i(zy)$, $z \in \Omega_i$. Then for $0 < |h| < \varepsilon$, the supports of the mappings $R(y)X\varphi_i$, $R(y\exp hX)\varphi_i$ and $R(y)\varphi_i$ are all contained in Ω_i. Hence we have

$$\begin{aligned}
&\left\| R(y)X\varphi_i - \frac{1}{h}\big(U(y\exp hX)\varphi_i - U(y)\varphi_i\big)\right\|_{L^2(\Omega_i)} \\
&\quad = \left\| R(y)X\varphi_i - \frac{1}{h}\big(R(y\exp hX)\varphi_i - R(y)\varphi_i\big)\right\|_{L^2(\Omega_i)} \\
&\quad = \left\| R(y)X\varphi_i - \frac{1}{h}\big(R(y\exp hX)\varphi_i - R(y)\varphi_i\big)\right\|_{L^2(G)} \\
&\quad = \left\| X\varphi_i - \frac{1}{h}\big(R(\exp hX)\varphi_i - \varphi_i\big)\right\|_{L^2(G)} \\
&\quad \leqq h^2 \sup\big|K(h,x)\big|.
\end{aligned}$$

Therefore we obtain that

$$\frac{d}{dh}U(y\exp hX)\varphi_i\bigg|_{h=0} = R(y)X\varphi_i \quad \text{in } L^2(\Omega_i) \text{ for all } y \in W_i \text{ and } X \in \mathfrak{g}.$$

Hence the map $y \mapsto U(y)\varphi_i$ is a C^1 map from W_i to $L^2(\Omega_i)$. To repeat the above arguments, we can show that this map is of class C^∞. Now take $W = W_1 \cap \cdots \cap W_N$. We obtain that the map $y \mapsto U(y)\varphi = \sum_{i=1}^N U(y)\varphi_i$ is a C^∞ map from W to $L^2(\Omega)$. Since any bounded linear map is smooth, so for any $x_0 \in G$, the composition of the maps

$$y \overset{x_0\cdot}{\longmapsto} x_0^{-1}y \longmapsto U\big(x_0^{-1}y\big)\varphi \overset{U(x_0)\cdot}{\longmapsto} U(y)\varphi$$

is smooth on x_0W. It follows that φ is a C^∞ vector for U.

(ii) Since $dU(X)$ is a skew-adjoint operator on $L^2(\Omega)$, for $X \in \mathfrak{g}$, it remains to show that $dU(X)$ is an extension of X. Fix $X \in \mathfrak{g}$. For non-zero $h \in \mathbb{R}$ and $\varphi \in C_c^\infty(\Omega)$, put $Q_h\varphi = \frac{1}{h}(U(\exp hX)\varphi - \varphi)$. The second countability of G and Lemma 2.2 assert that

$$\lim_{h\to 0} Q_h\varphi(z) = X\varphi(z) \quad \text{a.e. for } z \in \Omega.$$

Part (i) implies that the space $C_c^\infty(\Omega)$ is contained in the domain of $dU(X)$, that is, the limit

$$\lim_{h\to 0}\frac{1}{h}\big(U(\exp hX)\varphi - \varphi\big) \quad \text{exists in } L^2(\Omega) \text{ for any } \varphi \in C_c^\infty(\Omega).$$

Hence, $dU(X)\varphi = X\varphi$ in $L^2(\Omega)$.

(iii) This is an immediate consequence of Jorgensen's result and of part (ii).

(iv) Let $\pi \in \operatorname{supp}\mu$ and $a, b = 1, \ldots, d_\pi$. By Corollary 3.6, we have shown that $U(x)\widetilde{\pi}_{ab} = \sum_{s=1}^{d_\pi} \pi_{sb}(x)\widetilde{\pi}_{as}$ for $x \in G$. Also notice that the map $x \mapsto \pi_{sb}(x)$ is of class C^∞ on G for $s, b = 1, \ldots, d_\pi$ because π is a finite-dimensional representation. Therefore $\widetilde{\pi}_{ab}$ is a C^∞-vector for U. □

Remark 4.2. In view of the proof of Theorem 4.1, in fact we have shown that the results (i)–(iii) of the theorem are still held when L is a connected Lie group (not necessarily compact) and $(\pi, L^2(\Omega))$ is a locally right regular representation of L.

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