Littlewood Pisot numbers

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Abstract

A Pisot number is a real algebraic integer, all of whose conjugates lie strictly inside the open unit disk; a Salem number is a real algebraic integer, all of whose conjugate roots are inside the closed unit disk, with at least one of them of modulus exactly 1. Pisot numbers have been studied extensively, and an algorithm to generate them is well known. Our main result characterises all Pisot numbers whose minimal polynomial is a Littlewood polynomial, one with \( \{+1, -1\}\)-coefficients, and shows that they form an increasing sequence with limit 2. It is known that every Pisot number is a limit point, from both sides, of sequences of Salem numbers. We show that this remains true, from at least one side, for the restricted sets of Pisot and Salem numbers that are generated by Littlewood polynomials. Finally, we prove that every reciprocal Littlewood polynomial of odd degree \( n \geq 3 \) has at least three unimodular roots.

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1. Introduction

A Pisot (or Pisot–Vijayaraghavan) number is a real algebraic integer \( x > 1 \), all of whose conjugates lie inside the open unit disk. A real algebraic integer \( x > 1 \) is a Salem number if all of its conjugate roots are inside the closed unit disk, and at least one of
these conjugate roots has modulus exactly 1. The set of all Pisot numbers is usually
denoted by $S$, and the set of all Salem numbers is denoted by $T$. Many results are
known about the set $S$. For example, $S$ is known to be closed [11], and its minimum
is known to be the largest root of $z^3 - z - 1$, which is approximately 1.3247179...[12].
Further, every point of $S$ is a limit of points of $T$, from both sides [11].

A Littlewood polynomial is a polynomial $f(z) = \sum_{k=0}^{d} a_k z^k$ whose coefficients $a_k$
satisfy $a_k = \pm 1$ for $0 \leq k \leq d$. Let $L_n$ denote the class of Littlewood polynomials of
degree exactly equal to $n$. Notice that the set of Littlewood polynomials is a subset of
the set of polynomials whose coefficients are all congruent to 1 mod 2,

$$D_2 = \left\{ \sum_{k=0}^{d} a_k z^k \in \mathbb{Z}[z] : a_k \equiv 1 \mod 2 \text{ for } 0 \leq k \leq d \right\}.$$ 

Following [3], we say that a real number $\gamma > 1$ is a Littlewood Pisot number if it is
a Pisot number and its minimal polynomial is a Littlewood polynomial. We define
a Littlewood Salem number slightly differently; a real number $\beta > 1$ is a Littlewood Salem number if it is a Salem number and it is the root of a Littlewood polynomial. Unlike [3], we do not require the Littlewood polynomial to be irreducible in this case. By Borwein et al. [4] we know that the smallest Littlewood Pisot number is the golden
ratio $(1 + \sqrt{5})/2$, which is a root of $z^2 - z - 1$.

In this paper, we prove the following results:

**Theorem 1.** The set of all Littlewood Pisot numbers forms a sequence \{\(\gamma_n\)\}, $n \geq 2$, where the minimal polynomial of $\gamma_n$ is given by

$$P_n(z) = z^n - z^{n-1} - \cdots - z^2 - z - 1.$$ 

The sequence \{\(\gamma_n\)\} is strictly increasing and has limit point 2.

Thus, for each $n \geq 2$, there exists precisely one polynomial in $L_n$ that is a minimal
polynomial for a Pisot number $\gamma_n$. As an aside, it is interesting to note [1] that \(\text{Inf } S''\)
is also equal to 2, where $S''$ is the second derived set of $S$.

In [3] it was shown that the smallest Littlewood Pisot number is a limit point, from
both sides, of Littlewood Salem numbers. The next theorem confirms a speculation
made at the end of that paper, showing that this is partially the case for all Littlewood
Pisot numbers.

**Theorem 2.** Every Littlewood Pisot number is a limit point, from below, of Littlewood Salem numbers.

The proof of the analogous theorem in [3] for the smallest Littlewood Pisot number
also established that the Littlewood polynomials that represent the required sequences
of Salem numbers are in fact irreducible. However, the Littlewood Salem polynomials
constructed in this paper are not in general irreducible. This is treated further at the end
of Section 4, where we give criteria for the irreducibility of the constructed Littlewood Salem polynomials.

If $p(z)$ is a polynomial in $\mathbb{C}[z]$ of degree $d$ then we define $p^*(z) = z^d p(1/z)$, and we say that $p(z)$ is reciprocal if $p(z) = wp^*(z)$, where $w$ is any complex number of modulus 1. If we restrict $p(z)$ to $\mathbb{R}[z]$ then we say that $p(z)$ is reciprocal if $p(z) = \pm p^*(z) = \pm z^d p(1/z)$. From a result of Erdélyi [7] we know that every reciprocal polynomial $F \in \mathcal{L}_n$ has at least one root on $S$, the unit circle in the complex plane.

Somewhat more generally, in [9] it is shown that if $p(z) = \sum_{k=0}^{2n} a_k z^k$ is a reciprocal polynomial of even degree in $\mathbb{R}[z]$ with $|a_n| \leq \max\{|a_k| : k \in \{0, 1, \ldots, n-1\}\}$, then $p$ has at least one unimodular root. In this paper we prove the following theorem, and then use it to make some observations concerning unimodular roots of reciprocal Littlewood polynomials of odd degree.

**Theorem 3.** For $n \geq 1$, let $F(z) = \sum_{k=0}^{2n+1} a_k z^k$ be a reciprocal polynomial in $\mathbb{C}[z]$ with $F(0) \neq 0$ and $|a_n| \leq |a_0|$. Then $F$ has at least three zeroes that lie on $S$.

The proofs of Theorems 1 and 2 appear in Sections 3 and 4, respectively, while the proof of Theorem 3 is in Section 5.

### 2. A short description of the algorithm

In [5], Boyd constructed an algorithm for determining all the Pisot numbers in an interval $[a, b]$ of the real line, extending earlier work by Dufresnoy and Pisot [6]. We briefly describe this method here, following [5].

Given $\lambda \in S$, let $p(z)$ be its (monic) minimal polynomial of degree $d$ and let $p^*(z) = z^d p(1/z)$. Then $p^*(z)$ has integer coefficients, $p^*(0) = 1$ and $p^*(z)$ has exactly one zero inside the open unit disk, namely $\lambda^{-1}$. If $r(z)$ is a polynomial with integer coefficients, different from $p^*(z)$, with $r(0) > 0$ and $|r(z)| \leq |p^*(z)|$ on $|z| = 1$, then $f(z) = r(z)/p^*(z)$ is a rational function said to be associated with $\lambda$. The choice of $r(z) = (\text{sgn } p(0)) p(z)$ satisfies these conditions, unless $p^*(z) = 1 - cz + z^2$, in which case $r(z) \equiv 1$ works. The set of such functions $f$ is denoted by $C$. Then $C$ is characterised by the following properties:

(a) $f$ is analytic in $|z| \leq 1$ except for a simple pole at $\lambda^{-1} < 1$,
(b) $|f(z)| \leq 1$ on the unit circle,
(c) $f(z) = u_0 + u_1 z + u_2 z^2 + \cdots$ for $|z| < \lambda^{-1}$, where all the $u_i$ are integers, and $u_0 \geq 1$.

The coefficient sequence $\{u_n\}$ of an element $f$ in $C$ is characterised by the following recursive system of inequalities:

\[
\begin{align*}
1 & \leq u_0, \\
\left(\frac{u_0^2}{2} - 1\right) & \leq u_1, \\
\frac{w_n(u_0, \ldots, u_{n-1})}{u_n} & \leq w_n^+(u_0, \ldots, u_{n-1}) & \text{for } n \geq 2.
\end{align*}
\]
Here $w_n$ and $w_n^+$ are values that are determined by $u_0, \ldots, u_{n-1}$. The inequalities $w_n \leq u_n \leq w_n^+$, $n \geq 2$ restrict $u_n$ to a finite set of choices, except in one case: if $u_0 = 1$, then $w_2^+ = \infty$.

To compute $w_n$ and $w_n^+$ recursively we do as follows. Assume that $u_0, u_1, \ldots, u_{n-1}$ are known. Let $D_n(z) = -z^n + d_1 z^{n-1} + \cdots + d_n$ and $E_n(z) = -z^n D_n(1/z)$, where we select $d_1, d_2, \ldots, d_n$ so that the first $n$ coefficients of the Maclaurin series for $D_n(z)/E_n(z)$ are the given $u_0, u_1, \ldots, u_{n-1}$. Then $w_n$ is the coefficient of $z^n$ in this series. That is,

$$
\frac{D_n(z)}{E_n(z)} = u_0 + u_1 z + \cdots + u_{n-1} z^{n-1} + w_n z^n + \cdots. \quad (2)
$$

Similarly for $w_n^+$, we let $D_n^+(z) = z^n + d_1^+ z^{n-1} + \cdots + d_n^+$ and $E_n^+(z) = z^n D_n^+(1/z)$, where we select $d_1^+, \ldots, d_n^+$ so that the first $n$ coefficients of the series for $D_n^+(z)/E_n^+(z)$ are the given $u_0, u_1, \ldots, u_{n-1}$. Again, $w_n^+$ is the coefficient of $z^n$ in this series. Here we have

$$
\frac{D_n^+(z)}{E_n^+(z)} = u_0 + u_1 z + \cdots + u_{n-1} z^{n-1} + w_n^+ z^n + \cdots. \quad (3)
$$

Consistent with the construction described above, we set $w_1 = u_0^2 - 1$ and $w_1^+ = 1 - u_0^2$. Thus, both $w_n$ and $w_n^+$ are defined for $n \geq 1$, although the inequality $u_1 \leq w_1^+$ does not necessarily hold.

The polynomials $D, E, D^+$ and $E^+$ satisfy many recurrence relations—see, for example, [1, Chapter 7]. Some of the more useful ones for our purposes are

$$
D_{n+1}(z) = (1 + z) D_n(z) - \frac{u_n - w_n}{u_{n-1} - w_{n-1}} z D_{n-1}(z) \quad \text{for } n \geq 2, \quad (4)
$$

$$
D_{n+1}^+(z) = (1 + z) D_n^+(z) - \frac{w_n^+ - u_n}{w_{n-1}^+ - u_{n-1}} z D_{n-1}^+(z) \quad \text{for } n \geq 4 \quad (5)
$$

and

$$
D_{n+1}^+ E_n - D_n E_{n+1}^+ = (u_n - w_n) z^n (1 + z), \quad (6)
$$

$$
D_{n+2}^+ E_n - D_{n+1}^+ E_{n+2} = (u_n - w_n^+) z^n (1 - z^2). \quad (7)
$$

Another concise recurrence relation [1,5] helps to compute the difference $w_{n+1}^+ - w_{n+1}$, namely

$$
w_{n+1}^+ - w_{n+1} = \frac{4(u_n^+ - u_n)(u_n - w_n)}{w_n^+ - w_n}. \quad (8)
$$
For each integer $u_0 \geq 1$, sequences $\{u_n\}$ of integers whose terms satisfy (1) form an infinite tree. The nodes of the tree at height $n$ are finite subsequences $(u_0, u_1, \ldots, u_n)$. If $u_n = w_n$ or $u_n = w_n^+$, then such a node has no successors, and if $w_n < u_n < w_n^+$, then its successors are all the subsequences $(u_0, u_1, \ldots, u_n, u_{n+1})$ where $w_{n+1} \leq u_{n+1} \leq w_{n+1}^+$. A node that has no successors is a terminal node, and such nodes are of three possible types:

1. $w_n < u_n < w_n^+$, but there are no integers in $[w_{n+1}, w_{n+1}^+]$,
2. $u_n = w_n (or w_n^+)$, but $D_n (or D_n^+)$ does not have integer coefficients,
3. $u_n = w_n (or w_n^+)$, and $D_n (or D_n^+)$ has integer coefficients.

This third type of terminal node is in one-to-one correspondence with the points of $S$, excluding the quadratics $(c + (c^2 - 4)^{1/2})/2$ for $c \geq 3$, which come from reciprocal quadratic polynomials of the form $1 - cz + z^2$. More specifically, if the terminal node $(u_0, u_1, \ldots, u_n)$ is of the third type, then the polynomial $D_n (or D_n^+)$ has a unique root $\tau_n > 1 (or \tau_n^+ > 1)$, with all the other roots lying inside the open unit disk. Since $D_n$ and $D_n^+$ are monic and have integer coefficients, this means that they are irreducible over $Z[z]$. Thus $D_n (or D_n^+)$ is the minimal polynomial for the Pisot number $\tau_n (or \tau_n^+)$.

A path to infinity in this tree corresponds to an infinite sequence $\{u_n\}$ which satisfies $w_n < u_n < w_n^+$ for all $n \geq 2$, and thus to limit points of the set $S$. Such limit points $x'$ arise as the simple pole $(x')^{-1}$ of a rational function $F(z) = P(z)/Q(z)$ in the derived set of $C$.

The numbers $\{u_n\}$ must satisfy additional constraints when we require the Pisot number $x$ to lie in a given real interval $[a, b] \subset [1, \infty]$. Define the real numbers $v_n$ and $v_n^+$ by

$$v_n = w_n^+ - (w_{n-1}^+ - u_{n-1}) \frac{(1 + a) D_n^+ (a)}{D_{n-1}^+ (a)} \quad \text{for } n \geq 4 \quad \text{(9)}$$

and

$$v_n^+ = w_n + (u_{n-1} - w_{n-1}) \frac{(1 + b) D_n (b)}{D_{n-1} (b)} \quad \text{for } n \geq 2 \quad \text{(10)}$$

Then Boyd [5] showed that $v_n \leq u_n \leq v_n^+$, so

$$0 \leq u_n - w_n \leq (u_{n-1} - w_{n-1}) \frac{(1 + b) D_n (b)}{D_{n-1} (b)} \quad \text{for } n \geq 2 \quad \text{(11)}$$

and

$$0 \leq w_n^+ - u_n \leq (w_{n-1}^+ - u_{n-1}) \frac{(1 + a) D_n^+ (a)}{D_{n-1}^+ (a)} \quad \text{for } n \geq 4 \quad \text{(12)}$$
3. The set of Littlewood Pisot numbers

We now apply the algorithm to the class of Littlewood polynomials. The following additional results will be useful in the proof of Theorem 1. The first of these is a classical result due to Cauchy (see, for example, [10]):

**Lemma 4.** Let \( f(z) = \sum_{k=0}^{n} a_k z^k \) be a polynomial with complex coefficients, with \( a_n \neq 0 \). If \( f(\zeta) = 0 \) then \( |\zeta| < 1 + \max\{|a_k/a_n| : 0 \leq k \leq n - 1\} \).

The proof of the next lemma can be found in [4], in the proof of the main theorem.

**Lemma 5.** Suppose \( f(z) \) is a monic, non-reciprocal polynomial with integer coefficients satisfying \( f \equiv \pm f^* \mod m \) for some integer \( m \geq 2 \). Let

\[
\frac{f(z)}{f^*(z)} = \sum_{i \geq 0} q_i z^i,
\]

so that \( q_i \in \mathbb{Z} \) for \( i \geq 0 \). Then, \( m \) divides \( q_i \) for \( i \geq 1 \).

**Proof of Theorem 1.** By Lemma 4 we know that all the zeroes of a Littlewood polynomial must lie in the circle \( |z| < 2 \). Thus, all Littlewood Pisot numbers must lie in the real interval \([1, 2]\).

We will use the algorithm described above to construct the Littlewood Pisot numbers. In what follows, we use the same notation from Section 2.

We claim the following:

1. \( D_k(z) = 1 + z + z^2 + z^3 + \cdots + z^{k-1} - z^k \) for \( k \geq 1 \),
2. \( w_k = 2^k - 2 \) for \( k \geq 1 \),
3. \( w_k^+ > 2^k + 2 \) for \( k \geq 3 \), and
4. \( v_k^+ = 2^k + 1 \) for \( k \geq 2 \).

Before we prove these, we make a few observations. By Lemma 5, taking \( m = 2 \), if \( D(z) \) is a non-reciprocal Littlewood polynomial, then the coefficients \( \{u_j \} \), \( j \geq 1 \) in the expansion \( D(z)/D^*(z) = \sum_{j=0}^{\infty} u_j z^j \) must all be divisible by 2. Thus, with the possible exception of \( u_0 \), all the coefficients are even integers. Of course, the conclusion remains the same if we consider \( \frac{D(z)}{D^*(z)} \), or even if \( D(z) \) is reciprocal, in which case \( D(z)/D^*(z) = 1 \), and again all coefficients except \( u_0 \) are even. Thus, in using the algorithm we can choose only those values of \( \{u_k \} \), \( k \geq 1 \) that are even integers.

Next, assuming the truth of the above claims for a moment, we see that since \( w_k \leq u_k \leq w_k^+ \), \( u_k \) can take the values \( w_k = 2^k - 2, 2^k, 2^k + 2, \ldots \). But we also know that \( u_k \leq v_k^+ \) and so this limits us to exactly two cases: \( u_k = u_k \) and we have a terminal node with \( D_k(z) \) corresponding to a Pisot number, or \( u_k = 2^k \) with \( w_k < u_k < w_k^+ \) in this case.

Now to the proof of the above claims. We proceed by induction on \( k \), and we work out the first few cases (\( k = 1, 2, 3 \)) as examples.
Since \( 1 \leq u_0 \) and we wish to restrict \( D_k(z) \) to the class \( \mathcal{L}_n \), we have that \( u_0 = 1 \). Thus \( D_1(z) = u_0 - z = 1 - z \) and \( E_1(z) = 1 - z \) as well. Also \( w_1 = u_0^2 - 1 = 0 \), and \( w_1^+ = 0 \). Similarly, \( D_1^+(z) = u_0 + z = 1 + z \) and thus \( E_1^+(z) = 1 + z \).

Next, suppose \( D_2(z) = -z^2 + d_1 z + d_2 \) so that \( E_2(z) = -d_2 z^2 - d_1 z + 1 \) and so, \( D_2^+(z) = 1 + u_1 z + w_2 z^2 + \cdots \). This means that \( -z^2 + d_1 z + d_2 = (-d_2 z^2 - d_1 z + 1)(1 + u_1 z + w_2 z^2 + \cdots) \). From this we compare coefficients to find that \( d_2 = 1 \), \( d_1 = u_1/2 \) and \( w_2 = u_1^2/2 \). Thus \( D_2(z) = -z^2 + u_1^2 z + 1 \). Suppose the roots of \( D_2(z) \) are \( \tau_2 \) and \( -\tau_2^{-1} \), where \( \tau_2 \) is Pisot, i.e., \( 1 < \tau_2 < 2 \). Then \( 0 < \tau_2 - \tau_2^{-1} < 3/2 \) implies that \( 0 < u_1/2 < 3/2 \), which implies that \( u_1 = 1 \) or \( u_1 = 2 \). We choose \( u_1 = 2 \) since it is even. Thus \( D_2 = -z^2 + z + 1 \) and also \( w_2 = 2 \).

Now we cannot compute \( w_2^+ \) in this manner, since \( u_0 = 1 \) (see the remark after (1)). Thus we use \( v_2^+ \), which we compute by replacing \([a, b]\) in (11) by the interval \([1, 2]\). This yields

\[
0 \leq u_n - w_n \leq (u_{n-1} - w_{n-1}) \frac{3}{2} \frac{D_n(2)}{D_{n-1}(2)} \quad \text{for } n \geq 2. \tag{13}
\]

Using (13) we have \( 0 \leq u_2 - 2 \leq 2 \frac{3}{2} \frac{D_2(2)}{D_1(2)} \), or \( 2 \leq u_2 \leq 5 = v_2^+ \). With \( u_2 = w_2 = 2 \) we get a Pisot number \( \gamma_2 \) associated with \( D_2(z) \). With \( u_2 = 4 \), we get \( w_2 < u_2 < w_2^+ = \infty \) and so we can continue.

From (4) we compute

\[
D_3(z) = (1 + z) D_2(z) - \frac{u_2 - w_2}{u_1 - w_1} z D_1(z)
\]

\[
= (1 + z)(1 + z - z^2) - z(1 - z) = 1 + z + z^2 - z^3.
\]

Thus \( E_3(z) = 1 - z - z^2 - z^3 \) and we have \( D_3(z)/E_3(z) = 1 + 2z + 4z^2 + w_3 z^3 + \cdots \). Cross-multiplying and comparing coefficients of \( z^3 \) we find that \( w_3 = 6 \).

Computing \( w_3^+ \) is slightly different; we first need to find \( D_3^+(z) \) using (6) and (7) from Section 2. We have,

\[
D_3^+(z) E_2(z) - D_2(z) E_3^+(z) = (u_2 - w_2) z^2 (1 + z)
\]

and

\[
D_3^+(z) E_1^+(z) - D_1^+(z) E_3^+(z) = (u_1 - w_1^+) z (1 - z^2)
\]

and solving simultaneously for \( D_3^+(z) \) we have \( D_3^+(z) = (1 - z) D_2(z) - z(1 + z) \), or, \( D_3^+(z) = 1 + z - 3z^2 + z^3 \). Thus \( E_3^+(z) = 1 - 3z - z^2 + z^3 \), and looking at \( D_3^+(z)/E_3^+(z) = 1 + 2z + 4z^2 + w_3^+ z^3 + \cdots \) we find that \( w_3^+ = 14 \), which is strictly greater than \( 2^k + 2 \) for \( k = 3 \). We see from (13) that \( 0 \leq u_3 - 6 \leq 3 \), or \( 6 \leq u_3 \leq 9 = v_3^+ \). Thus again, either \( u_3 = w_3 = 6 \), we have a terminal node and \( D_3(z) \) corresponds to a Pisot number \( \gamma_3 \), or \( u_3 = 8 \) with \( w_3 < u_3 < w_3^+ \) and we continue.
Thus far we have essentially gone through a basis step. We now proceed with our induction step.

Assume that we have reached a node \((u_0, \ldots, u_k)\) of height \(k\) in the search tree (see Section 2), and that claims 1–4 hold for this \(k\). In particular, this means that \(w_j < u_j = 2^j\) for all \(j < k\). Also, either \(u_k = w_k\) or \(u_k = 2^k\); the first choice results in a terminal node with a Littlewood Pisot number corresponding to \(D_k(z)\), while the second takes us to the nodes of height \(k + 1\). In that case, by (4) we have that

\[
D_{k+1}(z) = (1 + z)D_k(z) - \frac{u_k - w_k}{u_{k-1} - w_{k-1}} z D_{k-1}(z)
\]

\[
= (1 + z)(1 + z + \cdots + z^{k-1} - z^k) - \frac{2}{2} z(1 + z + \cdots + z^{k-2} - z^{k-1})
\]

\[
= 1 + z + z^2 + \cdots + z^k - z^{k+1}.
\]

And so,

\[
E_{k+1}(z) = 1 - z - z^2 - \cdots - z^{k+1}
\]

and

\[
\frac{D_{k+1}(z)}{E_{k+1}(z)} = 1 + 2z + 4z^2 + \cdots + 2^k z^k + w_{k+1} z^{k+1} + \cdots.
\]

We compute \(w_{k+1}\) by multiplying on both sides by \(E_{k+1}(z)\) and looking at the coefficients of \(z^{k+1}\). We have \(w_{k+1} - 2^k \cdots - 4 - 2 - 1 = -1\), and so \(w_{k+1} = 2^{k+1} - 2\). Using these values and (10), we can compute \(v_{k+1}^+\):

\[
v_{k+1}^+ = w_{k+1} + (u_k - w_k) \frac{3}{2} \frac{D_{k+1}(2)}{D_k(2)}
\]

\[
= 2^{k+1} - 2 + 2 \left(\frac{3}{2}\right) \left(\frac{-1}{-1}\right)
\]

\[
= 2^{k+1} + 1.
\]

Finally we need to show the bound on \(w_{k+1}^+\). Using our induction hypothesis \(w_k^+ > 2^k + 2\) and formula (8) we have that

\[
w_{k+1}^+ = w_{k+1} + 4(w_k^+ - u_k)(u_k - w_k)(w_k^+ - w_k)^{-1}
\]

\[
= 2^{k+1} - 2 + 4(w_k^+ - 2^k)(2)(w_k^+ - 2^k + 2)^{-1}.
\]

Now, \(w_k^+ > 2^k + 2\) means that \(8(w_k^+ - 2^k)(w_k^+ - 2^k + 2)^{-1} > 4\), and so, \(w_{k+1}^+ > 2^{k+1} + 2\).
Thus at each stage, for \( u_n = w_n = 2^n - 2 \), we have a Littlewood Pisot number \( \gamma_n \) which is a root of \( D_n(z) = 1 + z + z^2 + \cdots + z^{n-1} - z^n \). There are no others in \( \mathcal{L}_n \). Further, the path to infinity in this search tree corresponds to the limit point of the \( \gamma_n \) (see Section 2). This path is given by the sequence of \( u_n \) such that \( w_n < u_n < w_n^* \) for all \( n \geq 2 \), with \( u_n = 2^n \). This corresponds to the function \( f(z) = 1 + 2z + 4z^2 + \cdots + 2^n z^n + \cdots = \frac{1}{1-2z} \) which has a pole at \( \frac{1}{\gamma} = \frac{1}{2} \). Thus \( \gamma' = 2 \) is the only limit point of Littlewood Pisot numbers. Finally, noting the change of sign between \( D_n(\gamma_n-1) \) and \( D_n(2) \) it is easy to see that the sequence \( \{\gamma_n\} \) is strictly increasing. □

4. Littlewood Salem numbers

We now prove Theorem 2, generalizing the method used in [3] to establish a similar statement for the smallest Littlewood Pisot number.

**Proof of Theorem 2.** In what follows, we alter our notation slightly for convenience. Let

\[
f_m(z) = z^{m-1} - \sum_{k=0}^{m-2} z^k
\]

and let \( \gamma_m \) be the Pisot number that has \( f_m(z) \) as its minimal polynomial. As in [3], for each \( m \geq 3 \) and \( n \geq 1 \) we define the sequence of Littlewood polynomials

\[
A_{m,n}(z) = z^{2mn} + f_m(z) \sum_{k=0}^{2n-1} z^{mk}.
\]

We prove that for each \( m \), the polynomials \( A_{m,n}(z) \) yield a sequence of Salem numbers that approach \( \gamma_m \) from below. Let

\[
P_{m,n}(z) = \sum_{k=0}^{2n} z^{mk},
\]

\[
Q_{m,n}(z) = \left( \sum_{k=0}^{m-2} z^k \right)^{\frac{2n-1}{k}} \left( \sum_{k=0}^{2n-1} z^{mk} \right)
\]

and let

\[
p_{m,n}(t) = e(-mnt)P_{m,n}(e(t)),
\]

\[
q_{m,n}(t) = e(-(mn - 1)t)Q_{m,n}(e(t)),
\]
where \( e(t) = e^{2\pi i t} \). Then it is easy to check that 
\[ A_{m,n}(z) = P_{m,n}(z) - zQ_{m,n}(z) \]
is a reciprocal polynomial, and that 
\[ a_{m,n}(t) = e(-mnt)A_{m,n}(e(t)) = p_{m,n}(t) - q_{m,n}(t) \]
where both \( p_{m,n}(t) \) and \( q_{m,n}(t) \) are real-valued, periodic functions with period 1. Further, 
\( p_{m,n}(t) \) has \( 2mn \) simple zeroes in the interval \((0, 1)\) at the points
\[ S_p = \left\{ \frac{k}{2mn + m} \, | \, 1 \leq k \leq 2mn + m \text{ and } (2n + 1) \nmid k \right\}; \]
and \( q_{m,n}(t) \) has \( 2mn - 2 \) zeroes in the same interval, at the points \( S_{q1} \cup S_{q2} \), where
\[ S_{q1} = \left\{ \frac{j}{2mn} \, | \, 1 \leq j \leq 2mn \text{ and } 2n \nmid j \right\} \text{ and } \]
\[ S_{q2} = \left\{ \frac{l}{m - 1} \, | \, 1 \leq l \leq m - 2 \right\}, \]
respectively. It is easily verified that the inequalities
\[ \frac{k}{2mn + m} < \frac{k - jk}{2mn} \leq \frac{k + 1}{2mn + m} < \frac{k + 1 - jk}{2mn}, \tag{14} \]
where \( jk = \lfloor k/(2n + 1) \rfloor \), hold for \( 1 \leq k \leq (2n + 1)m - 1 \). As \( k \) varies in this range, \( jk \) varies from 0 to \( m - 1 \). Equality in (14) is attained precisely when \( k \equiv 2n \pmod{2n + 1} \) and in this case we have that
\[ \frac{k}{2mn + m} < \frac{k - jk}{2mn} = \frac{jk + 1}{m} = \frac{k + 1}{2mn + m} < \frac{k + 2}{2mn + m}. \tag{15} \]
so no values in \( S_{q1} \) occur between these two points in \( S_p \).

Now we consider the elements of \( S_{q2} \) in relation to those of \( S_p \). The inequalities
\[ \frac{(2n + 1)(j + 1) + r}{2mn + m} \leq \frac{j + 1}{m - 1} \leq \frac{(2n + 1)(j + 1) + r + 1}{2mn + m}, \tag{16} \]
where
\[ r = \left\lfloor \frac{(2n + 1)(j + 1)}{m - 1} \right\rfloor, \]
can be checked for \( 0 \leq j \leq m - 3 \). All fractions in the sequence \( S_{q2} \) appear in (16) above. Notice that \( 0 \leq r \leq 2n \), and so \( (2n + 1)(j + 1) + r \equiv 0 \pmod{2n + 1} \) only when \( 0 \leq j < \frac{m - 2n - 2}{2n + 1} \) and \( n \leq (m - 2)/2 \). Similarly \( (2n + 1)(j + 1) + r + 1 \equiv 0 \pmod{2n + 1} \) only when \( \frac{2mn - 4n - 1}{2n + 1} \leq j \leq m - 3 \) and \( n \leq (m - 2)/2 \). When \( n > (m - 2)/2 \), the two congruences above are never satisfied.
We now use the remarks above to describe the sequence of roots of both \( p_{m,n}(t) \) and \( q_{m,n}(t) \) in the interval \((0, 1)\). When \( t = 0 \) the two functions are strictly positive. As \( t \) increases, \( p_{m,n}(t) \) has the first zero at \( t = 1/(2mn + m) \), after which every interval between two consecutive roots of \( p_{m,n}(t) \) contains exactly one root of \( q_{m,n}(t) \) \((14)\), except for the following two types of intervals:

(A) By \((15)\), when \( m - 2n^2 - 2 < 2n + 1 \) \( m \leq n \leq (m - 2)/2 \), there are no roots of \( q_{m,n}(t) \) between the roots \((2n + 1)/2m + m \) and \((2n + 1)(j+1)/2m + m \) of \( p_{m,n}(t) \). Note that when \( n > (m - 2)/2 \) we have the same result, but for all values of \( 0 \leq j \leq m - 2 \).

(B) By \((16)\), when \( m - 2n - 2 < 2n + 1 \) \( n \leq (m - 2)/2 \), there are two roots of \( q_{m,n}(t) \) between the roots \((2n + 1)(j+1+r)/2m + m \) and \((2n + 1)(j+1+r+1)/2m + m \) of \( p_{m,n}(t) \). These roots of \( q_{m,n}(t) \) are \((2n + 1)(j+1+r)/2m \) and \( j+1 \) from \( S_q1 \) and \( S_q2 \), respectively. Note that these two roots do not have to be distinct. When \( n > (m - 2)/2 \) we have the same result, but for all values of \( 0 \leq j \leq m - 3 \).

As a result, \( a_{m,n}(t) = p_{m,n}(t) - q_{m,n}(t) \) has at least one root between each pair of consecutive roots of \( p_{m,n}(t) \) except possibly in the two types of intervals above.

Notice that \((2n + 1)(j+1) + 1 \leq (2n + 1)(j+1) + r < (2n + 1)(j+1) + 2n \) for \( m - 2n^2 - 2 < j < 2n + 1 \) when \( n \leq (m - 2)/2 \), and for \( 0 \leq j \leq m - 3 \) when \( n > (m - 2)/2 \). This means that as \( t \) increases from 0 to 1, the occurrence of intervals of type (A) and (B) alternate, beginning and ending with an interval of type (A). From this we conclude that \( a_{m,n}(t) \) has two roots in intervals of type (B) above. Since the number of occurrences of intervals of type (A) is one more than that of type (B), we see that \( a_{m,n}(t) \) has at least \( 2mn - 2 \) roots in the interval \((0, 1)\). Thus, the polynomial \( A_{m,n}(z) \) has at least \( 2mn - 2 \) zeroes on the unit circle.

Since \( A_{m,n}(0) = 1 \) \( a_{m,n}(1) = 1 - 2n(m - 2) < 0 \), it follows that \( A_{m,n}(z) \) has a real root in the interval \((0, 1)\), and, since \( A_{m,n}(z) \) is reciprocal, one real root \( z_{m,n} \) in \((1, \infty)\) as well. This accounts for all \( 2mn \) roots of \( A_{m,n}(z) \) and we conclude that \( z_{m,n} \) is a Littlewood Salem number.

We next wish to show that for each \( m \), the sequence \( \{z_{m,n}\}_{n=1}^{\infty} \) converges to \( \gamma_m \) from below. Since \( A_{m,n}(z) = A_{m,n}^*(z) = 1 + z f_m(z) \sum_{k=0}^{2n-1} z^m k \), it follows that \( A_{m,n}(\gamma_m) = 1 \). Recalling that \( A_{m,n}(1) < 0 \), we conclude that \( z_{m,n} < \gamma_m \) for all \( n \). Using the identity

\[
A_{m,n+1}(z) = A_{m,n}(z) + z^{2mn} \left[ z^{2m} - 1 + f_m^*(z)(z^m + 1) \right]
\]

and the reciprocity of \( A_{m,n+1}(z) \), we find that

\[
A_{m,n+1}(z_{m,n}) = 1 - z_{m,n}^{2m} + f_m^*(1/z_{m,n})(z_{m,n}^m + z_{m,n}^{2m}).
\]

Now, \( 1 - z_{m,n}^{2m} < 0 \) and \( z_{m,n}^m + z_{m,n}^{2m} > 0 \). Further, since \( f_m^*(1) = 2 - m < 0 \), \( f_m^*(1/\gamma_m) = 0 \) and \( 1/\gamma_m < 1/z_{m,n} < 1 \), we have that \( f_m^*(1/z_{m,n}) < 0 \). This means that \( A_{m,n+1}(z_{m,n}) < 0 \), and so we conclude that \( z_{m,n+1} > z_{m,n} \) for all \( n \). The sequence \( \{z_{m,n}\}_{n=1}^{\infty} \) is strictly increasing and bounded above; in other words, the sequence is convergent.
Finally, by writing
\[ A_{m,n}(z) = z^{2mn} \left[ 1 + \frac{f_m^*(z)}{z^m - 1} \right] + \frac{f_m^*(z)}{1 - z^m} \]
for \( z \in (-1, 1) \), we see that as \( n \to \infty \), \( A_{m,n}(z) \) converges uniformly to \( \frac{f_m^*(z)}{1 - z^m} \) on compact subsets of \((-1, 1)\). Since \( f_m^*(z) \) has only the one zero \( 1/\gamma_m \) in \((-1, 1)\), it follows that \( \lim_{n \to \infty} \frac{1}{z_m^n} = 1/\gamma_m \), or that \( \{z_m,n\}_{n=1}^\infty \) converges to \( \gamma_m \). Thus the polynomials \( A_{m,n}(z) \) yield the required sequence of Littlewood Salem numbers. □

We now consider the irreducibility of the polynomials \( A_{m,n}(z) \). Suppose that \( g(z) \in \mathbb{Z}[z] \) divides \( A_{m,n}(z) \), but \( g(z_m,n) \neq 0 \). Then all roots of \( g \) lie in the closed unit disk, and by a theorem of Kronecker [2, Chapter 3] we can conclude that \( g \) must be a product of cyclotomic polynomials. We describe these cyclotomic factors more precisely in the following theorem:

**Theorem 6.** Let \( A_{m,n}(z) \) be defined as above. If \( d > 2 \) is odd, then \( A_{m,n} \) is divisible by \( \Phi_d \) if and only if \( d \mid \gcd(m - 1, 2n + 1) \). If \( d > 2 \) is even, then \( A_{m,n} \) is divisible by \( \Phi_d \) if and only if \( m \) is even and \( d \mid 2 \gcd(m - 2, 4n + 1) \).

Notice that we do not need to consider the cases \( d = 1 \) or 2, since both \( A_{m,n}(1) \) and \( A_{m,n}(-1) \) are not zero. Also, since \( A_{m,n} \) is in \( D_2 \), by Borwein et al. [3, Lemma 2.3] we know that if \( \Phi_d(z) \) divides \( A_{m,n}(z) \), then \( d \mid 4mn + 2 \).

**Proof of Theorem 6.** We first assume that for some odd \( d > 2 \), \( \Phi_d(z) \) divides \( A_{m,n}(z) \). Then, \( d \mid 2mn + 1 \) and so \( \Phi_d(z) \) must divide

\[ A_{m,n}(z) + \frac{z^{2mn+1} - 1}{z - 1} = 2 \frac{z^{2mn+m} - 1}{z^m - 1} = 2 \left( \prod_{r \mid 2mn+m, r \nmid m} \Phi_r(z) \right). \]

Thus, \( d \) must divide \( 2mn + m \) and so \( d \mid m - 1 \). This means \( \gcd(d, m) = 1 \) and so \( d \mid 2n + 1 \) as well, and we have that \( d \) must divide \( \gcd(m - 1, 2n + 1) \). Conversely, suppose that \( d > 2 \) is any divisor of \( \gcd(m - 1, 2n + 1) \). Since \( 2n + 1 \) is odd, \( d \) must be odd as well, and since \( d \mid m - 1 \), we know that \( d \) cannot divide \( m \). Now, as in the proof of Theorem 2, we can write

\[ A_{m,n}(z) = \sum_{k=0}^{2n} z^{mk} - z \left( \sum_{k=0}^{m-2} z^k \right) \left( \sum_{k=0}^{2n-1} z^{mk} \right) = \prod_{r \mid 2mn+m, r \nmid m} \Phi_r(z) - z \prod_{s \mid m-1, s \neq 1} \Phi_s(z) \prod_{t \mid 2mn, t \nmid m} \Phi_t(z) \]

and thus \( \Phi_d \) must divide \( A_{m,n} \).
Next, suppose that $\Phi_d(z)$ divides $A_{m,n}(z)$ for some even $d > 2$. Then $\frac{d}{2}$ must divide $2mn + 1$, and so $\gcd(2, \frac{d}{2}) = 1$, which means that $\Phi_d(z) = \Phi_{d/2}(-z)$. Thus $\Phi_{d/2}(z)$ must divide $(z^{2mn+1} - 1)/(z-1) - A_{m,n}(-z)$. If $m$ is odd, then a simple calculation shows that

$$\frac{z^{2mn+1} - 1}{z-1} - A_{m,n}(-z) = 2z^2 \prod_{r | m, r \neq 1} \Phi_r(z) \prod_{s | 2mn, s \neq 2m} \Phi_s(z) \prod_{t | m-2, t \neq 1} \Phi_t(-z).$$

Once again, since $\gcd(2, m - 2) = 1$, we have that $\Phi_t(-z) = \Phi_{2t}(z)$, and because $\frac{d}{2}$ is odd, we must have that $\frac{d}{2}$ divides $m$ or $2mn$. In either case we find that $d \mid 2$, a contradiction. Thus $m$ must be even, and in this case we have

$$\frac{z^{2mn+1} - 1}{z-1} - A_{m,n}(-z) = 2z^2 \prod_{r | 2mn, r \neq m} \Phi_r(z) \prod_{s | m-2, s \neq 1} \Phi_s(z). \quad (17)$$

This means that $\frac{d}{2}$ must divide $2mn$ or $m - 2$. As before, $\frac{d}{2}$ cannot divide $2mn$, and so $\frac{d}{2}$ must divide $m - 2$, which means that $d$ must divide both $2m - 4$ and $8n + 2$. That is, $d \mid 2\gcd(m - 2, 4n + 1)$.

Finally, if $d > 2$ is any even divisor of $2\gcd(m - 2, 4n + 1)$, then $d = 2r$, where $r \mid m - 2$ and $r \mid 4n + 1$, which together mean that $r \mid 2mn + 1$. If, in addition, $m$ is even, then we can conclude from (17) that $\Phi_r(z)$ must divide $A_{m,n}(-z)$. Thus, since $\Phi_r(z) = \Phi_{2r}(-z)$, we have that $\Phi_d \mid A_{m,n}$. □

We can combine the two conditions in Theorem 6 to describe precisely when the polynomials $A_{m,n}$ are irreducible.

**Corollary 7.** When $m$ is even, $A_{m,n}(z)$ is irreducible if and only if $\gcd(m - 1, 2n + 1) = 1 = \gcd(m - 2, 4n + 1)$. When $m$ is odd, $A_{m,n}(z)$ is irreducible if and only if $\gcd(m - 1, 2n + 1) = 1$.

For each fixed $m \geq 3$, it is now simple to construct an infinite sequence of polynomials $A_{m,n}(z)$ that are irreducible. Suppose that $\mathcal{P}_1$ and $\mathcal{P}_2$ denote the sets of all odd prime divisors of $m - 1$ and $m - 2$, respectively. Let $N_1 = \prod_{p \in \mathcal{P}_1} p$ and $N_2 = \prod_{p \in \mathcal{P}_2} p$. If either set of primes is empty, then we take the usual convention and define the corresponding product to be 1. When $m$ is odd, we set $n = kN_1$ for $k = 1, 2, \ldots$, to construct a sequence of irreducible polynomials $A_{m,n}(z)$. Similarly, if $m$ is even, we select $n = kN_1N_2$. In particular, notice that when $m = 2^l + 1$ for $l \geq 1$, then $\mathcal{P}_1$ is empty, and $A_{m,n}(z)$ is irreducible for all $n \geq 1$. The case $l = 1$ was investigated in [3], where it was proved that $A_{3,n}(z)$ is irreducible for all $n \geq 1$. 
5. Unimodular roots of reciprocal polynomials

We now consider Theorem 3, whose proof requires the following result:

**Lemma 8.** Suppose \( p(z) \) is a polynomial in \( \mathbb{C}[z] \), \( m \) is a positive integer and \( w \) is any complex number of modulus one. Then the number of roots of \( R_m(z) = wz^m p(z) \pm p^*(z) \) in the closed unit disk is greater than or equal to the number of roots of \( S_m(z) = z^m p(z) \) in the same region.

**Proof.** Notice that \( p(z) \) and \( p^*(z) \) have the same zeroes on \( \mathbb{S} \), and that these are also zeroes of both \( R_m(z) \) and \( S_m(z) \). These zeroes can be factored out from both \( R_m(z) \) and \( S_m(z) \), leaving new polynomials of the same form, but lacking those roots of \( p(z) \) that are on \( \mathbb{S} \). Thus, without loss of generality we can exclude these roots, and assume that \( p(z) \) has no zeroes of modulus 1.

For \( \varepsilon > 0 \), let \( f_\varepsilon(z) = (1 + \varepsilon) wz^m p(z) \pm p^*(z) \) and \( g_\varepsilon(z) = (1 + \varepsilon) wz^m p(z) \). Since \( |p(z)| = |p^*(z)| \) on \( \mathbb{S} \), we see that \( f_\varepsilon(z) \) has no zeroes of modulus one. Then for all \( z \in \mathbb{S} \),

\[
|f_\varepsilon(z) - g_\varepsilon(z)| = |p^*(z)| < (1 + \varepsilon)|p^*(z)| = (1 + \varepsilon)|p(z)| = |g_\varepsilon(z)|,
\]

and so by Rouché’s theorem, \( f_\varepsilon \) and \( g_\varepsilon \) have the same number of zeroes inside the open unit disk. But clearly \( g_\varepsilon \) has the same roots as \( S_m \), and so \( f_\varepsilon \) and \( S_m \) have the same number of zeroes in the open unit disk. Since the roots of \( f_\varepsilon \) vary continuously with \( \varepsilon \), it follows that for \( \varepsilon = 0 \) we have that the number of zeroes of \( R_m \) of modulus \( \leq 1 \) is at least as large as the number of zeroes of \( S_m \) in the open unit disk. Finally, we note that by assumption \( p \) (and thus \( S_m \)) has no zeroes on \( \mathbb{S} \), and so we may conclude that \( R_m \) has at least as many zeroes as \( S_m \) in the closed unit disk. \( \square \)

**Proof of Theorem 3.** Let \( F(z) = \sum_{k=0}^{2n+1} a_k z^k \) satisfy the conditions of the theorem, with \( F(z) = w F^*(z) \) and \( |w| = 1 \). Notice that the condition \( F(0) \neq 0 \) means that \( a_{2n+1} \neq 0 \), and so \( F \) has odd degree \( 2n + 1 \). By reciprocity, we see that \( \zeta \) is a root of \( F \) if and only if \( 1/\zeta \) is, and thus \( F \) must have at least one root on \( \mathbb{S} \). Suppose that it has only one root on \( \mathbb{S} \). Then, \( F \) has \( n + 1 \) roots inside the closed unit disk. Writing \( F(z) = wz^{n+1} p(z) + p^*(z) \) for \( p(z) = \sum_{k=0}^{n+1} a_k z^k \), by Lemma 8 we see that \( z^{n+1} p(z) \) has at most \( n + 1 \) roots inside the closed unit disk. That means \( p(z) \) has no roots in the closed unit disk, which is impossible since \( |a_n/a_0| \leq 1 \). Thus \( F(z) \) must have at least two roots on \( \mathbb{S} \), and since \( F \) is reciprocal and of odd degree, it follows that \( F \) must in fact have at least three roots of unit modulus. \( \square \)

Using these results, we can make the following observation. Suppose that \( F(z) \) is a reciprocal polynomial in \( \mathcal{L}_d \) for \( d = 2n + 1, n \geq 2 \), and that \( F \) has exactly three unimodular roots. Write \( F(z) = z^{n+1} f(z) \pm f^*(z) \), where \( f(z) \) is in \( \mathcal{L}_n \). Then, by Lemma 8, \( f \) must have exactly one root in the open unit disk, and this root must be real. By Theorem 1, we must have that \( f(z) = \pm P_n^*(z) \) where \( P_n(z) = z^n - z^{n-1} - \cdots - z^2 - z - 1 \) is the minimal polynomial of the unique Littlewood Pisot number of
degree $n$. Thus, reciprocal Littlewood polynomials of odd degree at least five, except possibly those of the above form, must have at least five unimodular roots.

Computational evidence suggests that this result is extremely conservative; indeed, the following related question appears in [8] as Problem 14.18—prove or disprove that $N(p_n) \to \infty$, where $N(p_n)$ denotes the number of real zeroes of

$$p_n(t) := \sum_{k=0}^{n} a_{k,n} \cos kt, \quad a_{k,n} = \pm 1, \quad k = 0, 1, 2, \ldots, n$$

in the period $[0, 2\pi)$.

The question of whether the number of unimodular roots of a reciprocal Littlewood polynomial goes to infinity with the degree, can be seen to be similar to the above problem as follows. Suppose that $F(z) \in \mathcal{L}_d$ satisfies $F(z) = F^*(z)$. Then,

$$\frac{1}{2} e^{-idt} F(e^{2it}) = \left\{ \begin{array}{ll} \pm \frac{1}{2} \pm \cos 2t \pm \cos 4t \cdots \pm \cos dt & \text{for even } d, \\
\pm \cos t \pm \cos 3t \cdots \pm \cos dt & \text{for odd } d. \end{array} \right.$$ 

Similarly, if $F(z) = -F^*(z)$, then,

$$ie^{-2idt} F(e^{4it}) \sin t = \left\{ \begin{array}{ll} \pm \cos 3t \pm \cos 5t \cdots \pm \cos (2d + 1)t & \text{for even } d, \\
\pm \cos t \pm \cos 3t \cdots \pm \cos (2d + 1)t & \text{for odd } d. \end{array} \right.$$ 

In the first case, the unimodular roots of $F(z)$ are precisely those roots of the corresponding cosine sum in the interval $[0, \pi)$. Similarly, in the second, the unimodular roots of $F(z)$ are those of the cosine sum in the interval $(0, \frac{\pi}{2})$.

In this spirit, we state the following conjecture concerning reciprocal Littlewood polynomials.

**Conjecture 9.** For any reciprocal polynomial $F$, let $r(F)$ denote the number of unimodular roots of $F$. Then, $\min_{F \in \mathcal{L}_n} r(F)$ goes to infinity with $n$.

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**References**
