Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Pure and Applied Algebra 197 (2005) 183–212
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Canonical map of low codimensional subvarieties

Valentina Beorchia[a], Ciro Ciliberto[b], Vincenzo Di Gennaro[b,*]

[a] *Dipartimento di Scienze Matematiche, Università di Trieste, via Valerio 12, 34100 Trieste, Italy*
[b] *Dipartimento di Matematica, Università di Roma "Tor Vergata", 00133 Roma, Italy*

Received 2 March 2004; received in revised form 10 August 2004
Communicated by A.V. Geramita

Abstract

Fix integers $a \geqslant 1$, b and c. We prove that for certain projective varieties $V \subset \mathbf{P}^r$ (e.g. certain possibly singular complete intersections), there are only finitely many components of the Hilbert scheme parametrizing irreducible, smooth, projective, low codimensional subvarieties X of V such that

$$h^0(X, \mathcal{O}_X(aK_X - bH_X)) \leqslant \lambda d^{\varepsilon_1} + c\left(\sum_{1 \leqslant h < \varepsilon_2} p_g(X^{(h)})\right),$$

where d, K_X and H_X denote the degree, the canonical divisor and the general hyperplane section of X, $p_g(X^{(h)})$ denotes the geometric genus of the general linear section of X of dimension h, and where λ, ε_1 and ε_2 are suitable positive real numbers depending only on the dimension of X, on a and on the ambient variety V. In particular, except for finitely many families of varieties, the canonical map of any irreducible, smooth, projective, low codimensional subvariety X of V, is birational.

MSC: Primary: 14C05; 14M07; 14M10; secondary: 14J99

0. Introduction

A famous theorem of Ellingsrud and Peskine [15] states that there are only finitely many components of the Hilbert scheme parametrizing smooth surfaces in $\mathbf{P}^4$ not of general type.

* Corresponding author.
E-mail addresses: beorchia@units.it (V. Beorchia), cilibert@axp.mat.uniroma2.it (C. Ciliberto), digennar@axp.mat.uniroma2.it (V. Di Gennaro).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2004.08.037

This paper has been followed by others in which suitable extensions have been presented (see [2–4,16,23]). More recently two of us gave further wide extensions of these results [8–10].

Going back to the original theorem of Ellingsrud and Peskine [15], Ellia and Folegatti [14] remarked that the technique of proof makes it possible to show a more general result. Namely they are able to prove boundedness for families of smooth surfaces in $\mathbf{P}^4$ with geometric genus bounded above by the sectional genus. This in turn implies boundedness for families of smooth surfaces in $\mathbf{P}^4$ with nonbirational canonical map.

The present paper is devoted to give a wide extension of Ellia–Folegatti's result (see Theorem 0.1), which we now will state.

Let V be an irreducible, possibly singular, projective variety over $\mathbf{C}$. Let $\mathscr{S}$ be a set of projective subvarieties of V. We will say that $\mathscr{S}$ is *bounded* if there is a closed immersion $V \subset \mathbf{P}^r$ such that

$$\sup\{deg(X) : X \in \mathscr{S}\} < +\infty.$$

This means that the varieties in $\mathscr{S}$ belong to finitely many components of the Hilbert scheme. In particular, this definition does not depend on the closed immersion.

In this paper we will prove the following:

Theorem 0.1. *Let $V \subset \mathbf{P}^r$ be an irreducible, projective variety of dimension m. Let $1 \leqslant n < m$ be an integer, and put $k = m - n$. Fix integers $a, b, c \in \mathbf{Z}$, with $a \geqslant 1$, and put $\varepsilon(a) = min\{\frac{n}{k} + 1, \frac{n}{k} + a - 1\}$. Assume that at least one of the following properties holds.*

(A) *$m = n+2$, $2 \leqslant n \leqslant 4$, V is smooth, $NS(V) \simeq \mathbf{Z}$ and any algebraic class in $H^{4n-8}(V, \mathbf{C})$ is a multiple of H_V^{2n-4}, where H_V is a hyperplane section of V;*

(B) *$m = n + 2$, $n \geqslant 4$ and, only when $a = 1$, $n \geqslant \frac{r+1}{2}$; for $i = -1, 0$ any algebraic class in $H_{2n+2i}(V, \mathbf{C})$ is a multiple of H_V^{2-i}, the general linear section $V^{(4)}$ of dimension 4 of V is smooth and $NS(V^{(4)}) \simeq \mathbf{Z}$;*

(C) *$m = n + 2$, $n \geqslant 5$, V is smooth, for $1 \leqslant i \leqslant 3$ any algebraic class in $H^{2i}(V, \mathbf{C})$ is a multiple of H_V^i, and there exist rational numbers $v_1, \ldots, v_n$ such that $c_i(T_V) = v_i H_V^i$ in $H^{2i}(V, \mathbf{C})$ for any $1 \leqslant i \leqslant n$;*

(D) *$n \geqslant \frac{m+2}{2}$ and, only when $a = 1$, $n \geqslant \frac{r+1}{2}$; for $i = -1, 0$ any algebraic class in $H_{2n+2i}(V, \mathbf{C})$ is a multiple of H_V^{k-i}; moreover, for some $2 \leqslant h \leqslant n$ with $h \geqslant k$, the general linear section $V^{(h+k)}$ of dimension $h + k$ of V is smooth, and either any algebraic class in $H^{2i}(V^{(h+k)}, \mathbf{C})$ is a multiple of $H^i_{V^{(h+k)}}$ for $i \in \{1, k\}$, or k is even and $H^{2i}(V^{(h+k)}, \mathbf{C}) \simeq \mathbf{C}$ for any $i = 1, \ldots, k - 1$;*

(E) *$n > \frac{3m-2}{4}$, V is smooth, $a \geqslant 2$, for $1 \leqslant i \leqslant 2k - 1$ any algebraic class in $H^{2i}(V, \mathbf{C})$ is a multiple of H_V^i, and there exist rational numbers $v_1, \ldots, v_n$ such that $c_i(T_V) = v_i H_V^i$ in $H^{2i}(V, \mathbf{C})$ for any $1 \leqslant i \leqslant n$.*

For any n-dimensional subvariety X of V denote by $X^{(h)}$ (K_X resp.) the general linear section of dimension h (the canonical divisor resp.) of X. Denote by $p_g(X^{(h)})$ the geometric genus of $X^{(h)}$. Put $H_X = X^{(n-1)}$ and $d = deg(X)$.

Then there exists a strictly positive real number $\lambda > 0$, depending only on n, a and the ambient variety V, such that the set of irreducible, smooth, projective subvarieties X of V *of dimension n satisfying the following inequality under the hypothesis* (A) *with* $n \neq 3$, *or* (C) *or* (E)

$$h^0(X, \mathcal{O}_X(aK_X - bH_X)) \leqslant \lambda d^{n/k+1} + c\left(\sum_{h=1}^{n-1} p_g(X^{(h)})\right), \tag{0.1a}$$

or the following inequality under the hypothesis (A) *with* $n = 3$

$$h^0(X, \mathcal{O}_X(aK_X - bH_X)) \leqslant \lambda d^2 + c(p_g(X^{(1)}) + p_g(X^{(2)})),$$

or the following inequality under the hypothesis (B) *or* (D)

$$h^0(X, \mathcal{O}_X(aK_X - bH_X)) \leqslant \lambda d^{\varepsilon(a)} + c\left(\sum_{1 \leqslant h < \varepsilon(a)-1} p_g(X^{(h)})\right), \tag{0.1b}$$

is bounded.

Moreover, *when V is smooth*, *under the hypothesis* (A) *with* $n \neq 3$, *or* (B) *with* $a \geqslant 2$, *or* (C), *or* (D) *with* $a \geqslant 2$, *or* (E), *the previous estimates are sharp in the following sense*: *there exists a real number* $\mu > \lambda$ *depending only on n*, *a and the ambient variety V*, *such that the set of irreducible*, *smooth*, *projective subvarieties X of V of dimension n satisfying the inequality*

$$h^0(X, \mathcal{O}_X(aK_X - bH_X)) \leqslant \mu d^{n/k+1} \tag{0.1c}$$

is not bounded.

In particular, using Theorem 0.1 for $a = b = 1$, we see that if the degree of the subvariety $X \subset V$ is large enough, then $h^0(X, \mathcal{O}_X(K_X - H_X)) > 0$, i.e. the linear system $|K_X - H_X|$ is not empty. This implies that the canonical linear system $|K_X|$ induces a birational map on X. Therefore we have the following

Corollary 0.2. *With the same assumption of Theorem* 0.1, *except for finitely many families of varieties*, *the canonical map of any irreducible*, *smooth*, *projective subvariety of V of dimension n*, *is birational*.

Theorem 0.1 has a rather wide range of applications, also to singular varieties. By Lefschetz Hyperplane Theorem, Poincaré duality and Barth Theorem, any smooth complete intersection fourfold V on a Grassmann variety or on a Lagrangian maximal Grassmannian variety or on a spinor variety [22], any smooth complete intersection $V \subset \mathbf{P}^r$ of dimension 5 or 6, any smooth fourfold in $\mathbf{P}^6$ and any smooth sixfold in $\mathbf{P}^8$ verifies the hypothesis (A).

When $n \geqslant 5$ (and, only for $a = 1$, when $n \geqslant (r+1)/2$), possibly singular complete intersections $V \subset \mathbf{P}^r$ of dimension $n+2$, with $dim(Sing(V)) \leqslant n-6$, verify the hypothesis (B) (see [11, Theorem (2.11), p. 144]). Moreover any hypersurface $V \subset \mathbf{P}^{n+3}$ of degree $t \geqslant 3$ defined by the equation

$$x_0^a x_1^{t-a} + x_1 x_2^{t-1} + \cdots + x_{n+1} x_{n+2}^{t-1} + x_{n+3}^t = 0,$$

with $n \geqslant 4$, $1 \leqslant a < t-1$ and $(a,t)=1$, has at most two singular points and satisfies the hypothesis (B) (see [11], Proposition (2.24), p. 148). This provides examples for the assumption (B) in the case $n=4$ too. By Barth Theorem again, any smooth subvariety $V \subset \mathbf{P}^r$ of dimension $n+2$, with $n \geqslant (r+2)/2$, satisfies the hypothesis (B).

As before one sees that any possibly singular complete intersection $V \subset \mathbf{P}^r$ of dimension m with $n > (m+2)/2$ and $dim(Sing(V)) \leqslant 2n-m-4$ (and, only for $a=1$, with $n \geqslant (r+1)/2$), and any smooth subvariety $V \subset \mathbf{P}^r$ of dimension m with $n \geqslant (r+2)/2$, satisfies the hypothesis (D). When $n=(m+2)/2$ (and, only for $a=1$, when $n \geqslant (r+1)/2$), by Noether–Lefschetz Theorem [5] this is true also for Noether–Lefschetz general complete intersections $V \subset \mathbf{P}^r$ of dimension m, with $h^{a,m-a}(V) \neq 0$ for some $a < m/2$ (for instance, any general hypersurface of even dimension $m \geqslant 4$ and degree $\geqslant 3$ [21]).

Smooth complete intersections $V \subset \mathbf{P}^r$ of dimension $m \geqslant 7$ ($m < (4n+2)/3$ resp.) satisfy the hypothesis (C) ((E) resp.).

As for the proof, the general strategy consists in determining a lower bound for the geometric genus of a subvariety $X \subset V$. Our methods are partly based on the technical developments of [8–10], to which we will often refer.

Under the hypothesis (A), the basic tools are inequality (1.5) proved in [9], Castelnuovo–Halphen's theory, and Miyaoka–Yau's inequality (see Section 1).

The line of the proof under the hypothesis (B) is the following. A Barth–Lefschetz type of argument proves that the Néron–Severi group of the subvarieties X of V has rank 1. Then, when $a \geqslant 2$, Kawamata–Viehweg Vanishing Theorem allows us to obtain a lower bound for $h^0(X, \mathcal{O}_X(aK_X - bH_X))$ (see Proposition 2.2, which should be compared with Kollár–Luo–Matsusaka estimate [19, p. 302, Theorem 2.15.9]). In order to make this lower bound explicit, and then deduce the boundedness for X, we need a general result, i.e. Theorem 2.1, concerning boundedness for subvarieties with bounded sectional genus, whose proof relies on inequality (1.5) and Castelnuovo theory, and does not need the assumption $n \geqslant (r+1)/2$. The previous argument does not work when $a=1$. In this case, using the hypothesis $n \geqslant (r+1)/2$, we may apply Larsen Theorem (as in Amerik paper [1]) and deduce that the Picard group of X is generated by the hyperplane section. Then Castelnuovo theory [13] enables us to bound $h^0(X, \mathcal{O}_X(K_X - bH_X))$ from below, and so, using Theorem 2.1 again, one may conclude in a similar way as in the case $a \geqslant 2$ (see Section 2).

Under the hypothesis (C), the Chern classes of the normal bundle of a smooth subvariety $X \subset V$ are multiples of the linear sections. By means of a somewhat delicate numerical analysis based on the Hirzebruch–Riemann–Roch Theorem and the previous Theorem 2.1, this allows us to bound from below the arithmetic genus of X in terms of the degree d of X. Using the Hyperplane Lefschetz Theorem one may estimate the difference between the geometric genus of X and the arithmetic genus, finally obtaining a lower bound for the geometric genus, from which one easily concludes (see Section 3).

Under the hypothesis (D), Theorem 0.1 follows in a similar manner as under the hypothesis (B), taking into account a general result, i.e. Theorem 4.1, which states boundedness for subcanonical subvarieties, and does not need the assumption $n \geqslant (r+1)/2$. For the proof of this result, the main tools are Chern classes computations like in [23] and [8] (see Section 4).

Under the hypothesis (E), Theorem 0.1 follows by combining the methods used in the proof under the hypothesis (C) and (D) (see Section 5).

We prove the sharpness of the estimates (0.1a) and (0.1b) in the sense of (0.1c), by considering the unbounded set consisting of the k-codimensional complete intersections of balanced type on V.

For instance, for a smooth complete intersection surface X of balanced type (u, u) in $\mathbf{P}^4$, we have $p_g(X) \leqslant \frac{7}{12}d^2$. On the other hand, our analysis proves that smooth surfaces $X \subset \mathbf{P}^4$ with $p_g(X) \leqslant \frac{1}{12}d^2$ are bounded (see Section 1, (1.2)).

Under the hypothesis (A) with $n = 3$, or (B) with $a = 1$, or (D) with $a = 1$, our methods do not enable us to obtain the expected sharp estimate. In any case we give explicit estimate for the constant λ, and our analysis may give, in principle, explicit bounds for the degree d in terms of the given data. We decided not to dwell on this here.

It would be interesting to investigate whether Theorem 0.1 is sharp in a stronger sense. For instance, it implies boundedness for smooth surfaces of degree d in $\mathbf{P}^4$ with geometric genus $\leqslant \lambda d^2$, with $\lambda < 1/6$ (see Section 1, in particular (1.2)). A nice question is therefore whether the family of smooth surfaces in $\mathbf{P}^4$ with geometric genus $\leqslant d^2/6$ is bounded or not.

Notation. Let Y be any smooth, irreducible, projective variety over $\mathbf{C}$. We will denote by T_Y the tangent bundle of Y, and by K_Y a canonical divisor of Y. If $\mathscr{E}$ is any sheaf on Y we denote by $\chi(\mathscr{E})$ its Euler–Poincaré characteristic and by $c_i(\mathscr{E})$ its Chern classes. We denote by $p_g(Y)$ the geometric genus of Y. As usual $NS(Y)$ will be the Néron–Severi group of Y. We denote numerical equivalence by using the symbol $\equiv$. If $Z \subset Y$ is a subvariety, we denote by $N_{Z,Y}$ the normal sheaf of Z in Y. When $Y \subset \mathbf{P}^r$ we denote by H_Y the general hyperplane section of Y. Moreover, if $dim(Y) = l$ and $0 \leqslant j \leqslant l$, we denote by $Y^{(j)}$ the intersection of Y with a general linear subspace $\mathbf{P}^{r-l+j} \subset \mathbf{P}^r$. In particular $dim(Y^{(j)}) = j$, and $Y^{(l-1)} = H_Y$. We say that $Y \subset \mathbf{P}^r$ is *numerically subcanonical* if $K_Y = eH_Y$ in $H^2(Y, \mathbf{Q})$ for some $e \in \mathbf{Q}$. We say that $Y \subset \mathbf{P}^r$ is *subcanonical* if $K_Y = eH_Y$ in $Pic(Y)$, for some $e \in \mathbf{Z}$.

If x is a real number, we denote by $[x]$ the integral part of x. If $\mathscr{S}$ is a set and $f : \mathscr{S} \to [0, +\infty[$ is a numerical function, we say that a function $\phi : \mathscr{S} \to [0, +\infty[$ is $\mathrm{O}(f)$, and we write $\phi = \mathrm{O}(f)$, if $|\phi(\xi)| \leqslant Cf(\xi)$ for all $\xi \in \mathscr{S}$, where C is a constant > 0.

1. The proof of Theorem 0.1 under the hypothesis (A)

We start by proving Theorem 0.1 in the case $n = 2$, under the hypothesis (A). We need the following result:

Theorem 1.1. *Let $V \subset \mathbf{P}^r$ be an irreducible, projective variety of dimension $n + 2 \geqslant 4$. Assume that the general linear section $V^{(4)}$ of dimension 4 of V is smooth and such that $NS(V^{(4)}) \simeq \mathbf{Z}$. Fix an integer s, and a real number*

$$\lambda < \frac{1}{(n+1)!s^n}.$$

For any projective n-fold X contained in V, put $d = deg(X)$ and let $p_g(X)$ be the geometric genus of X. Then the set of irreducible, smooth, projective, codimension two subvarieties X of V, contained in some reduced, projective subvariety of $\mathbf{P}^r$ of dimension $n + 1$ and degree

$\leqslant s$, *and such that*

$$p_g(X) \leqslant \lambda d^{n+1}, \tag{1.1a}$$

is bounded.

Proof of Theorem 1.1. Using the same argument as in the proof of [10], Theorem 4.1, p. 487, one proves that

$$p_g(X) = \frac{d^{n+1}}{(n+1)!s^n} + \mathrm{O}(d^n) \tag{1.1b}$$

for any irreducible, smooth, projective, codimension two subvarieties X of V of degree d, contained in some reduced, projective subvariety of $\mathbf{P}^r$ of dimension $n+1$ and degree s (see [10], p. 491, line 7 from below). Our Theorem (1.1) follows comparing (1.1a) with (1.1b). □

We are in position to prove Theorem 0.1 in the case $n = 2$. To this purpose, let $V \subset \mathbf{P}^r$ be a smooth fourfold as in Theorem 0.1. We may assume V is nondegenerate. First we examine the case $a = 1$ and $b = 0$. Put $t = deg(V)$. Fix any real number λ such that

$$\lambda < \frac{1}{6t}. \tag{1.2}$$

Let $X \subset V$ be any smooth projective surface such that

$$h^0(X, \mathcal{O}_X(K_X)) \leqslant \lambda d^2 + cp_g(H_X). \tag{1.3}$$

We have

$$\chi(\mathcal{O}_X) \leqslant 1 + \lambda d^2 + cp_g(H_X). \tag{1.4}$$

Now we use the crucial inequality (see [9, p. 277] (2))

$$d^2/t + q(2g-2) + \mathrm{O}(d) \leqslant 2(K_X^2 - 6\chi(\mathcal{O}_X)), \tag{1.5}$$

where $g = p_g(H_X)$ and $K_V = qH_V$ in $H^2(V, \mathbf{C})$. By [9] we may assume that X is of general type. Hence, using Miyaoka–Yau inequality (compare with [14])

$$K_X^2 \leqslant 9\chi(\mathcal{O}_X)$$

and (1.4) and (1.5), we get

$$\left(\frac{1}{t} - 6\lambda\right) d^2 + (2q - 6c)g + \mathrm{O}(d) \leqslant 0. \tag{1.6}$$

If $2q - 6c \geqslant 0$ then d is bounded by (1.2). So we only have to examine the case $2q - 6c < 0$. From (1.6) we get

$$g \geqslant d^2 \frac{1 - 6t\lambda}{t(6c - 2q)} + \mathrm{O}(d). \tag{1.7}$$

Put

$$\frac{1}{2\alpha} = \frac{1-6t\lambda}{t(6c-2q)},$$

and

$$s_0 = max\{t+1,\ [\alpha]+1\}.$$

Notice that, by (1.3) and Castelnuovo's bound on the geometric genus of a projective curve, we have

$$p_g(X) \leqslant \lambda d^2 + cp_g(H_X) = \mathrm{O}(d^2).$$

Then, by Theorem 1.1 we may assume that there is no irreducible and reduced 3-fold $T \subset \mathbf{P}^r$ containing X and of degree $< s_0$. Hence, by [6], H_X is nondegenerate and not contained in any surface in $\mathbf{P}^{r-1}$ of degree $< s_0$. By [7] we deduce

$$g \leqslant d^2/2s_0 + \mathrm{O}(d). \tag{1.8}$$

From (1.7) and (1.8) we obtain

$$\frac{d^2}{2}\left(\frac{1}{\alpha} - \frac{1}{s_0}\right) + \mathrm{O}(d) \leqslant 0,$$

which implies that d is bounded because $s_0 > \alpha$. This concludes the proof of the boundedness in Theorem 0.1, under the hypothesis (A), with $n=2$, $a=1$ and $b=0$.

Now we turn to the case $n=2$ under the hypothesis (A), with $a=1$ and any fixed b. As before, fix any real number λ as in (1.2). Let $X \subset V$ be a smooth surface such that $h^0(X, \mathcal{O}_X(K_X - bH_X)) \leqslant \lambda d^2 + cp_g(H_X)$. From the Poincaré residue sequence (compare with [12], proof of Corollary (2.2) (a), and with [3], p. 329)

$$0 \to \omega_X(-1) \to \omega_X \to \omega_{H_X}(-1) \to 0,$$

we deduce

$$p_g(X) \leqslant \lambda d^2 + (b+c)p_g(H_X).$$

And so the boundedness of d follows from the previous analysis of the case $a=1$, $b=0$.

Next, we consider the case $a \geqslant 1$. Fix any real number λ as in (1.2), and let $X \subset V$ be a smooth surface such that $h^0(X, \mathcal{O}_X(aK_X - bH_X)) \leqslant \lambda d^2 + cp_g(H_X)$. Again by the previous analysis, we may assume $p_g(X) > 0$. Hence we have

$$h^0(X, \mathcal{O}_X(K_X - bH_X)) \leqslant h^0(X, \mathcal{O}_X(aK_X - bH_X)).$$

Therefore the boundedness of d follows from the analysis of the case $a=1$.

Finally, to prove the sharpness of estimate (0.1a) under the hypothesis (A) with $n=2$, consider the set $\mathscr{S}$ of smooth surfaces complete intersection on V of type (u,u). Clearly, $\mathscr{S}$ is not bounded. In order to estimate $h^0(X, \mathcal{O}_X(aK_X - bH_X))$ for $X \in \mathscr{S}$, first notice that $d = tu^2$ (t = degree of V), and by the adjunction formula we have $K_X = (q+2u)H_X$ in $H^2(X, \mathbf{C})$, where $K_V = qH_V$ in $H^2(V, \mathbf{C})$. From [19], p. 301, (2.15.8.6), we have

$$h^0(X, \mathcal{O}_X(aK_X - bH_X)) \leqslant \frac{[(a(q+2u)-b)H_X^2]^2}{H_X^2} + 2 = \frac{4a^2d^2}{t} + \mathrm{O}(d^{3/2}).$$

This proves (0.1c) and completes the proof of Theorem 0.1 under the hypothesis (A) with $n = 2$.

Next we are going to prove Theorem 0.1 in the case $n = 3$, under the hypothesis (A). First we examine the case $a = 1$ and $b = 0$. To this purpose, taking into account Castelnuovo's bound for the genus of a projective curve, it suffices to prove the boundedness of the set of smooth 3-folds $X \subset V$ such that

$$p_g(X) \leqslant \lambda d^2 + cp_g(H_X), \tag{1.9}$$

where λ is *any* fixed real number, and c is any fixed integer $\geqslant 1$. We follow the proof of Theorem 0.1 in the case $n = 3$, in [10]. First notice that using (2.2) of [10] and Hyperplane Lefschetz Theorem one has

$$\begin{aligned}\chi(\mathcal{O}_X) &\geqslant 1 - h^1(X, \mathcal{O}_X) - p_g(X)\\ &\geqslant c(1 - h^1(X, \mathcal{O}_X) - p_g(H_X)) + (cp_g(H_X) - p_g(X) + 1 - c)\\ &\geqslant c(-\chi(\mathcal{O}_{H_X}) - 2(g-1)) + (cp_g(H_X) - p_g(X) + 1 - c),\end{aligned}$$

where $g = p_g(X^{(1)})$ denotes the linear sectional genus of X. As in the proof of (i) of Lemma 2.1 in [10], we deduce

$$\begin{aligned}(24\gamma - 12q + 24c)\chi(\mathcal{O}_{H_X}) \geqslant{}& d^2(2\gamma - q - 2)/t + (g-1)(2d/t + \mathrm{O}(1)) + \mathrm{O}(d)\\ &+ 24(cp_g(H_X) - p_g(X)),\end{aligned}$$

where γ is any integer such that the twisted tangent bundle $T_V(\gamma)$ is globally generated and $K_V = qH_V$ in $H^2(V, \mathbf{C})$. Using (g) of Lemma 2.1 in [10], and taking into account that we may assume $2\gamma - q + 2c > 0$, we obtain

$$\begin{aligned}0 \leqslant{}& (g-1)[\mu(g-1)/d - 2d/t + \mathrm{O}(1)] - 2d^2(2\gamma - q + c - 1)/t + \mathrm{O}(d)\\ &+ 24(p_g(X) - cp_g(H_X)),\end{aligned}$$

where $\mu = 8(2\gamma - q + 2c)$. From (1.9) we get

$$\begin{aligned}0 \leqslant{}& (g-1)[\mu(g-1)/d - 2d/t + \mathrm{O}(1)] - 2d^2[-12\lambda + (2\gamma - q + c - 1)/t]\\ &+ \mathrm{O}(d).\end{aligned} \tag{1.10}$$

By Theorem 1.1 in [10] we may assume $g > 1$. Moreover, as in the case $n=2$, using Theorem 1.1 and (1.9), we may also assume

$$g \leqslant d^2/2s_0 + \mathrm{O}(d),$$

where

$$s_0 = max\{t+1,\ [\mu t/4]+1\}.$$

Therefore

$$(g-1)[\mu(g-1)/d - 2d/t + \mathrm{O}(1)] \leqslant 0$$

for $d > \mathrm{O}(1)$, and by (1.10) we get

$$2d^2[-12\lambda + (2\gamma - q + c - 1)/t] + \mathrm{O}(d) \leqslant 0,$$

which proves the boundedness of d, because we may choose γ such that $-12\lambda + (2\gamma - q + c - 1)/t > 0$. This concludes the proof of Theorem 0.1 under the hypothesis (A), in the case $n=3$, when $a=1$ and $b=0$.

As in the case $n=2$, one reduces the proof of the general case $a \geqslant 1$ and $b, c \in \mathbf{Z}$, to the case $a=1$ and $b=0$. This concludes the proof of Theorem 0.1 under the hypothesis (A), in the case $n=3$.

Now we are going to prove the theorem in the case $n=4$, under the hypothesis (A). First we examine the case $a=1$ and $b=0$. Fix any positive real number λ such that

$$\lambda < \frac{1}{1440t^2}. \tag{1.11}$$

As before, taking into account Castelnuovo's bound for the genus of a projective curve, it suffices to prove the boundedness of the set of smooth projective 4-folds $X \subset V$ such that

$$p_g(X) \leqslant \lambda d^3 + c(p_g(X^{(2)}) + p_g(X^{(3)})), \tag{1.12}$$

where c is any fixed integer $\geqslant 1$. We follow the proof of Theorem 0.1 in the case $n=4$, in [10]. Arguing as above, we obtain

$$\begin{aligned}\chi(\mathcal{O}_X) &\leqslant 1 + p_g(X^{(2)}) + p_g(X)\\ &\leqslant (c+1)(1 + p_g(X^{(2)}) + p_g(X^{(3)})) + (p_g(X) - cp_g(X^{(2)}) - cp_g(X^{(3)}))\\ &\leqslant (c+1)(2\chi(\mathcal{O}_{X^{(2)}}) + 2(g-1) + 2 - \chi(\mathcal{O}_{X^{(3)}})) + (p_g(X) - cp_g(X^{(2)})\\ &\quad - cp_g(X^{(3)})),\end{aligned}$$

where $g = p_g(X^{(1)})$ denotes the linear sectional genus of X. As in the proof of (m) of Lemma 3.1 in [10], we deduce

$$\begin{aligned}&60(2\gamma - q + 2(c+1))\chi(\mathcal{O}_{X^{(3)}})\\ &\quad\leqslant (-9d/t + \mathrm{O}(1))\chi(\mathcal{O}_{X^{(2)}}) + (g-1)[d(9q/2 - 9 - 10\gamma)/t + \mathrm{O}(1)]\\ &\qquad - d^3/(12t^2) + \mathrm{O}(d^2) + 120(p_g(X) - cp_g(X^{(2)}) - cp_g(X^{(3)})),\end{aligned} \tag{1.13}$$

where γ is any integer such that $T_V(\gamma)$ is globally generated and $K_V = qH_V$ in $H^2(V, \mathbf{C})$. Now put

$$\alpha = 2g - 2 - 3d, \quad v = 2\gamma - q, \quad \beta = \frac{5(2\gamma - q + 2(b+1))}{2(1 + vd/\alpha)}.$$

Notice that we may assume $\gamma > \mathrm{O}(1)$. By Theorem 1.1 in [10] we also may assume

$$g \geqslant \mathrm{O}(d) \quad \text{and} \quad d \geqslant \mathrm{O}(1). \tag{1.14}$$

Hence we may assume $0 < vd/\alpha < 1/2$, from which we get

$$0 < 5(2\gamma - q + 2(b+1))/3 < \beta < 5(2\gamma - q + 2(b+1))/2. \tag{1.15}$$

Combining inequality (i) of Lemma 3.1 in [10], i.e.

$$\begin{aligned} 24(1 + vd/\alpha)\chi(\mathcal{O}_{X^{(3)}}) \geqslant (g-1)(2d/t + \mathrm{O}(1)) \\ + \chi(\mathcal{O}_{X^{(2)}})(-6d^2/(\alpha t) + \mathrm{O}(1)) - 36(\chi(\mathcal{O}_{X^{(2)}}))^2/ \\ \alpha - d^4/(4\alpha t^2) + \mathrm{O}(d^2) \end{aligned}$$

(notice that the term $36(\chi(\mathcal{O}_{X^{(2)}}))^2/\alpha$ corrects a misprint in [10]), with (1.12) and (1.13), we get

$$A(\chi(\mathcal{O}_{X^{(2)}}))^2 + B\chi(\mathcal{O}_{X^{(2)}}) + C \geqslant 0,$$

where $A = 36$, $B = 6d^2/t - 9d\alpha/(\beta t) + \alpha\mathrm{O}(1)$, and

$$\begin{aligned} C = \alpha(g-1)[d(9q/2 - 9 - 10\gamma)/(\beta t) - 2d/t + \mathrm{O}(1)] \\ + d^4/(4t^2) + \alpha d^3(120\lambda - 1/(12t^2))/\beta + \alpha\mathrm{O}(d^2). \end{aligned}$$

Using (1.11), (1.14) and (1.15) one sees that $B < 0$ and $C < 0$. At this point, taking into account Theorem 1.1, in order to prove the boundedness of d, one may proceed exactly as in the proof of Theorem 0.1 in [10], in the case $n = 4$, under the hypothesis (A) (see [10], p. 486, line 18 from above). This concludes the proof of Theorem 0.1 in the case $n = 4$, under the hypothesis (A), when $a = 1$, $b = 0$ and c is any integer. Next, as before, one reduces the general case $a \geqslant 1$ and $b,\ c \in \mathbf{Z}$, to the case $a = 1$ and $b = 0$.

One proves the sharpness in the sense of (0.1c) in a similar way as in the case $n = 2$.

This concludes the proof of Theorem 0.1 in the hypothesis (A).

Remark 1.16. From the previous proof of Theorem 0.1 under the hypothesis (A), we see that in the case $n = 3$, for any fixed integers $a \geqslant 1$, b, c, and for *any* fixed real number λ, the set of irreducible, smooth, projective subvarieties X of V of dimension 3 such that $h^0(X, \mathcal{O}_X(aK_X - bH_X)) \leqslant \lambda d^2 + c(p_g(X^{(1)}) + p_g(X^{(2)}))$, is bounded.

2. The proof of Theorem 0.1 under the hypothesis (B)

We begin with the following preliminary result, concerning boundedness for subvarieties with bounded sectional genus (compare with Theorem 1.1 in [10]).

Theorem 2.1. *Let $V \subset \mathbf{P}^r$ be an irreducible, projective variety of dimension $m = n + 2 \geqslant 4$ and degree t. Denote by $V^{(4)}$ the general linear section of V of dimension 4. For any subvariety $X \subset V$ of dimension n, denote by $X^{(h)}$ the general linear section of X of* *dimension h* ($1 \leqslant h \leqslant n$), *by d the degree of X, and by g the geometric genus of* $X^{(1)}$. *Fix a real number* λ *such that*

$$0 < \lambda < \frac{1}{2\sqrt{2t}}. \tag{2.1a}$$

Assume that $V^{(4)}$ *is smooth, and that its Néron–Severi group has rank* 1. *Then the set of irreducible, projective subvarieties X of V of dimension n such that* $X^{(2)}$ *is smooth and*

$$g \leqslant \lambda d^{3/2},$$

is bounded.

Proof of Theorem 2.1. We may assume $n = 2$. Let $X \subset V$ be any smooth projective surface such that $g \leqslant \lambda d^{3/2}$. By [10] we may assume that X is of general type. Therefore $\chi(\mathcal{O}_X)) \geqslant 0$, and from (1.5) we get

$$d^2/t - 2K_X^2 + \mathrm{O}(d^{3/2}) \leqslant 0. \tag{2.1b}$$

Now consider the orthogonal decomposition $K_X = D + eH_X$ in $NS(X) \otimes \mathbf{Q}$, with $D \cdot H_X = 0$ and $D^2 \leqslant 0$. Since $e = (2g - 2 - d)/d$, $g \leqslant \lambda d^{3/2}$ and we may assume $d > \mathrm{O}(1)$, we have

$$|e| \leqslant 2\lambda d^{1/2}.$$

It follows that

$$K_X^2 = (D + eH_X)^2 \leqslant e^2 d \leqslant 4\lambda^2 d^2.$$

Using (2.1b) we have

$$\left(\frac{1}{t} - 8\lambda^2\right) d^2 + \mathrm{O}(d^{3/2}) \leqslant 0,$$

which, taking into account (2.1a), proves the boundedness of d. □

We are in position to prove Theorem 0.1 under the hypothesis (B), with $a = 1$. To this aim, fix any real number λ such that

$$0 < \lambda < \frac{1}{n!\sqrt{(2t)^n}}. \tag{2.2}$$

Notice that, taking into account Castelnuovo–Harris bound for the geometric genus of a projective variety [18], it suffices to prove that the set of smooth, projective subvarieties X of V of dimension n with

$$h^0(X, \mathcal{O}_X(K_X - bH_X)) \leqslant \lambda d^{n/2}, \tag{2.3}$$

is bounded. Using our hypothesis on the homology of the ambient variety V, we know that $X = \alpha H_V^2$ in $H_{2n}(V, \mathbf{C})$, for some $\alpha \in \mathbf{Q}$. Hence we can consider the following natural commutative diagram

$$\begin{array}{ccc} & & A^1(X) \\ & \overset{j}{\swarrow} & \downarrow {\scriptstyle \alpha H_X^2} \\ A^3(V) & \xrightarrow{\rho} & A^3(X) \end{array} \tag{2.4}$$

where $A^3(V) \subset H_{2n-2}(V, \mathbf{C})$ denotes the space of algebraic classes of codimension 3 of V, and similarly for X, ρ is the natural restriction map, and j is the map which takes a cycle on X and considers it as a cycle on V. Notice that, by our hypothesis, $A^3(V) \simeq \mathbf{C}$. Moreover the vertical map is injective by Hard Lefschetz: this is where we use $n \geqslant 4$. It follows that j is an isomorphism, i.e. $A^1(X) \simeq NS(X) \otimes \mathbf{C}$ has dimension 1. When $a = 1$, we assume $n \geqslant (r+1)/2$ and so we may apply Larsen Theorem [20]. With the same argument developed in [1, Proposition 8, p. 69], we deduce that the Picard group of X is generated by the hyperplane section. It follows that

$$K_X = eH_X$$

in $Pic(X)$, with $e = (2g - 2 - (n-1)d)/d$. Therefore, by (2.3) we have

$$h^0(X, \mathcal{O}_X(K_X - bH_X)) = h^0(X, \mathcal{O}_X(e - b)) \leqslant \lambda d^{n/2}. \tag{2.5}$$

On the other hand, by Castelnuovo Theory (see Proposition (3.23), p. 117 in [13]) we know that

$$h^0(X, \mathcal{O}_X(e - b)) \geqslant (e - b)^n / n! \tag{2.6}$$

(by Theorem 2.1 we may assume $e - b > 0$). Combining (2.5) with (2.6) we obtain

$$g \leqslant \frac{(n!\lambda)^{1/n}}{2} d^{3/2} + \mathrm{O}(d), \tag{2.7}$$

and the boundedness of d follows by (2.2) and Theorem 2.1. This concludes the proof of Theorem 0.1 under the hypothesis (B), for $a = 1$.

Now we are going to prove Theorem 0.1 under the hypothesis (B), with $a \geqslant 2$. We need the following preliminary result, which relies on Kawamata–Viehweg Vanishing Theorem, and allows us to bound $h^0(X, \mathcal{O}_X(aK_X - bH_X))$ from below only assuming that the Néron–Severi group of X has rank 1 (compare with (2.5) and (2.6) above).

Proposition 2.8. *Let $X \subset \mathbf{P}^r$ be a projective, irreducible and smooth variety of general type, of dimension $n \geqslant 2$ and degree d. Let D be a big and nef divisor on X. Assume that for suitable integers σ and τ, the general hyperplane section H_X is numerically equivalent to τD, and the canonical divisor K_X is numerically equivalent to σD. Fix an integer $l > \sigma$ and define α and β by dividing $l - \sigma = \alpha\tau + \beta$, $0 \leqslant \beta < \tau$. Put*

$$\begin{aligned} \gamma(n, d, \sigma, \tau, l) = {} & (n-2)(\alpha - n + 1)[\alpha\tau^2(n-1) + n\tau(\sigma - 2l)] \\ & + n(n-1)l(l - \sigma - \tau(n-2)). \end{aligned}$$

Then one has

$$h^0(X, \mathcal{O}_X(lD)) \geqslant d \binom{\alpha - 1}{n - 2} \gamma(n, d, \sigma, \tau, l)/2\tau^2 n(n-1). \tag{2.8a}$$

Remark 2.9. (i) In (2.8a) we assume $\binom{\alpha-1}{n-2}=1$ for $n=2$, and $\binom{\alpha-1}{n-2}=0$ for $n>2$ and $\alpha<n-1$.

(ii) Proposition 2.8 should be compared with Kollár–Luo–Matsusaka estimate [19], p. 302, Theorem 2.15.9, which, in our context, is not as good as inequality (2.8a).

Proof of Proposition 2.8. First we examine the case $n=2$. Since $K_X \equiv \sigma D$, and $l>\sigma$, by Kawamata–Viehweg Vanishing Theorem we have $h^i(X, \mathcal{O}_X(lD))=0$ for any $i>0$. Hence by Riemann–Roch Theorem we get $h^0(X, \mathcal{O}_X(lD)) = \chi(\mathcal{O}_X) + l(l-\sigma)D^2/2$. Since X is of general type, then $\chi(\mathcal{O}_X)\geqslant 0$ and so we deduce

$$h^0(X, \mathcal{O}_X(lD)) \geqslant l(l-\sigma)D^2/2 = d\gamma(2, d, \sigma, \tau, l)/4\tau^2. \tag{2.10}$$

This proves Proposition 2.8 for $n=2$.

Now we are going to examine the case $n>2$. We may assume $\alpha\geqslant n-1$. By Kawamata–Viehweg Vanishing Theorem we have

$$h^1(X, \mathcal{O}_X(lD - j_1 H_X)) = 0 \quad \text{for any } 0\leqslant j_1 \leqslant \alpha-1.$$

Therefore, from the hyperplane section exact sequence

$$0 \to \mathcal{O}_X(lD-(j_1+1)H_X) \to \mathcal{O}_X(lD - j_1 H_X) \to \mathcal{O}_X(lD - j_1 H_X)\otimes \mathcal{O}_{X^{(n-1)}} \to 0,$$

we get

$$h^0(X, \mathcal{O}_X(lD)) \geqslant \sum_{j_1=0}^{\alpha-2} h^0(X^{(n-1)}, \mathcal{O}_X(lD - j_1 H_X)\otimes \mathcal{O}_{X^{(n-1)}}).$$

On the other hand, by Kawamata–Viehweg Vanishing Theorem again, we also have, for any $0\leqslant j_1\leqslant \alpha-2$,

$$h^0(X^{(n-1)}, \mathcal{O}_X(lD - j_1 H_X)\otimes \mathcal{O}_{X^{(n-1)}}) = h^0(X^{(n-1)}, \mathcal{O}_{X^{(n-1)}}((l-j_1\tau)D^{(n-1)})),$$

where $X^{(n-1)}=H_X$ denotes the general hyperplane section of X and $D^{(n-1)}$ denotes the restriction of D to $X^{(n-1)}$. It follows that

$$h^0(X, \mathcal{O}_X(lD)) \geqslant \sum_{j_1=0}^{\alpha-2} h^0(X^{(n-1)}, \mathcal{O}_{X^{(n-1)}}((l-j_1\tau)D^{(n-1)})).$$

Iterating the previous argument for the successive linear sections of X, we get

$$h^0(X, \mathcal{O}_X(lD)) \geqslant \sum_{j_1=0}^{\alpha-2}\sum_{j_2=0}^{\alpha-3-j_1}\cdots\sum_{j_{n-2}=0}^{\alpha-(n-1)-j_1-j_2-\cdots-j_{n-3}} \times h^0(X^{(2)}, \mathcal{O}_{X^{(2)}}((l-(j_1+j_2+\cdots+j_{n-2})\tau)D^{(2)})), \tag{2.11}$$

where $X^{(2)}$ denotes the general linear section of X of dimension 2, and $D^{(2)}$ the restriction of D on $X^{(2)}$. Notice that $X^{(2)}$ is of general type because X is. Hence from (2.10) we have

$$\begin{aligned} &h^0(X^{(2)}, \mathcal{O}_{X^{(2)}}((l-(j_1+j_2+\cdots+j_{n-2})\tau)D^{(2)})) \\ &\quad \geqslant d[l-(j_1+j_2+\cdots+j_{n-2})\tau][l-(j_1+j_2+\cdots+j_{n-2})\tau \\ &\qquad -\sigma-(n-2)\tau]/2\tau^2. \end{aligned} \tag{2.12}$$

Combining (2.11) with (2.12) we obtain (2.8a). □

We are in position to prove Theorem 0.1 under the hypothesis (B), with $a \geqslant 2$. Fix any real number λ such that

$$0 < \lambda < \frac{(a-1)^{n-1}(2a+n-2)}{2n!\sqrt{(2t)^n}}. \tag{2.13}$$

As in the case $a=1$, it suffices to prove that the set of smooth, projective subvarieties X of V of dimension n with

$$h^0(X, \mathcal{O}_X(aK_X - bH_X)) \leqslant \lambda d^{(n+2)/2}, \tag{2.14}$$

is bounded. As in the case $a=1$, one sees that $NS(X)$ has rank 1. Therefore, for some ample divisor D on X, and suitable integers σ and τ, we have $H_X \equiv \tau D$, and $K_X \equiv \sigma D$. Put $e=\sigma/\tau$ and notice that $e=(2g-2-(n-1)d)/d$. By Theorem 2.1 we may assume $e > \mathrm{O}(1)$, in particular X is of general type. Moreover, if we put $l = a\sigma - b\tau$ then we have $l > \sigma$ because $a \geqslant 2$. Also, by Kawamata–Viehweg Vanishing Theorem, we have $h^0(X, \mathcal{O}_X(aK_X - bH_X)) = h^0(X, \mathcal{O}_X(lD))$. Hence we can apply Proposition 2.8 and from (2.8a) we get

$$h^0(X, \mathcal{O}_X(aK_X - bH_X)) \geqslant \frac{d}{2n!}[(a-1)^{n-1}(2a+n-2)e^n + \mathrm{O}(e^{n-1})]. \tag{2.15}$$

Now fix any real number μ such that $\lambda < \mu < \frac{(a-1)^{n-1}(2a+n-2)}{2n!\sqrt{(2t)^n}}$ (compare with (2.13)), and put

$$\lambda_1 = \sqrt[n]{\frac{n!\mu}{2^{n-1}(a-1)^{n-1}(2a+n-2)}}. \tag{2.16}$$

We have $0 < \lambda_1 < \frac{1}{2\sqrt{2t}}$ (see (2.1a)). Hence, from Theorem 2.1, we may assume $g > \lambda_1 d^{3/2}$. From (2.15) we obtain

$$\begin{aligned} h^0(X, \mathcal{O}_X(aK_X - bH_X)) \geqslant{} & \frac{(a-1)^{n-1}}{n!} 2^{n-1}(2a+n-2)\lambda_1^n d^{(n+2)/2} \\ & + \mathrm{O}(d^{(n+1)/2}). \end{aligned} \tag{2.17}$$

In view of 2.16, comparing 2.17 with 2.14 we deduce that d is bounded.

In a similar way as under the hypothesis (A), with $n \neq 3$, considering codimension two balanced complete intersections on V (when V is smooth), one proves the sharpness of Theorem 0.1 under the hypothesis (B), when $a \geqslant 2$, in the sense of (0.1c).

This concludes the proof of Theorem 0.1 under the hypothesis (B), when $a \geqslant 2$.

Remark 2.18. (i) As already remarked in (2.3), we notice that, taking into account Castelnuovo–Harris bound for the geometric genus of a projective variety [18], in order to prove Theorem 0.1 under the hypothesis (B), it suffices to prove the existence of a positive real number λ such that the set of smooth, projective subvarieties X of V of dimension n with $h^0(X, \mathcal{O}_X(aK_X - bH_X)) \leqslant \lambda d^{n/2}$, is bounded. The corresponding claim holds true also under the hypothesis (D) (compare with (4.3.2) in Section 4). But this is not true under the hypothesis (A) (and (C) and (E), see below).

(ii) With obvious modification, the proof of Theorem 0.1 under the hypothesis (B) with $a = 1$ works also for $a \geqslant 2$.

3. The proof of Theorem 0.1 under the hypothesis (C)

As in the proof of Theorem 0.1 under the hypothesis (A), in order to prove Theorem 0.1 under the hypothesis (C), we may assume $a = 1$ and $b = 0$. This said, the proof consists in showing the existence of a suitable constant $\mu > 0$ such that

$$p_g(X) - c\left(\sum_{h=1}^{n-1} p_g(X^{(h)})\right) \geqslant \mu d^{(n+2)/2},$$

for any smooth and n-dimensional subvariety $X \subset V$ of degree $d > O(1)$. In order to prove this, first we compare the geometric genus of X with the arithmetic genus. This is done in Corollary (3.5) below. To prove this result, we need some preliminaries. First we prove Lemma (3.1) and Proposition (3.2) which allow us to control the difference between the geometric genus and the arithmetic genus in terms of the arithmetic genera of the linear sections. Next we need Lemma (3.3) and Lemma (3.4) to obtain a numerical control of these arithmetic genera.

Lemma 3.1. *Let $Y \subset \mathbf{P}^r$ be an irreducible, smooth, projective variety of dimension $m \geqslant 2$. For any $1 \leqslant j \leqslant m$, denote by $Y^{(j)}$ the general linear section of Y of dimension j. Then we have*

$$h^j(Y, \mathcal{O}_Y) \leqslant p_g(Y^{(j)}).$$

Proof. Use Hyperplane Lefschetz Theorem and induction on m. □

Proposition 3.2. *Let $Y \subset \mathbf{P}^r$ be an irreducible, smooth, projective variety of dimension $m \geqslant 1$. Then there exist integers $c_0, \ldots, c_{m-1}$ and $d_0, \ldots, d_{m-1}$, depending only on m, and such that*

$$c_0 + \sum_{j=1}^{m-1} c_j \chi(\mathcal{O}_{Y^{(j)}}) \leqslant p_g(Y) + (-1)^{m+1}\chi(\mathcal{O}_Y) \leqslant d_0 + \sum_{j=1}^{m-1} d_j \chi(\mathcal{O}_{Y^{(j)}}).$$

Proof. The case $m = 1$ being trivial, we may assume $m \geqslant 2$ and argue by induction on m. We have

$$p_g(Y) + (-1)^{m+1}\chi(\mathcal{O}_Y) = h^m(Y, \mathcal{O}_Y) + (-1)^{m+1}\sum_{j=0}^{m}(-1)^j h^j(Y, \mathcal{O}_Y)$$
$$= (-1)^{m+1}\sum_{j=0}^{m-1}(-1)^j h^j(Y, \mathcal{O}_Y).$$

By Lemma (3.1) we deduce

$$-\left(1 + \sum_{j=1}^{m-1} p_g(Y^{(j)})\right) \leqslant p_g(Y) + (-1)^{m+1}\chi(\mathcal{O}_Y) \leqslant 1 + \sum_{j=1}^{m-1} p_g(Y^{(j)}),$$

and our claim follows by using the induction hypothesis on the general linear sections $Y^{(1)}, \ldots, Y^{(m-1)}$. □

Lemma 3.3. *Let $V \subset \mathbf{P}^r$ be an irreducible, smooth, projective variety of dimension $n + 2 \geqslant 7$. Assume that for $1 \leqslant i \leqslant 3$, any algebraic class in $H^{2i}(V, \mathbf{C})$ is a multiple of H_V^i, where H_V is the hyperplane section of V. Let $X \subset V$ be an irreducible, smooth, projective subvariety of dimension n, with degree d, and assume that $d > \mathrm{O}(1)$. Then*

$$c_1^2(N_{X,V}) \geqslant 2c_2(N_{X,V}) > 0.$$

Proof. We already know that, by our hypothesis on the cohomology of V, X is numerically subcanonical (see (2.4) before). Hence we have $K_X = eH_X$ in $H^2(X, \mathbf{C})$, where $e = (2g - 2 - (n-1)d)/d$ (as usual, g denotes the linear genus of X). Therefore we have $c_1(N_{X,V}) = n_1 H_X$ in $H^2(X, \mathbf{C})$, where

$$n_1 = e - q,$$

with $K_V = qH_V$ in $H^2(V, \mathbf{C})$. On the other hand, from the self-intersection formula, we have $c_2(N_{X,V}) = n_2 H_X^2$ in $H^4(X, \mathbf{C})$, where

$$n_2 = d/t$$

($t =$ degree of V). Hence we only have to prove that

$$n_1^2 \geqslant 2n_2.$$

To this purpose, fix a constant $\gamma \gg 0$ such that the twisted tangent bundle $T_V(\gamma)$ is globally generated. Then also $N_{X,V}(\gamma)$ is, and therefore we have the positivity of its Segre class (see [17])

$$-s_5(N_{X,V}(\gamma)) = c_1^5(N_{X,V}(\gamma)) - 4c_1^3(N_{X,V}(\gamma))c_2(N_{X,V}(\gamma)) + 3c_1(N_{X,V}(\gamma))c_2^2(N_{X,V}(\gamma)) \geqslant 0. \quad (3.3a)$$

Now notice that $c_1(N_{X,V}(\gamma)) = (n_1 + 2\gamma)H_X$, and $c_2(N_{X,V}(\gamma)) = (n_2 + \gamma n_1 + \gamma^2)H_X^2$. By Theorem 2.1 we may choose the constant γ independently of X, so that

$$n_1 + 2\gamma > 0 \quad \text{and} \quad n_2 + \gamma n_1 + \gamma^2 > 0. \tag{3.3b}$$

Therefore previous inequality (3.3a) yields

$$y^2 - 4y + 3 \geqslant 0,$$

where

$$y = (n_1 + 2\gamma)^2/(n_2 + \gamma n_1 + \gamma^2).$$

It follows that either $y \leqslant 1$ or $y \geqslant 3$. From the positivity

$$-s_3(N_{X,V}(\gamma)) = [(n_1 + 2\gamma)^3 - 2(n_1 + 2\gamma)(n_2 + \gamma n_1 + \gamma^2)] \cdot H_X^3 \geqslant 0,$$

we have $2(n_2 + \gamma n_1 + \gamma^2) \leqslant (n_1 + 2\gamma)^2$, i.e. $y \geqslant 2$. The above argument implies that $y \geqslant 3$, i.e.

$$(3n_2 - n_1^2) - \gamma n_1 - \gamma^2 \leqslant 0.$$

Now suppose $n_1^2 \leqslant 2n_2$. From the previous inequality we have

$$n_1^2/2 - \gamma n_1 - \gamma^2 \leqslant 0.$$

We obtain $n_1 \leqslant \mathrm{O}(1)$ which, by Theorem 2.1, implies that d is bounded. This is in contrast with our hypothesis $d > \mathrm{O}(1)$. Hence we must have $n_1^2 \geqslant 2n_2$, and this concludes the proof of Lemma (3.3). □

Lemma 3.4. *Let $V \subset \mathbf{P}^r$ be an irreducible, smooth, projective variety of dimension $n + 2 \geqslant 7$. Assume that, for $1 \leqslant i \leqslant 3$, any algebraic class in $H^{2i}(V, \mathbf{C})$ is a multiple of H_V^i, where H_V is the hyperplane section of V. Moreover assume that, for $1 \leqslant i \leqslant n$, there exist rational numbers $v_1, \dots, v_n$ such that $c_i(T_V) = v_i H_V^i$ in $H^{2i}(V, \mathbf{C})$. For any irreducible, smooth, projective subvariety $X \subset V$ of dimension n, with degree d and linear genus g, put $n_1 = (2g - 2 - (n-1)d)/d + v_1$. Assume $d > \mathrm{O}(1)$. Then $n_1 > 0$ and, for any $1 \leqslant h \leqslant n$, one has*

$$|\chi(\mathcal{O}_{X^{(h)}})| \leqslant d\,\mathrm{O}(n_1^h).$$

Proof. Fix an integer $h \in \{1, \dots, n\}$. By Hirzebruch–Riemann–Roch Theorem we know that $\chi(\mathcal{O}_{X^{(h)}})$ is equal to the degree of the top Todd class of $X^{(h)}$. This is the degree of a certain linear combination, with coefficients depending only on h, of monomials like

$$c_1^{\alpha_1}(T_{X^{(h)}}) \cdot c_2^{\alpha_2}(T_{X^{(h)}}) \dots c_h^{\alpha_h}(T_{X^{(h)}}),$$

with $\alpha_1, \dots, \alpha_h$ non-negative integers, and $h = \sum_{j=1}^{h} j\alpha_j$. From the natural exact sequence

$$0 \to T_{X^{(h)}} \to T_{V^{(h+2)}} \otimes \mathcal{O}_{X^{(h)}} \to N_{X^{(h)},V^{(h+2)}} \to 0,$$

we may compute the Chern classes of $X^{(h)}$ in terms of the Chern classes of $T_{V^{(h+2)}} \otimes \mathcal{O}_{X^{(h)}}$, and the Chern classes of $N_{X^{(h)},V^{(h+2)}}$. With the same notation as in the proof of Lemma (3.3), we have

$$c_1(N_{X^{(h)},V^{(h+2)}}) = n_1 H_{X^{(h)}} \quad \text{and} \quad c_2(N_{X^{(h)},V^{(h+2)}}) = n_2 H^2_{X^{(h)}}.$$

Taking into account our hypothesis on the Chern classes of *V*, it follows that $\chi(\mathcal{O}_{X^{(h)}})$ is equal to a certain linear combination, with coefficients depending only on *h*, of monomials like

$$dn_1^{\alpha_1} n_2^{\alpha_2} v_1^{\beta_1} v_2^{\beta_2} \dots v_h^{\beta_h},$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2, \dots, \beta_h$ non-negative integers such that $\alpha_1 + 2\alpha_2 + \sum_{j=1}^h j\beta_j \leqslant h$. From Lemma (3.3) we know that if $d > \mathrm{O}(1)$ then

$$n_1^2 \geqslant 2n_2 > 1.$$

Hence, for a suitable constant *K*, we have

$$|dn_1^{\alpha_1} n_2^{\alpha_2} v_1^{\beta_1} v_2^{\beta_2} \dots v_h^{\beta_h}| \leqslant K d n_1^h.$$

This proves that $|\chi(\mathcal{O}_{X^{(h)}})| \leqslant d\mathrm{O}(n_1^h)$. $\square$

As a consequence of Proposition (3.2), Lemma (3.3) and Lemma (3.4), we obtain the following.

Corollary 3.5. *Let $V \subset \mathbf{P}^r$ be an irreducible, smooth, projective variety of dimension $n+2 \geqslant 7$. Assume that, for $1 \leqslant i \leqslant 3$, any algebraic class in $H^{2i}(V, \mathbf{C})$ is a multiple of H_V^i, where H_V is the hyperplane section of V. Moreover assume that, for $1 \leqslant i \leqslant n$, there exist rational numbers $v_1, \dots, v_n$ such that $c_i(T_V) = v_i H_V^i$ in $H^{2i}(V, \mathbf{C})$. Fix an integer $c \geqslant 0$. For any irreducible, smooth, projective subvariety $X \subset V$ of dimension n, with degree d and linear genus g, put $n_1 = (2g - 2 - (n-1)d)/d + v_1$. Assume $d > \mathrm{O}(1)$. Then $n_1 > 0$ and*

$$p_g(X) - c\left(\sum_{h=1}^{n-1} p_g(X^{(h)})\right) \geqslant (-1)^n \chi(\mathcal{O}_X) + d\mathrm{O}(n_1^{n-1}).$$

We are in position to prove Theorem 0.1 under the hypothesis (C). To this purpose, let $X \subset V$ be a smooth subvariety of codimension 2. By Corollary (3.5), in order to bound $p_g(X) - c(\sum_{h=1}^{n-1} p_g(X^{(h)}))$ from below, it is enough to bound $(-1)^n \chi(\mathcal{O}_X)$ from below. We keep the notation we introduced in the proof of Lemma (3.3). As in the proof of this lemma, using Hirzebruch–Riemann–Roch Theorem, one may write

$$(-1)^n \chi(\mathcal{O}_X) = (-1)^n d(U + R),$$

where *U* is a linear combination, with coefficients depending only on *n*, of monomials like

$$n_1^{\alpha_1} n_2^{\alpha_2},$$

with α_1, α_2 non-negative integers such that $\alpha_1 + 2\alpha_2 = n$, and R is a linear combination, with coefficients depending only on n, of monomials like

$$n_1^{\alpha_1} n_2^{\alpha_2} v_1^{\beta_1} v_2^{\beta_2} \ldots v_h^{\beta_h},$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2, \ldots, \beta_h$ non-negative integers such that $\alpha_1 + 2\alpha_2 + \sum_{j=1}^{h} j\beta_j = n$ and $\alpha_1 + 2\alpha_2 < n$. As in the proof of Lemma (3.3) one sees that

$$(-1)^n dR = d\mathrm{O}(n_1^{n-1}).$$

Summing up we obtain

$$p_g(X) - c\left(\sum_{h=1}^{n-1} p_g(X^{(h)})\right) \geqslant (-1)^n dU + d\mathrm{O}(n_1^{n-1}). \tag{3.6}$$

Now define v and ρ by dividing

$$n = 2v + \rho, \quad 0 \leqslant \rho \leqslant 1,$$

and denote by

$$x_0, x_1, \ldots, x_v$$

the integers, depending only on n, such that

$$(-1)^n U = \frac{1}{(n+2)!}\left(\sum_{j=0}^{v} x_j n_1^{n-2j} n_2^j\right). \tag{3.7}$$

Now we need the following numerical lemma. We will prove it later.

Lemma 3.8. *With the same notation as before, denote by*

$$q(y) = \sum_{j=0}^{v} x_j y^{v-j},$$

and by $q^{(j)}(y)$ its j-th derivative. Then one has $q^{(j)}(2) > 0$, for any $0 \leqslant j \leqslant v$.

As a consequence we have the following.

Corollary 3.9. *There exists a rational number $\varepsilon > 0$, depending only on n, such that, if we put*

$$q_\varepsilon(y) = q(y) - \varepsilon y^v,$$

then one has $q_\varepsilon^{(j)}(2) > 0$, for any $0 \leqslant j \leqslant v$. In particular, if $y \geqslant 2$ then $q_\varepsilon(y) \geqslant q_\varepsilon(2)$.

Now, continuing our computation, from (3.6) and (3.7), and taking into account Lemmas (3.3), (3.8) and Corollary (3.9), we may write

$$
\begin{aligned}
&p_g(X) - c\left(\sum_{h=1}^{n-1} p_g(X^{(h)})\right)\\
&\quad\geqslant \frac{d}{(n+2)!} n_1^{\rho} n_2^{\nu}\left[q_\varepsilon\left(\frac{n_1^2}{n_2}\right) + \varepsilon\left(\frac{n_1^2}{n_2}\right)^{\nu}\right] + d\mathrm{O}(n_1^{n-1})\\
&\quad= \frac{d}{(n+2)!} n_1^{\rho} n_2^{\nu} q_\varepsilon\left(\frac{n_1^2}{n_2}\right) + \frac{d}{(n+2)!}\varepsilon n_1^{n} + d\mathrm{O}(n_1^{n-1})\\
&\quad= \frac{d}{(n+2)!} n_1^{\rho} n_2^{\nu} q_\varepsilon\left(\frac{n_1^2}{n_2}\right) + \frac{d}{(n+2)!} n_1^{n-1}[\varepsilon n_1 + \mathrm{O}(1)]\\
&\quad\geqslant \frac{d}{(n+2)!} n_1^{\rho} n_2^{\nu} q_\varepsilon\left(\frac{n_1^2}{n_2}\right)\\
&\quad\geqslant \frac{d}{(n+2)!} n_1^{\rho} n_2^{\nu} q_\varepsilon(2) \geqslant \frac{d}{(n+2)!} n_2^{\nu+\rho/2} q_\varepsilon(2) = \frac{q_\varepsilon(2)}{(n+2)!\sqrt{t^n}} d^{n/2+1}.
\end{aligned}
$$

Since $\frac{q_\varepsilon(2)}{(n+2)!\sqrt{t^n}}$ is a constant > 0, the previous inequality proves Theorem 0.1 under the hypothesis (C) (as before, considering codimension two balanced complete intersections on V, one proves the sharpness in the sense of (0.1c)).

It remains to prove the numerical Lemma (3.8). To this aim, keep all the notation we introduced before. Since the polynomial $q(y)$ depends only on n, in order to compute it, we may use a complete intersection $X \subset V$ of type (u, v). In this case we have

$$n_1 = u + v \quad \text{and} \quad n_2 = uv.$$

Moreover

$$\chi(\mathcal{O}_X) = p_V(0) - p_V(-u) - p_V(-v) + p_V(-u - v)$$

($p_V(l)$ = Hilbert polynomial of V). We may compute the term $(-1)^n \mathrm{d}U$ assuming that all Chern classes of V are 0. By Hirzebruch–Riemann–Roch Theorem again, this implies that we may assume

$$p_V(l) = t\frac{l^{n+2}}{(n+2)!}$$

(t = degree of V). Since $d = uvt$ we obtain

$$(-1)^n dU = \frac{d}{(n+2)!}\left[\frac{(u+v)^{n+2} - u^{n+2} - v^{n+2}}{uv}\right].$$

In other words, the coefficients $x_0, x_1, \ldots, x_v$ of the polynomial $q(y)$ are defined by the identity

$$\frac{(u+v)^{n+2}-u^{n+2}-v^{n+2}}{uv}=\sum_{j=0}^{v} x_j(u+v)^{n-2j}(uv)^j. \tag{3.10}$$

Now, if we put

$$y=\frac{(u+v)^2}{uv},$$

we have

$$q(y)=y^{v+1}-\frac{u^{n+2}+v^{n+2}}{(u+v)^{\rho}(uv)^{v+1}}. \tag{3.11}$$

Fix a real number $3/2 \leqslant y \leqslant 5/2$. Then we have $y=\frac{(u+v)^2}{uv}$ with $u=\frac{[y-2+\sqrt{-1}\sqrt{y(4-y)}]}{2}$ and $v=1$. Notice that since $|u|=1$, then we have

$$u=exp(\sqrt{-1}\theta)$$

for some real number θ. In particular we have

$$y-2=2\cos\theta \quad \text{and} \quad \sqrt{y(4-y)}=2\sin\theta. \tag{3.12}$$

From (3.11) we get

$$q(y)=y^{v+1}-\frac{u^{n+2}+1}{(u+1)^{\rho}u^{v+1}}, \tag{3.13}$$

and so, taking into account that if $y=2$ then $u=\sqrt{-1}$, we deduce

$$q(2)=\begin{cases} 2^{v+1} & \text{if } n\equiv 0 \quad mod(4),\\ 2^{v+1}+(-1)^{(n-1)/4} & \text{if } n\equiv 1 \quad mod(4),\\ 2^{v+1}+2(-1)^{(n+6)/4} & \text{if } n\equiv 2 \quad mod(4),\\ 2^{v+1}+(-1)^{(n+5)/4} & \text{if } n\equiv 3 \quad mod(4).\end{cases}$$

This proves that $q(2)=q^{(0)}(2)>0$.

Moreover, from (3.10), one sees that $x_0=n+2$, and therefore $q^{(v)}(2)>0$.

It remains to evaluate the derivatives $q^{(j)}(2)$, for $1\leqslant j\leqslant v-1$.

First we examine the case n is even. Hence $n=2v$ and $\rho=0$. In this case, from (3.13), we may write

$$\begin{aligned} q(y) &= y^{v+1}-\left(u^{v+1}+\frac{1}{u^{v+1}}\right)\\ &= y^{v+1}-[exp(\sqrt{-1}(v+1)\theta)+exp(-\sqrt{-1}(v+1)\theta)] \end{aligned}$$

and so we get

$$q(y)=y^{v+1}-2\cos(v+1)\theta. \tag{3.14}$$

One may write $\cos(v+1)\theta$ as a polynomial in $\cos\theta$ of degree $v+1$, i.e.

$$\cos(v+1)\theta = \sum_{l=0}^{v+1} \beta_l (\cos\theta)^l,$$

with suitable integer numbers $\beta_0, \dots, \beta_{v+1}$. By (3.12) we obtain the following polynomial identity

$$q(y) = y^{v+1} - 2\left[\sum_{l=0}^{v+1} \beta_l \frac{(y-2)^l}{2^l}\right].$$

From this formula we deduce, for $1 \leqslant j \leqslant v-1$,

$$q^{(j)}(2) = \frac{j!}{2^{j-1}}\left[2^v \binom{v+1}{j} - \beta_j\right]. \tag{3.15}$$

Now we need the following.

Sublemma. *With the same notation as before, assume* $v \geqslant 0$. *Then for any* $0 \leqslant l \leqslant v+1$ *one has*

$$|\beta_l| \leqslant 2^l \binom{v+1}{l}.$$

Proof of the Sublemma. The case $0 \leqslant v \leqslant 1$ being trivial, we may assume $v \geqslant 2$ and argue by induction on v. From the identity

$$\cos(v+1)\theta = 2\cos\theta\cos v\theta - \cos(v-1)\theta \tag{3.16}$$

and the induction hypothesis, we deduce that

$$\beta_{v+1} = 2^v, \quad \beta_v = 0, \quad \text{and} \quad |\beta_0| \leqslant 1.$$

Hence we only have to estimate $|\beta_l|$ for $1 \leqslant l \leqslant v-1$. To this purpose, put

$$\cos v\theta = \sum_{l=0}^{v} \gamma_l (\cos\theta)^l \quad \text{and} \quad \cos(v-1)\theta = \sum_{l=0}^{v-1} \delta_l (\cos\theta)^l.$$

From (3.16) we have $\beta_l = 2\gamma_{l-1} - \delta_l$. Therefore, using the induction hypothesis, we get

$$|\beta_l| \leqslant 2|\gamma_{l-1}| + |\delta_l| \leqslant 2^l \binom{v}{l-1} + 2^l \binom{v-1}{l} \leqslant 2^l \binom{v+1}{l}. \qquad \square$$

Continuing the computation of $q^{(j)}(2)$ from (3.15) and using the sublemma, we get

$$q^{(j)}(2) \geqslant 2j!\binom{v+1}{j}(2^{v-j} - 1) > 0.$$

This concludes the proof of Lemma (3.8) in the case n even.

Finally assume n is odd, hence $n = 2v + 1$ and $\rho = 1$. In this case, from (3.13), we have

$$q(y) = y^{v+1} - (-1)^{v+1} - \left[\sum_{l=1}^{v+1} (-1)^{v+l-1}(u^l + u^{-l})\right].$$

Therefore we get

$$q(y) = y^{v+1} + (-1)^{v+2} - 2\left[\sum_{l=1}^{v+1} (-1)^{v+l-1} \cos l\theta\right]. \tag{3.17}$$

A direct computation proves Lemma (3.8) for $2 \leqslant v \leqslant 4$. Hence we may assume $v > 4$ and argue by induction on v. From (3.17) we may write

$$q(y) = r(y) + \left\{ y^{v-1} + (-1)^v - 2\left[\sum_{l=1}^{v-1} (-1)^{v+l-3} \cos l\theta\right]\right\},$$

where

$$r(y) = [y^{v+1} - 2\cos(v+1)\theta] - [y^{v-1} - 2\cos(v\theta)].$$

By induction hypothesis, all the derivatives $q^{(j)}(2) - r^{(j)}(2)$ are $\geqslant 0$ for any $j \geqslant 0$. Therefore we only have to prove that $r^{(j)}(2) > 0$ for $1 \leqslant j \leqslant v-1$. This follows by rewriting $r(y)$ as

$$r(y) = [y^{v+1} - 2\cos(v+1)\theta] - [y^v - 2\cos(v\theta)] + y^v - y^{v-1},$$

and using a similar computation as in the case n even (compare with (3.14)). This concludes the proof of Lemma (3.8).

Remark 3.18. From (3.10) one obtains explicit formulae for $x_0, x_1, \ldots, x_v$, i.e. one has, for $0 \leqslant j \leqslant v$,

$$x_j = (-1)^j \binom{n-j+2}{j+1} + \sum_{l=0}^{j-1} (-1)^{j+1+l}(l+1)\binom{n-j+2}{j-1-l}.$$

In particular we see that

$$x_0 = n + 2,$$

and deduce $q(y)$ for low n. For example we have

$$q(y) = \begin{cases} 7y^2 - 14y + 7 & \text{if } n = 5, \\ 8y^3 - 20y^2 + 16y - 2 & \text{if } n = 6, \\ 9y^3 - 27y^2 + 30y - 9 & \text{if } n = 7, \\ 10y^4 - 35y^3 + 50y^2 - 25y + 2 & \text{if } n = 8. \end{cases}$$

4. The proof of Theorem 0.1 under the hypothesis (D)

First we prove a boundedness result for subcanonical subvarieties (see Theorem 4.1 and Corollary 4.3 below), which does not need the assumption $n \geqslant \frac{r+1}{2}$ appearing in the hypothesis (D).

Theorem 4.1. *Let $V \subset \mathbf{P}^r$ be an irreducible, projective variety of dimension m and degree t. Let n be an integer with $n < m \leqslant 2n$, and put $k = m - n$. Assume that for some $1 \leqslant h \leqslant n$ with $h \geqslant k$, the general linear section $V^{(h+k)}$ of dimension $h + k$ of V is smooth, and that either any algebraic class in $H^{2i}(V^{(h+k)}, \mathbf{C})$ is a multiple of $H^i_{V^{(h+k)}}$, for $i \in \{1, k\}$, or k is even and $H^{2i}(V^{(h+k)}, \mathbf{C}) \simeq \mathbf{C}$ for any $i = 1, \ldots, k - 1$. For any subvariety $X \subset V$ of dimension n and any $1 \leqslant h \leqslant n$, denote by $X^{(h)}$ the general h-dimensional linear section of X. Put $d = deg(X)$, $g = p_g(X^{(1)})$, and fix a real number λ such that*

$$0 < \lambda < \frac{1}{2\sqrt[k]{t}}. \tag{4.1a}$$

Then the set of irreducible, projective subvarieties X of V of dimension n such that $X^{(h)}$ is smooth, numerically subcanonical, and such that

$$g \leqslant \lambda d^{(k+1)/k},$$

is bounded.

Proof of Theorem 4.1. We may assume $n = h$ by taking $V = V^{(h+k)}$. Therefore V is smooth. Let $X \subset V$ be any smooth projective subvariety of dimension n, of degree d, numerically subcanonical, and such that $g \leqslant \lambda d^{(k+1)/k}$. We have

$$K_X = eH_X$$

in $H^2(X, \mathbf{Q})$, with $e = (2g - 2 - (n - 1)d)/d$. Since we may assume $d > \mathrm{O}(1)$, we have

$$|e| \leqslant 2\lambda d^{1/k}. \tag{4.1b}$$

Now let q and γ be rational numbers such that $K_V = qH_V$ in $H^2(V, \mathbf{C})$ and $T_V(\gamma)$ is globally generated. Then also $N_{X,V}(\gamma)$ is globally generated. Since

$$c_1(N_{X,V}(\gamma)) = (k\gamma + e - q)H_X, \tag{4.1c}$$

by [17] we deduce $0 \leqslant k\gamma + e - q$. Notice that taking $\gamma > \mathrm{O}(1)$ we may assume

$$1 \leqslant k\gamma + e - q. \tag{4.1d}$$

We need the following lemma. We will prove it later.

Lemma 4.2. *With the same assumption as before, for any $i = 1, \ldots, k$ one has*

$$d(k\gamma + e - q)^{n-k}\mathrm{O}(d^{(i-1)/k}) \leqslant (\gamma H_X)^{k-i} c_i(N_{X,V}) c_1(N_{X,V}(\gamma))^{n-k} \leqslant d(k\gamma + e - q)^{n-k}[\gamma^{k-i}e^i + \mathrm{O}(d^{(i-1)/k})]. \tag{4.2a}$$

Using the previous Lemma 4.2 for $i = k$, we have

$$c_k(N_{X,V})c_1(N_{X,V}(\gamma))^{n-k} \leqslant d(k\gamma + e - q)^{n-k}[e^k + \mathrm{O}(d^{(k-1)/k})]. \tag{4.2b}$$

On the other hand, if any algebraic class in $H^{2k}(V, \mathbf{C})$ is a multiple of H_V^k, then $X = (d/t)H_V^k$ in $H^{2k}(V, \mathbf{C})$, and therefore by the self-intersection formula we get

$$c_k(N_{X,V}) = (d/t)H_X^k.$$

By (4.1c) we deduce

$$c_k(N_{X,V})c_1(N_{X,V}(\gamma))^{n-k} = d^2(k\gamma + e - q)^{n-k}/t. \tag{4.2c}$$

If k is even and $H^{2i}(V, \mathbf{C}) \simeq \mathbf{C}$ for any $i = 1, \ldots, k-1$, Lefschetz Hyperplane Theorem and Hodge–Riemann bilinear relations imply that the intersection form on $H^{2k}(V^{(2k)}, \mathbf{R})$ is positive definite. Hence we have

$$(X^{(k)} - (d/t)H^k_{V^{(2k)}})^2 \geqslant 0.$$

Using the self-intersection formula again we get

$$c_k(N_{X^{(k)}, V^{(2k)}}) = (X^{(k)})^2 \geqslant d^2/t.$$

Using (4.1c) it follows that

$$\begin{aligned} c_k(N_{X,V})c_1(N_{X,V}(\gamma))^{n-k} &= (k\gamma + e - q)^{n-k} c_k(N_{X,V})H_X^{n-k} \\ &= (k\gamma + e - q)^{n-k} c_k(N_{X^{(k)}, V^{(2k)}}) \\ &\geqslant d^2(k\gamma + e - q)^{n-k}/t, \end{aligned} \tag{4.2d}$$

which, by (4.2c), holds in *any* case.

Summing up, from (4.1b), (4.1d), (4.2b) and (4.2d) we obtain

$$d/t \leqslant e^k + \mathrm{O}(d^{(k-1)/k}) \leqslant (2\lambda)^k d + \mathrm{O}(d^{(k-1)/k}).$$

Therefore we have

$$\left[\frac{1}{t} - (2\lambda)^k\right] d + \mathrm{O}(d^{(k-1)/k}) \leqslant 0,$$

which, taking into account (4.1a), proves the boundedness of d. This concludes the proof of Theorem 4.1.

Now we are going to prove Lemma 4.2. To this purpose, we argue by induction on i. When $i = 1$, from (4.1c) we have

$$(\gamma H_X)^{k-1} c_1(N_{X,V}) c_1(N_{X,V}(\gamma))^{n-k} = \gamma^{k-1} d(e-q)(k\gamma + e - q)^{n-k},$$

which, taking into account (4.1d), proves Lemma 4.2 for $i = 1$. Assume then $2 \leqslant i \leqslant k$. From the formula

$$c_i(N_{X,V}(\gamma)) = \sum_{j=0}^{i} \binom{k-j}{i-j} (\gamma H_X)^{i-j} c_j(N_{X,V}),$$

intersecting with $(\gamma H_X)^{k-i} c_1(N_{X,V}(\gamma))^{n-k}$, we get

$$
\begin{aligned}
&(\gamma H_X)^{k-i} c_i(N_{X,V}) c_1(N_{X,V}(\gamma))^{n-k} \\
&\quad = (\gamma H_X)^{k-i} c_i(N_{X,V}(\gamma)) c_1(N_{X,V}(\gamma))^{n-k} \\
&\qquad - \sum_{j=1}^{i-1} \binom{k-j}{i-j} (\gamma H_X)^{k-j} c_j(N_{X,V}) c_1(N_{X,V}(\gamma))^{n-k} \\
&\qquad - \binom{k}{i} \gamma^k d(k\gamma + e - q)^{n-k}.
\end{aligned} \tag{4.2e}
$$

Using induction and (4.1b) we have

$$
\binom{k-j}{i-j} (\gamma H_X)^{k-j} c_j(N_{X,V}) c_1(N_{X,V}(\gamma))^{n-k} = d(k\gamma + e - q)^{n-k} \mathrm{O}(d^{j/k}) \tag{4.2f}
$$

for any $1 \leqslant j \leqslant i-1$. Notice that, in order to obtain the equality in (4.2f), we have to use both inequalities in (4.2a). The equality (4.2f) enables us to control, from above and from below, the terms in (4.2e) which appear in the sum. And in fact, using (4.2e) and (4.2f), we get

$$
\begin{aligned}
&(\gamma H_X)^{k-i} c_i(N_{X,V}) c_1(N_{X,V}(\gamma))^{n-k} \\
&\quad = (\gamma H_X)^{k-i} c_i(N_{X,V}(\gamma)) c_1(N_{X,V}(\gamma))^{n-k} + d(k\gamma + e - q)^{n-k} \mathrm{O}(d^{(i-1)/k}).
\end{aligned} \tag{4.2g}
$$

Now notice that since $N_{X,V}(\gamma)$ is globally generated, by [17] we have

$$
0 \leqslant (\gamma H_X)^{k-i} c_i(N_{X,V}(\gamma)) c_1(N_{X,V}(\gamma))^{n-k}.
$$

Hence the left-hand side inequality in (4.2a) holds. On the other hand one has (use [17] and the proof of Proposition in [23])

$$
c_i(N_{X,V}(\gamma)) c_1(N_{X,V}(\gamma))^{n-i} \leqslant c_1(N_{X,V}(\gamma))^n,
$$

from which, using (4.1b), (4.1c) and (4.1d), we deduce

$$
\begin{aligned}
&(\gamma H_X)^{k-i} c_i(N_{X,V}(\gamma)) c_1(N_{X,V}(\gamma))^{n-k} \\
&\quad = c_i(N_{X,V}(\gamma)) \gamma^{k-i} (k\gamma + e - q)^{n-k} H_X^{n-i} \\
&\quad = \gamma^{k-i} (k\gamma + e - q)^{i-k} c_i(N_{X,V}(\gamma)) (k\gamma + e - q)^{n-i} H_X^{n-i} \\
&\quad = \gamma^{k-i} (k\gamma + e - q)^{i-k} c_i(N_{X,V}(\gamma)) c_1(N_{X,V}(\gamma))^{n-i} \\
&\quad \leqslant \gamma^{k-i} (k\gamma + e - q)^{i-k} c_1(N_{X,V}(\gamma))^n = \gamma^{k-i} d(k\gamma + e - q)^{n+i-k} \\
&\quad = d(k\gamma + e - q)^{n-k} [\gamma^{k-i} e^i + \mathrm{O}(d^{(i-1)/k})].
\end{aligned}
$$

Comparing the previous inequality with (4.2g), we get the second inequality in (4.2a). This concludes the proof of Lemma 4.2.

Corollary 4.3. *Let $V \subset \mathbf{P}^r$ be an irreducible, projective variety of dimension m and degree t. Let n be an integer with $n < m \leqslant 2n$, and put $k = m - n$. Assume that for some $1 \leqslant h \leqslant n$ with $h \geqslant k$, the general linear section $V^{(h+k)}$ of dimension $h + k$ of V is smooth, and that* *either any algebraic class in* $H^{2i}(V^{(h+k)}, \mathbf{C})$ *is a multiple of* $H^i_{V^{(h+k)}}$*, for* $i \in \{1, k\}$*, or k is even and* $H^{2i}(V^{(h+k)}, \mathbf{C}) \simeq \mathbf{C}$ *for any* $i = 1, \ldots, k-1$.

Fix integers $a, b, c \in \mathbf{Z}$ *with* $a \geqslant 1$*, and put* $\varepsilon(a) = min\{\frac{n}{k}+1, \frac{n}{k}+a-1\}$*. For any n-dimensional subvariety X of V denote by* $X^{(h)}$ (K_X *resp.*) *the general linear section of dimension h* (*the canonical divisor resp.*) *of X. Denote by* $p_g(X^{(h)})$ *the geometric genus of* $X^{(h)}$*. Put* $H_X = X^{(n-1)}$ *and* $d = deg(X)$.

Then there exists a strictly positive real number $\lambda > 0$*, depending only on n, a and the ambient variety V, such that the set of irreducible, smooth, subcanonical, projective subvarieties X of V of dimension n such that*

$$h^0(X, \mathscr{O}_X(aK_X - bH_X)) \leqslant \lambda d^{\varepsilon(a)} + c\left(\sum_{1 \leqslant h < \varepsilon(a)-1} p_g(X^{(h)})\right)$$

is bounded.

Proof of Corollary 4.3. First we analyze the case $a = 1$. Fix any real number λ such that

$$0 < \lambda < \frac{1}{n!\sqrt[k]{t^n}}, \tag{4.3a}$$

where $t = deg(V)$. As in the proof of Theorem 0.1 under the hypothesis (B) (see Section 2), one sees that in order to prove the claim, it suffices to prove that the set of smooth, projective, subcanonical subvarieties *X* of *V* of dimension *n* with

$$h^0(X, \mathscr{O}_X(K_X - bH_X)) \leqslant \lambda d^{n/k}, \tag{4.3b}$$

is bounded. Using a similar argument as in the proof of (2.7), one sees that for such subvarieties *X* one has

$$g \leqslant \frac{(n!\lambda)^{1/n}}{2} d^{(k+1)/k} + \mathrm{O}(d).$$

So the boundedness of *d* follows by (4.3a) and Theorem 4.1.

The case $a \geqslant 2$ follows using Proposition 2.8 in a similar manner as in the proof of Theorem 0.1 under the hypothesis (B), with $a \geqslant 2$. □

We are in position to prove Theorem 0.1 in the hypothesis (D). Fix any smooth subvariety *X* of *V* of dimension *n*. Using our hypothesis on the homology of the ambient variety *V*, the inequality $n \geqslant (m+2)/2$, and a Barth–Lefschetz type of argument (see diagram (2.4)), one sees that $NS(X) \otimes \mathbf{C}$ has dimension 1.

When $a = 1$, we assume $n \geqslant (r+1)/2$, and so we may apply Larsen Theorem [20]. With the same argument developed in [1], Proposition 8, p. 69, we deduce that the Picard group of *X* is generated by the hyperplane section. In particular *X* is subcanonical. At this point, Theorem 0.1 in the hypothesis (D) with $a = 1$ is a consequence of Corollary 4.3.

When $a \geqslant 2$, using Theorem 4.1 instead of Theorem 2.1, one proves Theorem 0.1 in the hypothesis (D) using a similar argument as in the proof of Theorem 0.1 in the hypothesis (B) with $a \geqslant 2$. □

Remark 4.4. As a further consequence of Theorem 4.1, we have that *the set of smooth, projective, numerically subcanonical and not of general type subvarieties of dimension n in* $\mathbf{P}^{2n}$, *is bounded*.

5. The proof of Theorem 0.1 under the hypothesis (E)

We keep the notation introduced in Section 4, and assume the hypothesis (E). Fix any real number λ such that

$$0 < \lambda < \frac{(a-1)^{n-1}(2a+n-2)}{2n!t^{n/k}}. \tag{5.1}$$

Let $X \subset V$ be a smooth subvariety of dimension n and degree d, and assume that

$$h^0(X, \mathcal{O}_X(aK_X - bH_X)) - c\left(\sum_{h=1}^{n-1} p_g(X^{(h)})\right) \leqslant \lambda d^{n/k+1}. \tag{5.2}$$

As in (2.15), using Kawamata–Viehweg Vanishing Theorem, one proves that, when $a \geqslant 2$,

$$h^0(X, \mathcal{O}_X(aK_X - bH_X)) \geqslant \frac{d}{2n!}(a-1)^{n-1}(2a+n-2)n_1^n + d\mathrm{O}(n_1^{n-1}),$$

where $n_1 = e - q$, $e = (2g-2-(n-1)d)/d$, and $K_V = qH_V$ in $H^2(V, \mathbf{C})$ (notice that, by Theorem (4.1), we may assume $n_1 \geqslant \varepsilon d^{1/k}$, for a suitable constant $\varepsilon > 0$). Therefore we have

$$\begin{aligned} &h^0(X, \mathcal{O}_X(aK_X - bH_X)) - c\left(\sum_{h=1}^{n-1} p_g(X^{(h)})\right) \\ &\quad \geqslant \frac{d}{2n!}(a-1)^{n-1}(2a+n-2)n_1^n + d\mathrm{O}(n_1^{n-1}) - c\left(\sum_{h=1}^{n-1} p_g(X^{(h)})\right). \end{aligned} \tag{5.3}$$

Now using a Barth–Lefschetz type of argument (see diagram (2.4)), our hypothesis on the codimension k of X and on the cohomology of V implies that, for any $1 \leqslant i \leqslant k$,

$$c_i(N_{X,V}) = n_i H_X^i,$$

in $H^{2i}(X, \mathbf{C})$, for a suitable rational number n_i. Using (4.1c) and the second inequality in (4.2a), we deduce

$$(\gamma H_X)^{k-i}(n_i H_X^i)((k\gamma + e - q)H_X)^{n-k} \leqslant d(k\gamma + e - q)^{n-k}[\gamma^{k-i}e^i + \mathrm{O}(d^{(i-1)/k})].$$

Simplifying the factor $(k\gamma + e - q)^{n-k}$, and taking into account that $H_X^n = d$ and that $n_1 = e - q$, we get

$$n_i \leqslant (n_1 + q)^i + \mathrm{O}(d^{(i-1)/k}) = \mathrm{O}(n_1^i).$$

Similarly, using the first inequality in (4.2a), we get $n_i \geqslant \mathrm{O}(n_1^i)$. In other words, for any $1 \leqslant i \leqslant k$, we have

$$|n_i| \leqslant \mathrm{O}(n_1^i).$$

Using Proposition 3.2 and arguing as in the proof of Lemma 3.4, it follows that

$$p_g(X^{(h)}) \leqslant d\mathrm{O}(n_1^h)$$

for any $1 \leqslant h \leqslant n$. Hence we have

$$\sum_{h=1}^{n-1} p_g(X^{(h)}) \leqslant d\mathrm{O}(n_1^{n-1}).$$

Therefore, from (5.3), we get

$$h^0(X, \mathcal{O}_X(aK_X - bH_X)) - c\left(\sum_{h=1}^{n-1} p_g(X^{(h)})\right) \geqslant \frac{d}{2n!}(a-1)^{n-1}(2a+n-2)n_1^n + d\mathrm{O}(n_1^{n-1}). \quad (5.4)$$

Now fix any real number μ such that $\lambda < \mu < \frac{(a-1)^{n-1}(2a+n-2)}{2n!t^{n/k}}$ (compare with (5.1)), and put

$$\lambda_1 = \sqrt[n]{\frac{n!\mu}{2^{n-1}(a-1)^{n-1}(2a+n-2)}}. \quad (5.5)$$

We have $0 < \lambda_1 < \frac{1}{2\sqrt[k]{t}}$ (see (4.1a)). Hence, from Theorem 4.1, we may assume $n_1 > 2\lambda_1 d^{1/k}$. From (5.4) we obtain

$$h^0(X, \mathcal{O}_X(aK_X - bH_X)) \geqslant \frac{(a-1)^{n-1}}{n!}2^{n-1}(2a+n-2)\lambda_1^n d^{n/k+1} + \mathrm{O}(d^{(n+k-1)/k}). \quad (5.6)$$

In view of (5.5), comparing (5.6) with (5.2) we deduce that d is bounded.

One proves the sharpness of the estimate in a similar way as in the hypothesis (A). This concludes the proof of Theorem 0.1 under the hypothesis (E).

Acknowledgements

We would like to thank F. Zak for some useful discussion on the subject of this paper.

References

[1] E. Amerik, On codimension-two subvarieties of hypersurfaces, J. Reine Angew. Math. 483 (1997) 61–73.

[2] E. Arrondo, I. Sols, On congruences of lines in the projective space, Mem. Soc. Math. France (N.S.) 50 (1992) 1–96.

[3] R. Braun, G. Ottaviani, M. Schneider, F.O. Schreyer, Boundedness for non general type 3-folds in $\mathbf{P}^5$, in: V. Ancona, A. Silva (Eds.), Complex Analysis and Geometry, Proceedings of CIRM 1991, Plenum Press, New York, 1993, pp. 311–338.
[4] M.A. De Cataldo, A finiteness theorem for low codimensional nonsingular subvarieties of quadrics, Trans. Amer. Math. Soc. 349 (6) (1997) 2359–2370.
[5] J. Carlson, M. Green, P. Griffiths, J. Harris, Infinitesimal variations of Hodge structure (I), Compositio Math. 50 (1983) 109–205.
[6] L. Chiantini, C. Ciliberto, A few remarks on the lifting problem, Astérisque 218 (1993) 95–109.
[7] L. Chiantini, C. Ciliberto, V. Di Gennaro, The genus of projective curves, Duke Math. J. 70/2 (1993) 229–245.
[8] C. Ciliberto, V. Di Gennaro, Boundedness for low codimensional subvarieties, preprint (2001).
[9] C. Ciliberto, V. Di Gennaro, Boundedness for surfaces on smooth fourfolds, J. Pure Appl. Algebra 173 (2002) 273–279.
[10] C. Ciliberto, V. Di Gennaro, Boundedness for codimension two subvarieties, Int. J. Math. 13 (5) (2002) 479–495.
[11] A. Dimca, Singularities and Topology of Hypersurfaces, Springer Universitext, New York, 1992
[12] L. Ein, Subvarieties of generic complete intersections, Invent. Math. 94 (1988) 163–170.
[13] D. Eisenbud, J. Harris, Curves in projective space, Séminaire Math. Sup., Les Presses de l'Université de Montréal, Montréal, 1982.
[14] P. Ellia, C. Folegatti, On the canonical map of smooth surfaces in $\mathbf{P}^4$, preprint (2002).
[15] G. Ellingsrud, Ch. Peskine, Sur les surfaces lisses de $\mathbf{P}^4$, Invent. Math. 95 (1989) 1–11.
[16] L. Fania, G. Ottaviani, Boundedness for codimension two submanifolds of quadrics, Collect. Math. 49 (2–3) (1998) 293–315.
[17] W. Fulton, R. Lazarsfeld, Positive polynomials for ample vector bundles, Ann. Math. 118 (1983) 35–60.
[18] J. Harris, A bound on the geometric genus of projective varieties, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 8 (4) (1981) 35–68.
[19] J. Kollár, Rational Curves on Algebraic Varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete 3.Folge, Band 32, Springer, Berlin, 1999.
[20] M.E. Larsen, On the topology of complex projective manifolds, Invent. Math. 19 (1973) 251–260.
[21] A.N. Parshin, I.R. Shafarevich (Eds.), Algebraic Geometry III, Encyclopaedia of Math. Sciences, vol. 36, Springer, Berlin, 1998.
[22] P. Pragacz, J. Ratajski, A Pieri-type theorem for Lagrangian and odd orthogonal Grassmannians, J. Reine Angew. Math. 476 (1996) 143–189.
[23] M. Schneider, Boundedness of low codimensional submanifolds of projective space, Intern. J. Math. 3 (3) (1992) 397–399.