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Limit theorems for diffusions with a random potential

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Abstract

We study the long-time asymptotics of a multi-dimensional diffusion with a random potential satisfying a scaling property. We prove a subdiffusivity property.

Keywords: Subdiffusivity; Random media

1. Introduction

The aim of this paper is to derive limit theorems on the long-time asymptotics of multi-dimensional diffusions in a self-similar (or almost self-similar) random potential. Let $(B(x), x \in \mathbb{R}^n)$ be a random field indexed by $\mathbb{R}^n$, let Q be the law of B and let V be a (deterministic) function from $\mathbb{R}^n$ to $\mathbb{R}$. We define $W = V + B$. Let us consider the (formal) stochastic differential equation:

$$(*) \quad dX_t = d\beta_t - \tfrac{1}{2}\nabla W(X_t)\,dt, \qquad X_0 = 0,$$

where β is an n-dimensional Brownian motion, independent of W.

Intuitively speaking, we first solve equation $(*)$ for a given realization of W (see below for a rigorous definition of the word "solve" in this context). Thus we get a Markov process, say $(X_t^W, t \geqslant 0)$. We are interested in the average w.r.t. W of the law of X_t^W for large t. This problem was first considered by Th.Brox in his 1986 paper, when B is a one-dimensional Brownian motion and $V = 0$. In this case the drift term in $(*)$ is just white noise. Brox proved that the effect of the drift is to slow-down the diffusion: $(1/(\log t)^2)X_t^W$ converges in law to a probability measure μ, i.e. for any continuous bounded function f, $E_Q[E[f(X_t^W/(\log t)^2)]] \to \int f\,d\mu$ when $t \to +\infty$. This property is sometimes called subdiffusivity. The result of Brox has then been extended to a larger class of other one-dimensional random environments by Tanaka (1987) and Kawasu et al. (1989). In Mathieu (1994) we made an attempt to generalize these results to higher dimensions. It turns out that the method of Brox and Tanaka cannot be used for $n > 1$. Thus we introduced a different point

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of view which is based on a simple remark: provided that B has a scaling property, the problem of getting information on the law of X_t^W for large t is equivalent to a zero-white-noise problem, with now a deterministic potential. Many questions on zero-white-noise limits can be answered using the large deviation technique. In our case, since we deal with non-differentiable potentials, there is no large deviation principle. In order to overcome this difficulty, we were led to a different approach of the zero-white-noise problem based on semi-group techniques: this is the content of Mathieu (1995). We shall now apply these ideas to random potentials.

Let us mention that the model we shall consider has been introduced at a physical level by Marinari et al. (1983).

We make the following assumptions on B and V:

(H0) (Continuity) $V(0) = 0$ and Q. a.s. $B(0) = 0$. The function V is continuous and Q. a.s. the function $x \to B(x)$ is continuous.

(H1) (Scaling property) There exists a $\beta > 0$ s.t. for any $c > 0$, the random fields $((1/c)B(c^\beta x), x \in \mathbb{R}^n)$ and B have the same law.

(H2) Q. a.s. the connected components of the set $\{x \text{ s.t. } B(x) < 1\}$ are bounded.

(H3) $R^{-1/\beta} \sup_{|z| \leqslant R} |V(z)| \to 0$ when $R \to +\infty$.

These hypothesis are satisfied, for instance, if $n = 1$, and B is a Brownian motion, or a Bessel process, or a Brownian motion reflected at 0, and $V(x) = |x|^\alpha$ for $0 \leqslant \alpha \leqslant \frac{1}{2}$. When $n > 1$, we can choose for B Lévy's Brownian motion, or its modulus (see Mathieu (1994) for the proof of (H3)). Note that in these examples, the sample paths of W are not differentiable.

Since we do not assume that W is differentiable, we cannot rely on Ito's theory to solve equation $(*)$. In order to provide a solution to equation $(*)$, we use the same approach as in Mathieu (1994): Let f be a probability density on $\mathbb{R}^n$. Let $C(\mathbb{R}^n, \mathbb{R})$ be the set of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$. For any $w \in C(\mathbb{R}^n, \mathbb{R})$, let X^w be the solution in the sense of Dirichlet forms of the S.D.E. $dX_t^w = d\beta_t - (\frac{1}{2})\nabla w(X_t^w)\,dt$, with initial law f. Thus X^w is the Markov process with Dirichlet form $u \to (\frac{1}{2}) \int |\nabla u|^2 e^{-w}$. Let P_f^w be the law of X^w on $C(\mathbb{R}_+, \mathbb{R}^n)$. Let $(\mathcal{X}_t, t \geqslant 0)$ be the coordinate projections of $C(\mathbb{R}_+, \mathbb{R}^n)$. Still following Mathieu (1994), we denote by $\mathbb{P}_f$ the probability on $C(\mathbb{R}_+, \mathbb{R}^n)$ obtained by averaging P_f^w w.r.t. the law of W, i.e. $\mathbb{P}_f(F) = E_Q P_f^W(F)$, for any measurable bounded functional F. We shall use the notation $\mathbb{E}_f$ for the expectation w.r.t. $\mathbb{P}_f$.

Our aim is to express the law of $\mathcal{X}_t$ for large t in terms of the geometry of B and V. We shall need a few definitions from Mathieu (1994) or (1995): let $w \in C(\mathbb{R}^n, \mathbb{R})$. Let D be a bounded domain of $\mathbb{R}^n$. We call D a valley if there exists $a \in \mathbb{R}$ s.t. D is one of the connected components of the set $\{x \text{ s.t. } w(x) < a\}$. We say that D is a r-valley if it is a valley and if $\sup_D w - \inf_D w = r$, where $\sup_D w = \sup_{x \in D} w(x)$. We then call r the depth of D and denote it by $d_1(w, D)$.

More generally, we define the depths of w on D: for $r > 0$, let $f(r)$ be the number of r-valleys of w contained in D. For $i \geqslant 1$, let $d_i(w, D) = \inf\{r \text{ s.t. } f(r) < i\}$. For $r > 0$, let $\Delta_r(w)$ be the connected component of the set $\{x \text{ s.t. } w(x) < r\}$ that contains 0. Let $N_r(w)$ be the number of r-valleys of w contained in $\Delta_r(w)$, and let $(D_r^i(w), 1 \leqslant i \leqslant N_r(w))$ be the r-valleys of w contained in Δ_r.

We shall prove:

Theorem 1. *Let f be a probability density with compact support. There exists a family of functions on $\mathbb{R}_+ \times C(\mathbb{R}^n, \mathbb{R})$, $(a_i, i \in \mathbb{N}^*)$, s.t.*
– $a_i(t,w) = 0$ *for any* $t \in \mathbb{R}_+$ *whenever* $i > N_1(w)$,
– $a_i(t,w) \geqslant 0$ *for any* $t \in \mathbb{R}_+, i \geqslant 0, w \in C(\mathbb{R}^n, \mathbb{R})$,
– *for any t and i, the function $w \to a_i(t,w)$ is measurable,*
– $\sum_i a_i(t,w) = 1$ *for any w and t.*
Furthermore, for any bounded measurable function g

$$\mathbb{E}_f[g((1/(\log t)^\beta)\mathscr{X}_t)] - E_Q\left[\sum_i a_i(t,B) \int_{D_1^i(B)} g\ \mathrm{e}^{-(\log t)B} \Big/ \int_{D_1^i(B)} \mathrm{e}^{-(\log t)B}\right] \to 0 \tag{1}$$

when t tends to $+\infty$.

Comments. (1) implies that the family of the laws of $(1/(\log t)^\beta)\mathscr{X}_t$ under $\mathbb{P}_f$ is tight. Thus $(\log t)^\beta$ is the "right scale". In some examples, it is possible to deduce from (1) that $(1/(\log t)^\beta)\mathscr{X}_t$ converges in law and to compute the limit. First note that this limit will not depend on V.

If $B \geqslant 0$ Q. a.s., then $N_r = 1$ and thus (1) becomes

$$\mathbb{E}_f[g((1/(\log t)^\beta)\mathscr{X}_t)] - E_Q\left[\int_{\Delta_1(B)} g\ \mathrm{e}^{-(\log t)B} \Big/ \int_{\Delta_1(B)} \mathrm{e}^{-(\log t)B}\right] \to 0.$$

From this last formula one can get an expression for the limit in law of $(1/(\log t)^\beta)\mathscr{X}_t$ (see Mathieu, 1994). But in general, since we do not know how to calculate the a_i's, we can only get partial information on the limit of $(1/(\log t)^\beta)\mathscr{X}_t$. Note in particular that (1) implies that, for any domain D

$$\liminf_{t \to +\infty} \mathbb{P}_f[(1/(\log t)^\beta)\mathscr{X}_t \in D] \geqslant Q[\Delta_1(B) \subset D]. \tag{2}$$

Using the same technique as for the proof of Theorem 1, it is also possible to estimate other quantities related to the long-time behaviour of $\mathscr{X}_t$ such as the hitting times of a sphere whose radius tends to $+\infty$. This would lead to inequalities similar to (2). We shall only quote one result: for $r \in \mathbb{R}_+$, let $\sigma_r = \inf\{t \geqslant 0 \text{ s.t. } W(\mathcal{X}_t) = r\}$. It turns out that the limiting law of $(1/r)\log \sigma_r$ can be written explicitly in term of the geometry of B:

Theorem 2. *Let f be a probability density with compact support. The law of $(1/r) \log \sigma_r$ under $\mathbb{P}_f$ converges, when r tends to $+\infty$, to the law of $d_1(B, \Delta_1(B))$ under Q.*

Thus, in order to obtain more precise statements on the long-time behaviour of $\mathscr{X}_t$, it would be necessary to compute the "law" of $\Delta_1(B)$. This can be easily done if B is a one-dimensional Brownian motion. To our knowledge, this problem is unsolved for random fields indexed by $\mathbb{R}^n$, $n > 1$, such as Levy's Brownian motion.

Proof of Theorem 1. We follow the same method as in Mathieu (1994): for $\varepsilon > 0$ and $w \in C(\mathbb{R}^n, \mathbb{R})$, let $X^{\varepsilon,w}$ be the solution, in the sense of Dirichlet forms, of the S.D.E. $\mathrm{d}\, X_t^{\varepsilon,w} = \varepsilon \mathrm{d}\beta_t - \frac{1}{2}\nabla w(X_t^{\varepsilon,w})\mathrm{d}t$. As previously, let $P_f^{\varepsilon,w}$ be the law of $X^{\varepsilon,w}$, when the initial law is f, and let $\mathbb{P}_f^{\varepsilon}$ be the probability on $C(\mathbb{R}_+, \mathbb{R}^n)$ defined by $\mathbb{P}_f^{\varepsilon}(F) = E_Q P_f^{\varepsilon,W^{\varepsilon}}(F)$, where $W^{\varepsilon}(x) = \varepsilon^2 V(\varepsilon^{-2\beta}x) + B(x)$. Also let $f_{\varepsilon}(x) = \varepsilon^{-2\beta n} f(\varepsilon^{-2\beta}x)$.

It follows from the hypothesis H1 that the law of $(\varepsilon^{2\beta}\mathscr{X}_t, t \geqslant 0)$ under $\mathbb{P}_f$ coincides with the law of $(\mathscr{X}(\varepsilon^{2(2\beta-1)}t), t \geqslant 0)$ under $\mathbb{P}_{f_{\varepsilon}}^{\varepsilon}$ (see Lemma III.2.2 of Mathieu, 1994). Therefore, the statement of Theorem 1 is equivalent to:

$$\mathbb{E}_{f_{\varepsilon}}^{\varepsilon}[g(\mathscr{X}(\mathrm{e}^{1/\varepsilon^2}))] - E_Q\left[\sum_i a_i(\mathrm{e}^{1/\varepsilon^2}, B) \int_{D_1^i(B)} g\, \mathrm{e}^{-B/\varepsilon^2} \Big/ \int_{D_1^i(B)} \mathrm{e}^{-B/\varepsilon^2}\right] \to 0. \tag{3}$$

We shall in fact prove an almost sure version of (3): Let $\hat{C}(\mathbb{R}^n, \mathbb{R})$ be the set of continuous functions w, from $\mathbb{R}^n$ to $\mathbb{R}$ s.t. for any $a \in \mathbb{R}$ the connected components of the set $\{x \text{ s.t. } w(x) < a\}$ are bounded. For $w \in \hat{C}(\mathbb{R}^n, \mathbb{R})$, let $R_1(w)$ be the set of $r > 0$ such that:

– there exists $x \in \Delta_r(w)$ s.t. $w(x) < 0$ and

– there exists $\rho > r$ s.t. for every $i \leqslant N_r(w)$, there exists a ρ-valley, D, s.t. $D_r^i(w) \subset D$ and $d_2(w, D) < r$.

Note that if $r \in R_1(w)$ then $d_1(\Delta_r(w)) > r$.

Let w^{ε} be a family of functions in $C(\mathbb{R}, \mathbb{R}^n)$ s.t. $w^{\varepsilon} \to w$ uniformly on compact sets of $\mathbb{R}^n$.

We claim that, for any probability density f with compact support, if $r \in R_1(w)$

$$E_{f_{\varepsilon}}[g(X^{\varepsilon,w^{\varepsilon}}(\mathrm{e}^{r/\varepsilon^2}))] - \sum_i b_i(\varepsilon, w) \int_{D_r^i(w)} g\, \mathrm{e}^{-w/\varepsilon^2} \Big/ \int_{D_r^i(w)} \mathrm{e}^{-w/\varepsilon^2} \to 0 \tag{4}$$

for any bounded continuous function g, where the b_i's satisfy: $b_i(\varepsilon, w) \geqslant 0$, $b_i(\varepsilon, w) = 0$ if $i > N_r(w)$, $\sum_i b_i(\varepsilon, w) = 1$.

Proof of (4). The proof of (4) repeats arguments of Mathieu (1994, 1995), but since we had assumed in Mathieu (1995) that $w^{\varepsilon} = w$ and $\nabla w \in L_{\infty,\mathrm{loc}}$ we cannot directly apply Theorem 6. For a domain D, let $\tau^{\varepsilon,w^{\varepsilon}}(D) = \inf\{t > 0 \text{ s.t. } X_t^{\varepsilon,w^{\varepsilon}} \notin D\}$; We start with

Lemma 1. *Let D be a valley. For $\eta > 0$,*

$$P_{f_{\varepsilon}}(\varepsilon^2 \log \tau^{\varepsilon,w^{\varepsilon}}(D) \geqslant d_1(w, D) - \eta) \to 1.$$

Similarly, for any probability density with support contained in D, k, and for any $\eta > 0$,

$$P_k(\varepsilon^2 \log \tau^{\varepsilon,w^{\varepsilon}}(D) \geqslant d_1(w, D) - \eta) \to 1.$$

Proof. It is sufficient to show that $E_{f_{\varepsilon}}(\exp(-\mathrm{e}^{(d_1(w,D)-\eta)/\varepsilon^2}\tau^{\varepsilon,w^{\varepsilon}}(D))) \to 0$. If $w^{\varepsilon} = w$ for any ε, then the lemma follows from Theorem II.3 of Mathieu (1994). The general case can be handled identically.

Since $P_{f_{\varepsilon}}(\tau^{\varepsilon,w^{\varepsilon}}(\Delta_r) \geqslant e^{(r+\eta)/\varepsilon^2}) \to 1$, for $\eta > 0$ small enough (Remember that $d_1(w, \Delta_r) > r$), it is enough to prove (4) for the process $X^{\varepsilon,w^{\varepsilon}}$ reflected on the boundary of $\Delta_{\rho}(w)$.

Following Mathieu (1995), for any bounded domain D with a smooth boundary, we shall denote by $({}^r\mathcal{E}^{\varepsilon,w^\varepsilon,D}, {}^r\mathcal{F}^{\varepsilon,w^\varepsilon,D})$ the smallest closed extension of the Dirichlet form ${}^r\mathcal{E}^{\varepsilon,w^\varepsilon,D}(u) = (\varepsilon^2/2)\int_D |\nabla u|^2 \exp(-w^\varepsilon/\varepsilon^2)$, for $u \in C^\infty(\bar{D})$. ${}^r\mathcal{E}^{\varepsilon,w^\varepsilon,D}$ is the Dirichlet form of the Markov process $({}^rX_t^{\varepsilon,w^\varepsilon,D}, t \geqslant 0)$ obtained from $X^{\varepsilon,w^\varepsilon}$ by reflection on the boundary of D. We denote by $({}^rP_t^{\varepsilon,w^\varepsilon,D}, t\geqslant 0)$ its semi-group.

Let $\gamma^{\varepsilon,w^\varepsilon,D}(\mathrm{d}x) = \exp(-w^\varepsilon(x)/\varepsilon^2) I_{\bar{D}}(x)/\int_D \exp(-w^\varepsilon/\varepsilon^2)\mathrm{d}x$ be its invariant probability measure.

Let $(-\Lambda_i^{\varepsilon,w^\varepsilon}(D), \psi_i^{\varepsilon,w^\varepsilon,D})_{i\geqslant 0}$ be a spectral decomposition of the generator of ${}^r\mathcal{E}^{\varepsilon,w^\varepsilon,D}$, with the conventions: $\Lambda_0^{\varepsilon,w^\varepsilon}(D) = 0$, $\psi_0^{\varepsilon,w^\varepsilon,D} = 1$, $\Lambda_i^{\varepsilon,w^\varepsilon}(D) \leqslant \Lambda_{i+1}^{\varepsilon,w^\varepsilon}(D)$, $\int |\psi_i^{\varepsilon,w^\varepsilon,D}|^2 \gamma^{\varepsilon,w^\varepsilon,D} = 1$, $\int \psi_i^{\varepsilon,w^\varepsilon,D}\psi_j^{\varepsilon,w^\varepsilon,D}\gamma^{\varepsilon,w^\varepsilon,D} = 0$, when $i \neq j$.

Our basic tool is the following result on the asymptotics of $\Lambda_i^{\varepsilon,w^\varepsilon}(D)$: assume that D is a valley, then for any $i \geqslant 1$,

$$\varepsilon^2 \log \Lambda_i^{\varepsilon,w^\varepsilon}(D) \to -d_{i+1}(w,D), \tag{5}$$

(5) is proved in Theorem 2 of Mathieu (1995) when $w^\varepsilon = w$ for every ε. It can be easily extended to the general case: indeed assume that $\|w^\varepsilon - w\|_{L_\infty(\bar{D})} \leqslant \eta$. Since

$$\Lambda_1^{\varepsilon,w^\varepsilon}(D) = \inf_{u\in C^\infty(\bar{D})} \left(\int_{\bar{D}}\right) |\nabla u|^2 \exp(-w^\varepsilon/\varepsilon^2) \Big/ \int_{\bar{D}} |u|^2 \exp(-w^\varepsilon/\varepsilon^2)),$$

we have $|\Lambda_1^{\varepsilon,w^\varepsilon}(D) - \Lambda_1^{\varepsilon,w}(D)| \leqslant \exp(10\eta/\varepsilon^2)\Lambda_1^{\varepsilon,w}(D)$. Thus (5) follows at once for $i = 1$. The proof for the other eigenvalues is similar.

Lemma 2. *Let $D = D_r^1(w)$ and $t_\varepsilon = \exp(r/\varepsilon^2)$, where $r \in R_1(w)$. For any probability density k with support contained in D, for any continuous bounded function g*

$$E_k(g(X^{\varepsilon,w^\varepsilon}(t_\varepsilon))) - \int g\gamma^{\varepsilon,w,D} \to 0.$$

Proof. First assume that k is bounded.

Since $r \in R_1(w)$, we can choose $\rho > r$ and D' s.t. D' is a ρ-valley, $D \subset D'$ and $d_2(D') < r$. It follows from Lemma 1 that $P_k(\varepsilon^2 \log(\tau^{\varepsilon,w^\varepsilon}(D')) \geqslant \rho - \eta) \to 1$ for $\eta > 0$. Since $\rho > r$, $P_k(\varepsilon^2 \log(\tau^{\varepsilon,w^\varepsilon}(D')) \geq r + \eta) \to 1$ for some $\eta > 0$.
Therefore,

$$\text{(i)}\quad E_k(g(X^{\varepsilon,w^\varepsilon}(t_\varepsilon))) - E_k(g({}^rX^{\varepsilon,w^\varepsilon,D'}(t_\varepsilon))) \to 0. \tag{6}$$

On the other hand,

$$\begin{aligned}
&(E_k(g({}^rX^{\varepsilon,w^\varepsilon,D'}(t_\varepsilon))) - \int g\gamma^{\varepsilon,w,D'})^2 \\
&= \left(\int k\left({}^rP_{t_\varepsilon}^{\varepsilon,w^\varepsilon,D'} g - \int g\gamma^{\varepsilon,w^\varepsilon,D'}\right)\right)^2 \\
&\leqslant \mathrm{e}^{-2\Lambda_1^{\varepsilon,w^\varepsilon}(D')t_\varepsilon} \int (k/\gamma^{\varepsilon,w^\varepsilon,D'})^2 \gamma^{\varepsilon,w^\varepsilon,D'} \int g^2 \gamma^{\varepsilon,w^\varepsilon,D'}.
\end{aligned}$$

From (5) we know that $\varepsilon^2 \log(\Lambda_1^{\varepsilon,w^\varepsilon}(D')t_\varepsilon) \to r - d_2(w,D') > 0$. Since k is bounded, then $\limsup \varepsilon^2 \log \int (k/\gamma^{\varepsilon,w^\varepsilon,D'})^2 \gamma^{\varepsilon,w^\varepsilon,D'} < +\infty$, therefore

$$\text{(ii)}\quad E_k(g({}^rX^{\varepsilon,w^\varepsilon,D}(t_\varepsilon))) - \int g\gamma^{\varepsilon,w,D} \to 0. \tag{7}$$

Besides it is easy to see that $r > d_2(D')$ implies that

$$\text{(iii)}\quad \int g\gamma^{\varepsilon,w^\varepsilon,D'} - \int g\gamma^{\varepsilon,w^\varepsilon,D} \to 0 \tag{8}$$

(i)–(iii) yield the conclusion of the lemma.

The case of unbounded k can be treated by approximation and truncation.

Lemma 3. *Let $D = \Delta_r(w)$, $N = N_r(w)$ and $t_\varepsilon = \exp(r/\varepsilon^2)$, where $r \in R_1(w)$.*

$$E_{f_\varepsilon}[g({}^rX^{\varepsilon,w^\varepsilon,D}(t_\varepsilon)] - \sum_{0\leqslant i\leqslant N-1} b_i(\varepsilon,w)\int_{D_r^i(w)} g\,\mathrm{e}^{-w/\varepsilon^2} \Big/ \int_{D_r^i(w)} \mathrm{e}^{-w/\varepsilon^2} \to 0 \tag{9}$$

for any bounded continuous function g.

Proof. First assume that f is bounded. For $k \in L^2(\bar{D},\gamma^{\varepsilon,w^\varepsilon,D})$, let $\Phi^\varepsilon(k) = \sum_{i<N} \mathrm{e}^{-\Lambda_i^{\varepsilon,w^\varepsilon}(D)t_\varepsilon}(\int k\gamma^{\varepsilon,w^\varepsilon,D})\psi_i^{\varepsilon,w^\varepsilon,D}\gamma^{\varepsilon,w^\varepsilon,D}$. Thus $\Phi^\varepsilon(k)$ is a (nonnecessarily positive) bounded measure on $\bar{D}$. The idea of the proof is simple: first note that $\Phi^\varepsilon(k)$ is close to the law of ${}^rX^{\varepsilon,w^\varepsilon,D}(t_\varepsilon)$ with initial law $k(x)\,\mathrm{d}x$. Φ^ε maps $L^2(\bar{D},\gamma^{\varepsilon,w^\varepsilon,D})$ onto an N-dimensional subspace of the set of measures on $\bar{D}$. If the support of k is contained in $D_r^i(D)$, then, by the previous lemma, $\Phi^\varepsilon(k)$ is close to $\gamma^{\varepsilon,w,D_r^i(w)}$. The measures $(\gamma^{\varepsilon,w,D_r^i(w)}, 1\leqslant i\leqslant N)$ are linearly independent. Thus the range of Φ^ε is close to the vector space spanned by the $(\gamma^{\varepsilon,w,D_r^i(w)}, 1\leqslant i\leqslant N)$, which is exactly what Lemma 3 says. We now give the details of the proof:

By definition of Λ_i,

$$\begin{aligned}&|E_k(g({}^rX^{\varepsilon,w^\varepsilon,D}(t_\varepsilon))) - \int g\Phi^\varepsilon(k)|^2\\ &\leq \mathrm{e}^{-2\Lambda_N^{\varepsilon,w^\varepsilon}(D)t_\varepsilon} \int (k/\gamma^{\varepsilon,w^\varepsilon,D})^2\gamma^{\varepsilon,w^\varepsilon,D} \int g^2\gamma^{\varepsilon,w^\varepsilon,D}.\end{aligned}$$

From (5) and the definition of N, we know that $\lim \varepsilon^2 \log \Lambda_N^{\varepsilon,w^\varepsilon}(D)t_\varepsilon = r - d_{N+1}(w,D) > 0$. Therefore, for some $\eta > 0$, and for ε small enough:

$$|E_k(g({}^rX^{\varepsilon,w^\varepsilon,D}(t_\varepsilon))) - \int g\Phi^\varepsilon(k)|^2 \leq \exp(-\mathrm{e}^{\eta/\varepsilon^2}) \int (k/\gamma^{\varepsilon,w^\varepsilon,D})^2\gamma^{\varepsilon,w^\varepsilon,D} \int g^2\gamma^{\varepsilon,w^\varepsilon,D}. \tag{10}$$

For $i \leq N$, let k_i be a probability density with support contained in $D_r^i(D)$.

From Lemma 1 (applied to $D = \Delta_r(w)$) , we know that $P_{k_i}(\varepsilon^2 \log \tau^{\varepsilon,w^\varepsilon,D} \leqslant r+\eta) \to 0$, for $\eta < d_1(D) - r$. Hence

$$E_{k_i}(g({}^rX^{\varepsilon,w^\varepsilon,D}(t_\varepsilon))) - E_{k_i}(g(X^{\varepsilon,w^\varepsilon}(t_\varepsilon))) \to 0.$$

Lemma 2 implies that

$$E_{k_i}(g(X^{\varepsilon,w^\varepsilon}(t_\varepsilon))) - \int g\gamma^{\varepsilon,w,D_r^i(w)} \to 0.$$

Therefore, (7) implies that

$$\int g\gamma^{\varepsilon,w,D_r^i(w)} - \int g\Phi^\varepsilon(k_i) \to 0. \tag{11}$$

Hence $\int I_{D_r^i(w)}\Phi^\varepsilon(k_i) \to 1$ and $\int I_{D_r^j(w)}\Phi^\varepsilon(k_i) \to 0$ for any $i \neq j$.

From the Lemma of the Appendix it follows that, for ε small enough, the measures $(\Phi^\varepsilon(k_i), 1 \leqslant i \leqslant N)$ are linearly independent. On the other hand, Φ^ε maps $L_2(\bar{D}, \gamma^{\varepsilon,w^\varepsilon,D})$ into an N-dimensional subspace of the set of measures on $\bar{D}$. Therefore there exist coefficients $b_i(k,\varepsilon,w)$ s.t. $\Phi^\varepsilon(k) = \sum_{i\leqslant N} b_i(k,\varepsilon,w)\Phi^\varepsilon(k_i)$, in particular $\Phi^\varepsilon(f_\varepsilon) = \sum_{i\leqslant N} b_i(\varepsilon,w)\Phi^\varepsilon(k_i)$. Applying (7) to $k = f_\varepsilon$, we obtain that $E_{f_\varepsilon}(g({}^rX^{\varepsilon,w^\varepsilon,D}(t_\varepsilon))) - \int g\Phi^\varepsilon(f_\varepsilon) \to 0$, uniformly for $|g|$ bounded by 1. Hence, for any measurable subset of $\bar{D}$, A, $0 \leqslant \liminf \int_A \Phi^\varepsilon(f_\varepsilon) \leqslant \limsup \int_A \Phi^\varepsilon(f_\varepsilon) \leqslant 1$, uniformly in A. We can now apply the second part of the lemma of the Annex to deduce that $|b_i(\varepsilon,w)| \leqslant 2$, for any $i \leqslant N$ and ε small enough.

(7) and (8) imply that

$$\begin{aligned} &E_{f_\varepsilon}(g({}^rX^{\varepsilon,w^\varepsilon,D}(t_\varepsilon))) - \sum_{i\leqslant N} b_i(\varepsilon,w)\int g\gamma^{\varepsilon,w,D_r^i(w)} \\ &= E_{f_\varepsilon}(g({}^rX^{\varepsilon,w^\varepsilon,D}(t_\varepsilon))) - \int g\Phi^\varepsilon(f_\varepsilon) + \sum_{i\leqslant N} b_i(\varepsilon,w)\Big(\int g\Phi^\varepsilon(k_i) - \int g\gamma^{\varepsilon,w^\varepsilon,D_r^i(w)}\Big) \\ &\to 0. \end{aligned} \tag{12}$$

The case of unbounded k can be treated by approximation and truncation.

It is now easy to recover the conditions on the b_i's: since $b_i(\varepsilon,w) - P_{f_\varepsilon}({}^rX^{\varepsilon,w^\varepsilon,D}(t_\varepsilon) \in D_r^i(w)) \to 0$, we can replace $b_i(\varepsilon,w)$ by $b_i(\varepsilon,w)\vee 0$ in (9). Besides, since $\sum_i b_i(\varepsilon,w) \to 1$, it is also possible to replace $b_i(\varepsilon,w)$ by $b_i(\varepsilon,w)/\sum_j b_j(\varepsilon,w)$ in (9).

End of the proof of (4).

The only thing we have to check in order to complete the proof of (4) is that

$$E_{f_\varepsilon}(g({}^rX^{\varepsilon,w^\varepsilon,D}(t_\varepsilon))) - E_{f_\varepsilon}(g(X^{\varepsilon,w^\varepsilon}(t_\varepsilon))) \to 0.$$

But this is a consequence of the fact that $P_{f_\varepsilon}(\varepsilon^2 \log \tau^{\varepsilon,w^\varepsilon}(D) \geqslant d_1(w,D) - \eta) \to 1$ for $\eta > 0$ and $d_1(w,D) > r$, therefore $P_{f_\varepsilon}(\varepsilon^2 \log \tau^{\varepsilon,w^\varepsilon}(D) \geqslant r + \eta) \to 1$ for $\eta > 0$ small enough.

Proof of Theorem 1 (*Conclusion*). Our aim is to show (3). There are two cases to distinguish: either $B(x) \geqslant 0$ for any x, or there exists $x \in \mathbb{R}^n$ s.t. $B(x) < 0$.

Let $A_1 = \{w \in \hat{C}(\mathbb{R}^n, \mathbb{R})$ s.t. there exists x s.t. $w(x) < 0\}$ and

$A_2 = \{w \in \hat{C}(\mathbb{R}^n, \mathbb{R})$ s.t. for every x $w(x) \geqslant 0\}$. Let $R_2(w)$ be the set of $r > 0$ s.t. for every $x \in \Delta_r(w)$, $w(x) \geqslant 0$ and there exists $\rho > r$ s.t. there are no r-valleys in $\Delta_\rho(w) - \Delta_r(w)$. We claim that

Lemma 4. *For every $r > 0$, Q. a.s. $r \in R_1(B) \cup R_2(B)$.*

Proof. Let Q_1 – resp Q_2 – be the probability Q conditionally on A_1 – resp A_2. Note that the properties (H1)–(H3) are preserved under Q_1 or Q_2.

We shall prove that $Q_1(r \in R_1(B)) = 1$ and $Q_2(r \in R_2(B)) = 1$. Let $r_0(w) = \inf\{r \text{ s.t. } d_1(w, \Delta_r(w) > r\}$. It follows from the scaling property of B (H1) that $Q_1(r_0 = 0) = 1$. Therefore, (i) $Q_1(\text{for every } r,\ d_1(w, \Delta_r(B)) > r) = 1$.

We claim that for any $w \in \hat{C}(\mathbb{R}^n, \mathbb{R})$, for almost any r, for every r-valley D, there exists $\rho > r$ and a ρ-valley D' containing D, s.t. $d_2(w, D') < r$. Indeed let D be a η-valley. For $r \geqslant \eta$, let D_r be the unique r-valley containing D, and let $f(r) = d_2(w, D_r)$. For every r, $f(r) < r$. Therefore, if r is a point of continuity of f then there exists $\rho > r$ s.t. $f(\rho) < r$. Since f is increasing, f is continuous at almost any r. Since there is at most a denumerable number of valleys of w in $\mathbb{R}^n$, the claim is justified. In particular:

(ii) for almost any r, Q_1.a.s. there exists $\rho > r$ s.t. for $i \leqslant N_r(w)$, there exists a ρ-valley, D, s.t. $D_r^i(w) \subset D$ and $d_2(w, D) < r$.

(i) and (ii) imply that for almost any r, $Q_1(r \in R_1(B)) = 1$. But the scaling property of B implies that $Q_1(r \in R_1(B))$ does not depend on r. Therefore for any r, $Q_1(r \in R_1(B)) = 1$. The proof that $Q_2(r \in R_2(w)) = 1$ is similar: let $f(r) = d_2(\Delta_r(w))$. Note that if r is point of continuity of f then there exists a $\rho > r$ s.t. $d_2(\Delta_\rho(w)) < r$, and therefore there is only one r-valley contained in $\Delta_\rho(w)$: $\Delta_r(w)$. The points of discontinuity of f are denumerable. One then concludes as before.

End of the proof of (3).

(H2) and the scaling property imply that Q.a.s. for any $a \in \mathbb{R}$, the connected components of the set $\{x \text{ s.t. } B(x) < a\}$ are bounded, i.e. $B \in \hat{C}(\mathbb{R}^n, \mathbb{R})$ Q.a.s.

(H3) implies that $\varepsilon^2 V(\varepsilon^{-2\beta} x) \to 0$ uniformly on compact sets. Hence Q.a.s. $W^\varepsilon \to B$ uniformly on compact sets.

If $1 \in R_1(w)$ we can apply directly (4). Assume that $1 \in R_2(w)$, then $N_1(w) = 1$ and $D_1^1(w) = \Delta_1(w)$.

We claim that for any continuous bounded function g

$$E_{f_\varepsilon}[g(X^{\varepsilon, w^\varepsilon}(e^{1/\varepsilon^2})] - \int_{\Delta_1(w)} g\gamma^{\varepsilon, w, \Delta_1(w)} \to 0.$$

Since the proof is identical to the proof of Lemma 2, we omit it.

Thus in both cases (4) holds. Since $1 \in R_1(B) \cup R_2(B)$ a.s. (4) holds a.s. for $r = 1$ and therefore (3) holds.

Proof of Theorem 2. As for Theorem 1, it is sufficient to prove that $\varepsilon^2 \log \tau^{\varepsilon, w^\varepsilon}(\Delta_1(w))$ converges in law under P_{f_ε} to $d_1(w, \Delta_1(w))$, for any sequence of continuous functions $w^\varepsilon \in \hat{C}(\mathbb{R}^n, \mathbb{R})$ s.t. $w^\varepsilon \to w$ uniformly on compact sets of $\mathbb{R}^n$. Let $\eta > 0$. In Lemma 1 we have proved that $P_{f_\varepsilon}(\varepsilon^2 \log \tau^{\varepsilon, w^\varepsilon}(\Delta_1(w)) \geqslant d_1(w, \Delta_1(w)) - \eta) \to 1$.

If $w^\varepsilon = w$ for any ε, then it follows from Theorem 4 of Mathieu (1995), that $P_{f_\varepsilon}(\varepsilon^2 \log \tau^{\varepsilon, w}(\Delta_1(w)) \leq d_1(w, \Delta_1(w)) + \eta) \to 1$. The general case can be handled identically.

Appendix Lemma. *Let $k \in \mathbb{N}$. Let $\varepsilon \leqslant (3+2k)^{-1}$. Let $(\nu_i, 1 \leqslant i \leqslant k)$ be measures on a space E and $(A_i, 1 \leqslant i \leqslant k)$ be measurable subsets of E. Assume that for every $i \neq j$, $\nu_i(A_i) \geqslant 1-\varepsilon$ and $|\nu_i(A_j)| \leqslant \varepsilon$. Then (i) the ν_i's are linearly independent. (ii) Let $(\lambda_i, 1 \leqslant i \leqslant k)$ be real numbers. Assume that the measure $\mu = \sum_i \lambda_i \nu_i$ is bounded below by $-\varepsilon$ and bounded above by $1+\varepsilon$, then for any i, $|\lambda_i| \leqslant 2$.*

Proof. (i) Assume that $\sum_i \mu_i \nu_i = 0$. Without any restriction, we can assume that $\mu_1 \geqslant 0$ and $\mu_1 \geqslant |\mu_i|$ for any i. Then $\mu_1 \nu_1(A_1) \geqslant \mu_1(1-\varepsilon)$ and $|\mu_i \nu_i(A_1)| \leqslant \mu_1 \varepsilon$ for $i \neq 1$. Therefore, if $\mu_1 \neq 0$, then $k\varepsilon \geqslant 1-\varepsilon$, which is impossible.

(ii) Assume that $|\lambda_1| \geqslant |\lambda_i|$ for any i.

First assume that $\lambda_1 > 0$. Then $\lambda_1 \nu_1(A_1) \geqslant \lambda_1(1-\varepsilon)$, and $\lambda_i \nu_i(A_1) \geqslant -\varepsilon\lambda_1$ for $i \neq 1$. Since $\mu(A_1) \leqslant 1+\varepsilon$, $(-\varepsilon k + 1 - \varepsilon)\lambda_1 \leqslant 1+\varepsilon$ and therefore $\lambda_1 \leqslant 2$.

If $\lambda_1 < 0$, then $\lambda_1 \nu_1(A_1) \leqslant \lambda_1(1-\varepsilon)$, and $\lambda_i \nu_i(A_1) \leqslant -\varepsilon\lambda_1$ for $i \neq 1$. Since $\mu(A_1) \leqslant -\varepsilon$, $(-\varepsilon k + 1 - \varepsilon)\lambda_1 \geqslant -\varepsilon$ and therefore $\lambda_1 \geqslant -2$.

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