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A new proof of a theorem of Mansour and Sun

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Abstract

We give a new proof of a theorem of Mansour and Sun by using number theory and Rothe's identity.

It is well known that the number of ways of choosing k points, no two consecutive, from a collection of n points arranged on a cycle is $\frac{n}{n-k}\binom{n-k}{k}$ (see [11, Lemma 2.3.4]). A generalization of this result was obtained by Kaplansky [5], who proved that the number of k-subsets $\{x_1, \ldots, x_k\}$ of $\mathbb{Z}_n$ such that $|x_i - x_j| \notin \{1, 2, \ldots, p\}$ $(1 \le i < j \le k)$ is $\frac{n}{n-pk}\binom{n-pk}{k}$, where $n \ge pk + 1$. Some other generalizations and related problems were studied by several authors (see [2,6,7,9]). Very recently, Mansour and Sun [8] extended Kaplansky's result as follows.

Theorem 1. *Let $m, p, k \ge 1$ and $n \ge mpk + 1$. Then the number of k-subsets $\{x_1, \ldots, x_k\}$ of $\mathbb{Z}_n$ such that $|x_i - x_j| \notin \{m, 2m, \ldots, pm\}$ for all $1 \le i < j \le k$, denoted by $f_{m,n}$, is given by $\frac{n}{n-pk}\binom{n-pk}{k}$.*

Their proof needs to establish a recurrence relation and compute the residue of a Laurent series. Mansour and Sun [8] also asked for a combinatorial proof of Theorem 1. In this note, we shall give a new but not purely combinatorial proof of Theorem 1. Let p and k be fixed throughout. Let (a, b) denote the greatest common divisor of the integers a and b. We first establish the following three lemmas.

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Lemma 2. *Let $(a, m) = 1$ and let d be a positive integer. Then at least one of a, $a + m$, $a + 2m, \dots, a + (d - 1)m$ is relatively prime to d.*

Proof. If $(a, d) = 1$, we are done. Now assume that $(a, d) = p_1^{r_1} \dots p_s^{r_s}$ and $d = p_1^{l_1} \cdots p_t^{l_t}$, where $1 \le s \le t$ and $p_1, \dots, p_t$ are distinct primes and $r_1, \dots, r_s, l_1, \dots, l_t \ge 1$. We claim that $a + p_{s+1} \cdots p_t m$ is relatively prime to d. Indeed, since $(a, m) = 1$, we have $(p_1 \cdots p_s, m) = 1$ and therefore

$$(p_1 \cdots p_s, a + p_{s+1} \cdots p_t m) = (p_1 \cdots p_s, p_{s+1} \cdots p_t m) = 1,$$
$$(p_{s+1} \cdots p_t, a + p_{s+1} \cdots p_t m) = (p_{s+1} \cdots p_t, a) = 1.$$

This completes the proof. □

Lemma 3. *Let $(m, n) = d$. Then there exist integers a, b such that $(a, n) = 1$ and $am + bn = d$.*

Proof. Since $(m, n) = d$, we may write $m = m_1 d$ and $n = n_1 d$, where $(m_1, n_1) = 1$. Then there exist integers a and b such that $am_1 + bn_1 = 1$. It is clear that $(a, n_1) = 1$. Noticing that $(a + n_1 t)m_1 + (b - m_1 t)n_1 = 1$, by Lemma 2, we may assume that $(a, d) = 1$ and so $(a, n) = 1$. □

Lemma 4. *Let $m, n \ge 1$ and $(m, n) = d$. Then $f_{m,n} = f_{d,n}$.*

Proof. Let $\mathcal{A}_{m,n}$ denote the family of all k-subsets $\{x_1, \dots, x_k\}$ of $\mathbb{Z}_n$ such that $|x_i - x_j| \notin \{m, 2m, \dots, pm\}$ for all $1 \le i < j \le k$. Then $f_{m,n} = |\mathcal{A}_{m,n}|$. Since $(m, n) = d$, by Lemma 3, there exist integers a and b such that $(a, n) = 1$ and $am + bn = d$. Let a^{-1} be the inverse of $a \in \mathbb{Z}_n$. For any $X = \{x_1, \dots, x_k\} \in \mathcal{A}_{m,n}$, one has $Y = \{ax_1, \dots, ax_k\} \in A_{d,n}$. Conversely, for any $Y = \{y_1, \dots, y_k\} \in \mathcal{A}_{d,n}$, one can recover X by taking $X = \{a^{-1}y_1, \dots, a^{-1}y_k\}$. This proves that $X \mapsto Y$ is a bijection, and therefore $|\mathcal{A}_{m,n}| = |\mathcal{A}_{d,n}|$. □

Now we can give a proof of Theorem 1. By Lemma 4, it suffices to prove it for the case that n is divisible by m.

Proof of Theorem 1. Suppose $n = mn_1$. Let $\mathbb{Z}_{n,i} = \{i + mj : j = 0, \dots, n_1 - 1\}$. Then $|\mathbb{Z}_{n,i}| = n_1$ and $\mathbb{Z}_n = \biguplus_{i=0}^{m-1} \mathbb{Z}_{n,i}$. For any $X = \{x_1, \dots, x_k\} \subseteq \mathbb{Z}_n$ and $i = 0, \dots, m - 1$, define $X_i = X \cap \mathbb{Z}_{n,i}$ and $Y_i = \{j : j = 0, \dots, n_1 - 1 \text{ and } i + mj \in X_i\}$. Consider Y_i as a subset of $\mathbb{Z}_{n_1}$. It is easy to see that $X \in \mathcal{A}_{m,n}$ if and only if $Y_i \in \mathcal{A}_{1,n_1}$ for all $i = 0, \dots, m - 1$. Let $|Y_i| = |X_i| = k_i$. By the aforementioned Kaplansky result, we have the following expression:

$$f_{m,n} = \sum_{k_1+\cdots+k_m=k} \prod_{i=1}^{m} \frac{n_1}{n_1 - pk_i} \binom{n_1 - pk_i}{k_i}. \tag{1}$$

Note that $n \ge mpk + 1$, i.e., $n_1 \ge pk + 1$; the above expression is always well defined. Finally, by repeatedly using Rothe's identity

$$\sum_{k=0}^{n} \frac{xy}{(x + kz)(y + (n - k)z)} \binom{x + kz}{k} \binom{y + (n - k)z}{n - k} = \frac{x + y}{x + y + nz} \binom{x + y + nz}{n}$$

(see [1,3,4,10]), one sees that

$$f_{m,n} = \frac{n}{n - pk} \binom{n - pk}{k}. \quad \square$$

Remark 1. The idea of writing $\mathbb{Z}_n$ as a union of some pairwise non-intersecting subsets is the same as that in [8, Section 2]. However, we are unable to obtain such an expression for $f_{m,n}$ if $n \not\equiv 0 \pmod{m}$, as mentioned by Mansour and Sun [8]. This is why we need to establish Lemma 4. Our proof may be deemed a semi-bijective proof, and finding a purely bijective proof of Theorem 1 still remains open.

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