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GEOMETRY OF THE INTERSECTION RING OF THE MODULI SPACE OF FLAT CONNECTIONS AND THE CONJECTURES OF NEWSTEAD AND WITTEN

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We develop geometric techniques to study the intersection ring of the moduli space $\mathscr{S}_g(t_1, \ldots, t_n)$ of flat connections on a two-manifold Σ^g of genus g with n marked points $p_1, \ldots, p_n$. We find explicit homology cycles dual to generators of this ring, which allow us to prove recursion relations in g and n for their intersection numbers. The recursion relations in the genus g are related to generalizations of the Newstead Conjecture and of some recursion relations due to Donaldson. The recursion relations in the number n of marked points yield analogs of the recursion relations appearing in the work of Witten and Kontsevich on moduli spaces of punctured curves.

1. INTRODUCTION

The algebraic and symplectic geometry of the moduli space $\bar{\mathscr{S}}_g$ of semi-stable rank-two vector bundles on a Riemann surface Σ^g of genus g has been extensively studied in recent years. One recent focal point for this work has been provided by the Verlinde formula, originally arising in mathematical physics, which has motivated research leading to a better understanding of the structure of these spaces as symplectic or Kähler varieties.

Much less has been said or understood from this point of view about the moduli space $\mathscr{M}_g$ of curves of genus g. Like its counterpart $\bar{\mathscr{S}}_g$, $\mathscr{M}_g$ has a simple description in terms of characters of representations of the fundamental group of the underlying surface. For while the symplectic manifold underlying $\bar{\mathscr{S}}_g$ can be written as the space of characters $\mathrm{Hom}(\pi_1(\Sigma^g), SU(2))/SU(2)$, the symplectic manifold underlying $\mathscr{M}_g$ can be written as the quotient of a component of $\mathrm{Hom}(\pi_1(\Sigma^g), PSL(2, R))/PSL(2, R)$ by the action of the mapping class group. The circle of ideas centered on the Verlinde formula has not so far resulted in new insight into the structure of $\mathscr{M}_g$.

However, mathematical physics does provide clues to the study of moduli spaces of curves, just as it provided the Verlinde formula for moduli spaces of stable bundles. These clues are the so-called Witten conjectures [9] for recursion relations among the intersection numbers of certain cohomology classes in $\bar{\mathscr{M}}_{g,n}$, the Deligne–Mumford compactification of the moduli space of curves of genus g with n punctures. Recursion relations of this type were proved in the work of Kontsevich [7] for a moduli space of curves with marked boundaries. In this work the recursion relations appear as a result of complicated combinatorial computations [11].

The purpose of this paper is to show how some simple geometry again may underlie these computations of mathematical physics. We avoid the moduli space $\bar{\mathscr{M}}_{g,n}$ and concentrate our attention on moduli spaces of vector bundles $\mathscr{S}_g(t_1, \ldots, t_n)$, where we shall see that similar recursion relations can be found. Our aim in dealing with $\mathscr{S}_g(t_1, \ldots, t_n)$ is two-fold: first, we are able to develop our techniques in a setting where only compact smooth manifolds appear, and where various methods from symplectic geometry can be used unaltered. Second, our work will yield results of intrinsic interest on the geometry of the moduli space $\mathcal{S}_g(t_1, \dots, t_n)$: the analog of the so-called K-dV condition for $\bar{\mathcal{M}}_{g,n}$ will turn out to produce a geometric proof of the Newstead Conjecture and of its analogs for moduli of vector bundles of rank higher than two; while the analogs of the "Virasoro constraints" will be recursion relations for intersection numbers of $\mathcal{S}_g(t_1, \dots, t_n)$.

Let us state our main results. We concentrate here on rank-two vector bundles; we shall have more to say about higher-rank vector bundles in Section 5. We work with a compact, connected, oriented two-manifold Σ^g of genus g, and choose n distinct points $p_1, \dots, p_n \in \Sigma^g$. Corresponding to each p_i there can be chosen a generator c_i of $\pi_1(\Sigma^g - \{p_1, \dots, p_n\})$, conjugate to the curve traversing the boundary of a small disc containing p_i. Given this setting we may consider for $t_1, \dots, t_n \in \mathbb{R}$ the moduli space $\mathcal{S}_g(t_1, \dots, t_n)$ of representations $\rho \in \mathrm{Hom}(\pi_1(\Sigma^g - \{p_1, \dots, p_n\}), SU(2))$ such that $\mathrm{tr}\,\rho(c_i) = 2\cos \pi t_i$. This space is a smooth manifold if the t_i are chosen appropriately; if the t_i satisfy certain rationality conditions $\mathcal{S}_g(t_1, \dots, t_n)$ is related to a moduli space of vector bundles. As we shall see in Section 2.1, there exist n circle bundles $V_1, \dots, V_n$ on $\mathcal{S}_g(t_1, \dots, t_n)$ corresponding to each of the marked points $p_1, \dots, p_n$. The objects of our study will be the intersection numbers of the Chern classes of these circle bundles.

We shall present two ways to study these Chern classes. First, we shall see that the circle bundles V_i come equipped with natural connections, and hence the Chern classes $c_1(V_i)$ are represented by canonical forms f_i. But our main focus in this paper will be on *homology*: as we shall see there exist natural homology cycles dual to these Chern classes. These will enable us to perform computations in the cohomology ring by intersection theory.

Let us consider for simplicity the case of $\mathcal{S}_g(t)$ where $t \notin \mathbb{Z}$. To find a cycle dual to the Chern class of V_1, we find (for $g \geqslant 2$) a connected submanifold D of $\mathcal{S}_g(t)$ on whose complement the circle bundle possesses a section; then $c_1(V_1)$ will be proportional to the Poincare dual of D. As it turns out we can find such a cycle for any of the usual generators $a_1, \dots, a_g, b_1, \dots, b_g$ of $\pi_1(\Sigma^g - \{p_1\})$; given such a generator X, we define the cycle $D(X)$ as the image in $\mathcal{S}_g(t)$ of those representations ρ of $\pi_1(\Sigma^g - \{p_1\})$ where $[\rho(X), \rho(c_i)] = 1$. As X varies over the set $\{a_1, \dots, a_g, b_1, \dots, b_g\}$, we obtain $2g$ representatives of the homology class dual to $c_1(V_1)$. If $t \notin 2\mathbb{Z}$, $\mathcal{S}_g(t)$ contains no points corresponding to reducible representations, so that $\bigcap_{i=1}^{g} (D(a_i) \cap D(b_i)) = \emptyset$. Thus we obtain

THEOREM 1 (Newstead Conjecture). *Let $g \geqslant 2$. Then*

$$c_1(V_1)^{2g} = 0.$$

Several proofs of this conjecture have appeared in the literature [2, 6]; it can be seen to be a consequence of the Verlinde formula [8] or derived from knowledge of the volumes of the moduli spaces calculated in [2, 5, 10]. We prove some generalizations of this result for moduli spaces corresponding to surfaces with more marked points and to groups other than $SU(2)$; see Theorems 4.1 and 5.2.

In fact, our geometric cycles allow us to do quite a bit more, and to prove recursion relations in the genus g for the intersection numbers of classes of the type we consider above. To do this we first prove, in Proposition 3.5, that $c_1(V_1) = -\frac{1}{2}[D(X)]^*$, where $D(X)$ is oriented using the symplectic form $\omega \in \Omega^2(\mathcal{S}_g(t))$. Let $k \in \{1, \dots, g-1\}$, let $D_k = \bigcap_{i=1}^{k}(D(a_i) \cap D(b_i))$, and let $\iota: D_k \to \mathcal{S}_g(t)$ denote the inclusion. Then we have, for any $\Xi \in H^{6g-4k-4}(\mathcal{S}_g(t))$,

$$\int_{\mathcal{S}_g(t)} c_1(V_1)^{2k} \wedge \Xi = 2^{-2k} \int_{D_k} \iota^*(\Xi).$$

A particularly interesting case occurs where we let $\Xi = \omega^{3g-2k-2}$. We note that D_k is symplectomorphic to $\mathscr{S}_{g-k}(t) \times V$, where $\mathscr{S}_{g-k}(t)$ is equipped with a symplectic form we also denote by ω, and where V is given by the torus $(S^1 \times S^1)^k$, equipped with *twice* the usual symplectic form. Thus

THEOREM 2 (Donaldson recursion relation [2]).

$$\int_{\mathscr{S}_g(t)} c_1(V_1)^{2k} \wedge \omega^{3g-2k-2} = 2^{-k} k! \binom{3g-2k-2}{k} \int_{\mathscr{S}_{g-k}(t)} \omega^{3g-3k-2}.$$

Geometrically the cycle $D(a_i)$ corresponds to representations ρ of $\pi_1(\Sigma^g - \{p_1\})$ which restrict to representations into the maximal torus $U(1) \subset SU(2)$ on a subsurface of Σ^g which is given by a three-holed sphere bounded by curves homotopic to c_1, a_i, and $c_1 * a_i$ (see Fig. 1). A similar degeneration in the moduli of curves with one puncture (corresponding to representations of $\pi_1(\Sigma^g - \{p_1\})$ in $PSL(2, R)$ taking the generator c_1 to a *parabolic* element of $PSL(2, R)$) will correspond to stable curves with two internal nodes, corresponding to representations of $\pi_1(\Sigma^g - \{p_1\})$ in $PSL(2, R)$ carrying c_1 as well as a_i and $c_1 * a_i$ to parabolic elements of $PSL(2, R)$ (see Fig. 2). The analogy to eq. (2.34) of [9], or at least to Fig. 2(b) there, should be clear.

We proceed in a similar way to develop recursion relations in the number of marked points n. We now work with the moduli space $\mathscr{S}_g(t_1, \dots, t_n)$ and the Chern classes $r_i = c_1(V_i)$. To find homology classes dual to the r_i we proceed as above; we show in Section 3.2 that the circle bundles V_i possess sections on the complement of $D_i(X)$, where X can be chosen among the generators of $\pi_1(\Sigma^g - \{p_1, \dots, p_n\})$ denoted by c_j where $i \neq j$, and where, as above, $D_i(X)$ is the image in $\mathscr{S}_g(t_1, \dots, t_n)$ of those representations ρ of $\pi_1(\Sigma^g - \{p_1, \dots, p_n\})$ where $[\rho(X), \rho(c_i)] = 1$. The first thing to notice is that $D_i(c_j) = D_j(c_i)$, so that $D_i(c_j)$ cannot be connected since V_i and V_j are not powers of the same circle bundle. The manifold $D_i(c_j)$ in fact has two components we shall label $D_i(c_j)^+$ and $D_i(c_j)^-$; these have a natural orientation coming from the symplectic form. Using considerations from the theory of toric varieties we shall see that if t_i and t_j are sufficiently small, and if $t_i > t_j$, the

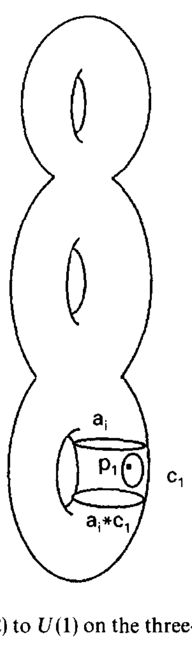

Fig. 1. The representation is reducible from $SU(2)$ to $U(1)$ on the three-holed sphere formed by a_1, c_1, and $a_1 * c_1$.

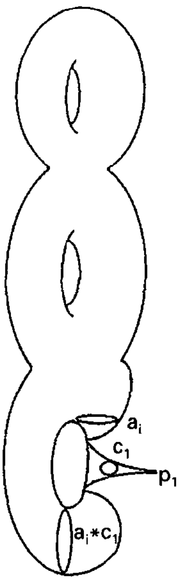

Fig. 2. A noded surface formed when a_i and c_1 correspond to commuting parabolic elements of $SL(2, R)$.

homology cycle dual to r_i is given by $\frac{1}{2}[D_i(c_j)^+] - \frac{1}{2}[D_i(c_j)^-]$ while that dual to r_j is given by $\frac{1}{2}[D_i(c_j)^+] + \frac{1}{2}[D_i(c_j)^-]$.

The role of the components $D_i(c_j)^+$ and $D_i(c_j)^-$ of $D_i(c_j)$ is best understood if we look at the geometry of the representations corresponding to points in $D_i(C_j)$. Such representations degenerate to representations into the maximal torus $U(1) \subset SU(2)$ on the three holed sphere bounded by curves homotopic to c_i, c_j, and $c_i * c_j$ (see Fig. 3). Such representations occur for two distinct possible values of tr $\rho(c_i * c_j)$, which must be either $2\cos(\pi(t_i + t_j))$ or $2\cos(\pi|t_i - t_j|)$, corresponding to the components $D_i(c_j)^+$ and $D_i(c_j^-)$ of $D_i(c_j)$, respectively. In fact we have symplectomorphisms

$$s_+ : \mathscr{S}_g(t_1, \ldots, \hat{t}_i, \ldots, \hat{t}_j, \ldots, t_n, t_i + t_j) \to D_i(c_j)^+$$

$$s_- : \mathscr{S}_g(t_1, \ldots, \hat{t}_i, \ldots, \hat{t}_j, \ldots, t_n, |t_i - t_j|) \to D_i(c_j)^-.$$

Let us now consider an integral of the form

$$\int_{\mathscr{S}_g(t_1, \ldots, t_n)} (r_1)^{k_1} \cdots (r_n)^{k_n} e^{x\omega}$$

where $x \in \mathbb{R}$. Our aim will be to use the explicit expression for the cycles $D_i(c_j)$ to express this integral in term of integrals of similar forms on moduli spaces corresponding to surfaces with fewer marked points. For convenience, we may as well take $i = n$, $j = n-1$. Let $\iota_\pm : D_i(c_j)^\pm \to \mathscr{S}_g(t_1, \ldots, t_n)$ denote the inclusions. Then

$$\int_{\mathscr{S}_g(t_1, \ldots, t_n)} (r_1)^{k_1} \cdots (r_n)^{k_n} e^{x\omega}$$

$$= \frac{1}{2} \int_{D_n(C_{n-1})^+} \iota_+^* \left((r_1)^{k_1} \cdots (r_{n-1})^{k_{n-1}} (r_n)^{k_n - 1} e^{x\omega}\right) - \frac{1}{2} \int_{D_n(C_{n-1})^-} \iota_-^* \left((r_1)^{k_1} \cdots (r_{n-1})^{k_{n-1}} (r_n)^{k_n - 1} e^{x\omega}\right).$$

On the other hand, we have the symplectomorphisms s_+ and s_- which identify $D_n(c_{n-1})^+$ and $D_n(C_{n-1})^-$ with the moduli spaces $\mathscr{S}_g(t_1, \ldots, t_{n-2}, t_{n-1} + t_n)$ and

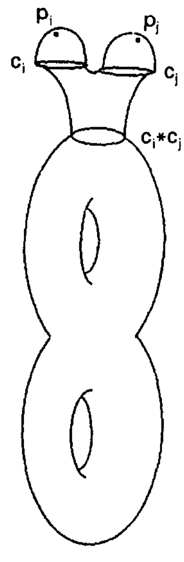

Fig. 3. The representation is reducible from $SU(2)$ to $U(1)$ on the three-holed sphere shown, formed by curves homotopic to c_i, c_j, and $c_i * c_j$.

$\mathscr{S}_g(t_1, \dots, t_{n-2}, |t_{n-1} - t_n|)$, which themselves are endowed with circle bundles and with cohomology classes which we denote by $r_m^{\pm}$; likewise denote the symplectic forms on these spaces by $\omega^{\pm}$. And in fact it will be shown that, where $t_n > t_{n-1}$,

$$(\iota_{\pm} \circ s_{\pm})^* r_m = r_m^{\pm} \text{ for } m \leqslant n-2$$

$$(\iota_{\pm} \circ s_{\pm})^* r_{n-1} = \pm r_{n-1}^{\pm}$$

$$(\iota_{\pm} \circ s_{\pm})^* r_n = r_{n-1}^{\pm}.$$

Thus we have the following recursion relation.

THEOREM 3. *Suppose $t_{n-1}, t_n > 0$ and $t_{n-1} + t_n < 1$. Then*

$$\int_{\mathscr{S}_g(t_1, \dots, t_n)} (r_1)^{k_1} \cdots (r_n)^{k_n} e^{x\omega} = \frac{1}{2} \int_{\mathscr{S}_g(t_1, \dots, t_{n-2}, t_{n-1} + t_n)} (r_1^+)^{k_1} \cdots (r_{n-2}^+)^{k_{n-2}} (r_{n-1}^+)^{k_{n-1} + k_n - 1} e^{x\omega^+}$$
$$- \frac{(-1)^{k_{n-1}}}{2} \int_{\mathscr{S}_g(t_1, \dots, t_{n-2}, |t_{n-1} - t_n|)} (r_1^-)^{k_1} \cdots$$
$$\cdots (r_{n-2}^-)^{k_{n-2}} (r_{n-1}^-)^{k_{n-1} + k_n - 1} e^{x\omega^-}.$$

These recursion relations are very similar to the Witten conjectures as presented in [9]. Recall those conjectures concerned the moduli space $\bar{\mathscr{M}}_{g,n}$ of curves of genus g with n punctures. According to Witten, there is, corresponding to each puncture p_i, a line bundle L_i on $\bar{\mathscr{M}}_{g,n}$ and a Chern class $x_i \in H^2(\bar{\mathscr{M}}_{g,n})$. We consider integrals of the form

$$\int_{\bar{\mathscr{M}}_{g,n}} (x_1)^{k_1} \cdots (x_n)^{k_n}$$

where the k_i are integers satisfying $\sum_i k_i = 3g + n - 3$. The Witten conjectures (in the form of the so-called "Virasoro constraints") then say, roughly, that

$$\int_{\bar{\mathcal{M}}_{g,n}} (x_1)^{k_1} \cdots (x_n)^{k_n} = \int_{\bar{\mathcal{M}}_{g,n-1}} (x_1)^{k_1 + k_n - 1} \cdots (x_{n-1})^{k_{n-1}} + \int_{\bar{\mathcal{M}}_{g,n-1}} (x_1)^{k_1} \cdots (x_2)^{k_2 + k_n - 1} \cdots (x_n)^{k_{n-1}}$$
$$+ \cdots +$$
$$+ \int_{\bar{\mathcal{M}}_{g,n-1}} (x_1)^{k_1} \cdots (x_n)^{k_n + k_{n-1} - 1} + \cdots,$$

where on the right-hand side of the equation, we have used the notation x_i to denote the appropriate classes on the moduli space $\bar{\mathcal{M}}_{g,n-1}$. The analogy to Theorem 3 is now clear. Geometrically the cycles $D_i(c_j)$ have their analogs in $\bar{\mathcal{M}}_{g,n}$ as representations corresponding to stable curves where the punctures corresponding to the ith and jth marked points reside on a three-punctured sphere; see Fig. 4.

The remainder of this paper is structured as follows. In Section 2 we define the moduli spaces and circle bundles in question, and construct the cohomology classes r_m we wish to study. In Section 3 we shall construct the homology classes dual to these r_m; these are the $D_m(X)$. The technical part of this section will be finding the precise relation between the dual of the $D_m(X)$ and the cohomology classes r_m: to do this we shall have to do some work using the methods of [5] and some ideas from the theory of toric varieties. The work of Section 3 is used in Section 4 to prove Theorems 1–3. Finally, in Section 5 we prove an analog of the Newstead Conjecture for moduli spaces corresponding to the higher-rank groups $SU(k)$ where $k > 2$.

2. MODULI SPACES AND CIRCLE BUNDLES

In this section, we construct the moduli spaces and circle bundles which will be the basic objects of study in this paper. The construction of the moduli spaces is more or less standard, and follows the lines of the construction given in [1, 4] for the case $n = 0$ and in, e.g. [5] where $n \neq 0$.

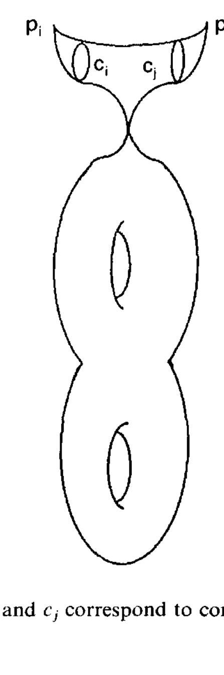

Fig. 4. A noded surface formed when c_i and c_j correspond to commuting parabolic elements of $SL(2, R)$.

2.1. Construction of the moduli spaces and bundles

Let Σ^g be a compact, connected, oriented two-manifold of genus $g \geqslant 2$, and let $p_1, \dots, p_n \in \Sigma^g$ be n distinct points in Σ^g. Then $\pi_1(\Sigma^g - \{p_1, \dots, p_n\})$ can be described as the quotient of the free group on the $2g+n$ standard generators $a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_n$ by the single relation $\prod_{i=1}^{2g} [a_i, b_i] = \prod_{j=1}^{n} c_j$; here each of the generators c_j can be chosen to correspond in a natural way to a point p_j. We take $G = SU(2)$ and denote by $T \subset G$ the subgroup of G consisting of diagonal matrices. Given $t_1, \dots, t_n \in \mathbb{R}$, we may consider the representation variety $R_g(t_1, \dots, t_n) = \{\rho \in \mathrm{Hom}(\pi_1(\Sigma^g - \{p_1, \dots, p_n\}), \; SU(2)) : \mathrm{tr}\, \rho(c_j) = 2\cos(\pi t_j)\}$, and its quotient $\mathscr{S}_g(t_1, \dots, t_n) = R_g(t_1, \dots, t_n)/G$ by the conjugation action of G. In order to guarantee the smoothness of $\mathscr{S}_g(t_1, \dots, t_n)$, we impose a condition on the t_j; this is expressed in the following definition.

Definition 2.1. A set $\{t_1, \dots, t_n\}$ of real numbers will be called *admissible* if $\sum_{j=1}^{n} \varepsilon_j t_j \notin \mathbb{Z}$ for all n-tuples $(\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 0, 1\}^n - \{(0, \dots, 0)\}$.

From now on we shall assume that the set $\{t_1, \dots, t_n\}$ is admissible; and the definition of $\mathscr{S}_g(t_1, \dots, t_n)$ shows we may as well take $t_1, \dots, t_n \in (0, 1)$. A variation on standard arguments shows that whenever $\{t_1, \dots, t_n\}$ is admissible, the quotient variety $\mathscr{S}_g(t_1, \dots, t_n)$ is a smooth symplectic manifold of dimension $6g + 2n - 6$; we denote the symplectic form on $\mathscr{S}_g(t_1, \dots, t_n)$ by $\omega^g_{t_1, \dots, t_n}$, or by ω where there is no possibility of confusion. As we shall see $\mathscr{S}_g(t_1, \dots, t_n)$ comes equipped with other two-forms as well.

These two-forms will arise naturally as the Chern classes of circle bundles on $\mathscr{S}_g(t_1, \dots, t_n)$. We construct these as follows. Let us denote by e^{it} the diagonal matrix $\begin{pmatrix} \mathrm{e}^{it} & 0 \\ 0 & \mathrm{e}^{-it} \end{pmatrix}$. Then, in analogy with the definition of $R_g(t_1, \dots, t_n)$, we define, for each $m = 1, \dots, n$, the variety $V^g_m(t_1, \dots, t_n)$ by $V^g_m(t_1, \dots, t_n) = \{\rho \in \mathrm{Hom}(\pi_1(\Sigma^g - \{p_1, \dots, p_n\}), SU(2)) : \mathrm{tr}\, \rho(c_j) = 2\cos(\pi t_j)$ and $\rho(c_m) = \mathrm{e}^{i\pi t_m}\}$. On the space $V^g_m(t_1, \dots, t_n)$ the group $T \subset G$ acts by conjugation, with global stabilizer $\mathbb{Z}_2 \subset T$ and with quotient $\mathscr{S}_g(t_1, \dots, t_n)$. The space $V^g_m(t_1, \dots, t_n)$ is therefore a circle bundle on $\mathscr{S}_g(t_1, \dots, t_n)$; let us denote the bundle projection by π_m. We then have n circle bundles $V^g_m(t_1, \dots, t_n) \to \mathscr{S}_g(t_1, \dots, t_n)$, and we may consider their Chern classes, which we denote by $r^g_m(t_1, \dots, t_n) \in H^2(\mathscr{S}_g(t_1, \dots, t_n))$ or simply r_m where there is no chance of confusion. It is the ring generated by these classes that we wish to study. Before going on to do so we make a few remarks.

2.2. Some remarks about the cohomology classes r_m

First, we note that we can actually find canonical forms f_m representing the classes r_m. To do this we construct an auxiliary space $U_g(t_1, \dots, t_n)$ as follows. Let $U_g(t_1, \dots, t_n) \subset G^{2g+n}$ be defined by

$$U_g(t_1, \dots, t_n) = \{A_i, B_i \in G, i = 1, \dots, g, \Gamma_j \in G, j = 1, \dots, n : \prod_{i=1}^{g} [A_i, B_i] = \prod_{j=1}^{n} \Gamma_j^{-1} \mathrm{e}^{i\pi t_j} \Gamma_j\}. \tag{1}$$

On the space $U_g(t_1, \dots, t_n)$ we may define n maps $U_g(t_1, \dots, t_n) \to G$ given by the Γ_j's; we denote these maps by abuse of notation by Γ_j. The space $U_g(t_1, \dots, t_n)$ also comes equipped with a natural action of $G \times T_1 \times \cdots \times T_n$, where $T_i \simeq T$, and where $(g, \xi_1, \dots, \xi_n) \in G \times T_1 \times \cdots \times T_n$ acts on $U_g(t_1, \dots, t_n)$ by

$$(g, \xi_1, \dots, \xi_n) \cdot \{A_i, B_i, \Gamma_j\} = \{g^{-1} A_i g, g^{-1} B_i g, \xi_j \Gamma_j g\}. \tag{2}$$

It is then clear that $U_g(t_1, \dots, t_n)/(G \times T_1 \times \cdots \times T_{m-1} \times T_{m+1} \times \cdots \times T_n) = V^g_m(t_1, \dots, t_n)$, while $U_g(t_1, \dots, t_n)/(G \times T_1 \times \cdots \times T_n) = \mathscr{S}_g(t_1, \dots, t_n)$.

Using the space $U_g(t_1, \dots, t_n)$ we can construct a natural connection form on the circle bundles $V^g{}_m(t_1, \dots, t_n)$. Recall first that if we are given a manifold M equipped with an S^1-action generated by a vector field ζ, a *connection* for this S^1 action is a one-form $\gamma \in \Omega^1(M)$ such that $\iota_\zeta \gamma = 1$ and $\mathscr{L}_\zeta \gamma = 0$ (here ι denotes interior product and $\mathscr{L}$ denotes lie derivative). In particular, if $M = G$, and we consider the left action of T on G, a connection form ζ for this action may be constructed out of the *right* invariant Maurer–Cartan form ζ_R: if we denote by $\langle , \rangle$ the Killing form on $\mathfrak{g}$, then

$$\zeta = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \zeta_R \right\rangle$$

is such a connection. It is then easy to see that $\tilde{\zeta}_m = \Gamma^*_m(\zeta) \in \Omega^1(U_g(t_1, \dots, t_n))$ is a connection one-form for the action of T_m on $U_g(t_1, \dots, t_n)$, which descends to the quotient $V^g_m(t_1, \dots, t_n)$ by right-invariance. We denote the resulting connection form on $V^g_m(t_1, \dots, t_n)$ by $\zeta_m \in \Omega^1(V^g_m(t_1, \dots, t_n))$; then

$$d\zeta_m = \pi^*_m(f_m), \tag{3}$$

where the two-form $f_m \in \Omega^2(\mathscr{S}_g(t_1, \dots, t_n))$ represents the Chern class r_m.

For our present work, another remark will be useful: we wish to see how the circle bundles $V^g_m(t_1, \dots, t_n)$ arise from the Duistermaat–Heckman theorem. To see this, we note that as in [5], the space $\mathscr{S}_g(t_1, \dots, t_n)$ can be obtained from the symplectic quotient of a moduli space associated to a two-manifold of genus $g + n$ obtained from Σ^g by deleting small discs about each of the points $p_1, \dots, p_n$ and attaching one-holed tori to the boundaries of the resulting surface. The circles bundles $V^g_m(t_1, \dots, t_n)$ can be seen to arise from the levels sets of the moment map in the usual way. The Duistermaat–Heckman theorem [3] then gives the following result when $(t_1, \dots, t_n) \in (0, 1)^n \subset \mathbb{R}^n$ is in the complement of a finite number of planes in $\mathbb{R}^n$:

PROPOSITION 2.2. *Suppose $(t_1, \dots, t_n)$ and $(t'_1, \dots, t'_n)$ are sufficiently close. Then the spaces $\mathscr{S}_g(t_1, \dots, t_n)$ and $\mathscr{S}_g(t'_1, \dots, t'_n)$ are diffeomorphic; under this diffeomorphism the cohomology classes $[\omega^g_{t_1, \dots, t_n}]$ and $[\omega^g_{t'_1, \dots, t'_n}]$ are related by*

$$[\omega^g_{t_1, \dots, t_n}] - [\omega^g_{t'_1, \dots, t'_n}] = \sum_{j=1}^{n} (t_j - t'_j) r_j. \tag{4}$$

This remark will be helpful in allowing us to calculate which element of $H_*(D_m(X))$ represents the dual of r_m.

3. DUAL HOMOLOGY CLASSES

In this section we shall construct cycles dual to the Chern classes r_m defined in Section 2. We shall actually find a collection of such cycles, which will depend on a choice of a generator of $\pi_1(\Sigma^g - \{p_1, \dots, p_n\})$. The two different types of standard generators given in Section 2 will give rise to two different types of cycles. To obtain a cycle of the first type we shall choose one of the generators called a_i or b_i; this type of cycle will enter into the proof of Theorems 1 and 2. The second type of cycle will be obtained by choosing a generator called c_j where $j \neq m$. This type of cycle will therefore be defined only when there is more than one marked point; it will enter into the proof of Theorem 3.

We construct these cycles by finding, for each choice of X among the set of standard generators $\{a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, \hat{c}_m, \dots, c_n\}$, a submanifold $D_m(X)$ of codimension two of the moduli space $\mathcal{S}_g(t_1, \dots, t_n)$ on the complement of which the circle bundle $V_m^g(t_1, \dots, t_n)$ possesses a section. Then $c_1(V_m^g(t_1, \dots, t_n))$ will be dual to an element of $\iota_* H_{6g+2n-8}(D_m(X))$, where $\iota: D_m(X) \to \mathcal{S}_g(t_1, \dots, t_n)$ denotes the inclusion. As we shall see, where X is one of the a_i or b_i, and when $g \geqslant 2$, the submanifold $D_m(X)$ will be connected, so that $c_1(V_m^g(t_1, \dots, t_n))$ must be proportional to $[D_m(X)]^*$; we shall reduce the computation of the constant of proportionality to the case of genus $g = 1$ where it may be performed using the methods of [5].

In the case of cycles of the second type, we shall see that $H_{6g+2n-8}(D_m(c_j)) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Indeed $D_m(c_j)$ cannot be connected since $D_m(c_j) = D_j(c_m)$, so that both $V_m^g(t_1, \dots, t_n)$ and $V_j^g(t_1, \dots, t_n)$ are trivial on the complement of $D_m(c_j)$; but they are not powers of the same circle bundle. Again we shall be able to reduce the study of $r_m = c_1(V_m^g(t_1, \dots, t_n))$ to the case of genus $g = 1$ and use the methods of [5] and some facts about toric varieties to find a cycle dual to r_m.

Recall that we have chosen once and for all standard generators $a_1, \dots, a_g$, $b_1, \dots, b_g, c_1, \dots, c_n$ for the fundamental group $\pi_1(\Sigma^g - \{p_1, \dots, p_n\})$. We shall associate a submanifold $D_m^g(X)(t_1, \dots, t_n)$ to each circle bundle $V_m^g(t_1, \dots, t_n)$ and each of these generators.

Definition 3.1. Let $X \in \{a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, \hat{c}_m, \dots, c_n\}$. Let $D_m^g(X)(t_1, \dots, t_n) \subset \mathcal{S}_g(t_1, \dots, t_n)$ denote the image in $\mathcal{S}_g(t_1, \dots, t_n)$ of those representations $\rho \in R_g(t_1, \dots, t_n)$ such that $[\rho(X), \rho(c_m)] = 1$.

When there is no possibility of confusion we shall write $D_m(X)$ for $D_m^g(X)(t_1, \dots, t_n)$.

The following lemma can be proved by standard methods; in Section 4 we shall produce a good characterization of $D_m(X)$.

LEMMA 3.2. *The subspace $D_m(X) \subset \mathcal{S}_g(t_1, \dots, t_n)$ is a smooth submanifold of $\mathcal{S}_g(t_1, \dots, t_n)$, which is connected if $g \geqslant 2$ and $X \in \{a_1, \dots, a_g, b_1, \dots, b_g\}$.*

The basic idea of this paper is to show that the circle bundle $V_m^g(t_1, \dots, t_n)$ is trivial on the complement $D_m(X)$. This is the content of the following proposition.

PROPOSITION 3.3. *The circle bundle $V_m^g(t_1, \dots, t_n)|_{\mathcal{S}_g(t_1, \dots, t_n) - D_m(X)}$ has a section.*

Proof. We recall that the circle bundle $V_m^g(t_1, \dots, t_n)$ was given by $V_m^g(t_1, \dots, t_n) = \{\rho \in \mathrm{Hom}(\pi_1(\Sigma^g - \{p_1, \dots, p_n\})\, SU(2)\colon\ \mathrm{tr}\,\rho(c_j) = 2\cos(\pi t_j)$ and $\rho(c_m) = e^{i\pi t_m}\}$, and that $\mathcal{S}_g(t_1, \dots, t_n)$ was the quotient of $V_m^g(t_1, \dots, t_n)$ by the conjugation action of T. A section $s\colon \mathcal{S}_g(t_1, \dots, t_n) - D_m(X) \to V_m^g(t_1, \dots, t_n)$ can therefore be given by choosing, for each $x \in \mathcal{S}_g(t_1, \dots, t_n) - D_m(X)$, a representation $\rho \in V_m^g(t_1, \dots, t_n)$ with $\pi_m(\rho) = x$. To do this we make use of the generator X; we define $s(x) \in V_m^g(t_1, \dots, t_n)$ by fixing the conjugation action of T on $\rho(X)$. To be concrete, we can take $s(x)$ to be the unique representation $\rho \in \pi_m^{-1}(x)$ with $\rho(X)$ having the form

$$\rho(X) = \begin{pmatrix} z & w \\ -w & \bar{z} \end{pmatrix}$$

where w is a *positive real number*. Such a representation will always exist in $\pi_m^{-1}(x)$ so long as $\pi_m^{-1}(x)$ contains no representation σ such that $\sigma(X)$ is diagonal; that is, so long as $[\sigma(X), \sigma(c_j)] \neq 1$, or $x \notin D_m(X)$.

3.1. Homology classes of the first type

We now restrict our attention to the submanifolds $D_m(a_i)$, $D_m(b_i)$, and to the case of surfaces of genus $g \geqslant 2$ where these submanifolds are connected. In order to obtain a homology class from the submanifolds $D_m(a_i)$ we must orient these submanifolds. Fortunately, the symplectic form provides us with a natural orientation; by the results of Donaldson [2], for example, we see that $\int_{D_m(a_i)} \omega^{3g+n-4} \neq 0$. Let us denote by $[D_m(a_i)]$ the homology cycle obtained by orienting $D_m(a_i)$ so that $\int_{D_m(a_i)} \omega^{3g+n-4} > 0$. The following is now immediate.

COROLLARY 3.4. *A homology class dual to r_m is given by a multiple $\alpha_{g,m}(t_1, \dots, t_n)[D_m(a_i)]$ of $[D_m(a_i)]$, or by a multiple $\beta_{g,m}(t_1, \dots, t_n)[D_m(b_i)]$ of $[D_m(b_i)]$.*

It is at once clear that $\alpha_{g,m}(t_1, \dots, t_n) = \beta_{g,m}(t_1, \dots, t_n)$, but as it stands now it might well happen that $\alpha_{g,m}(t_1, \dots, t_n)$ depends in some way on g, n, m, or on the values of the $t_1, \dots, t_n$. As we shall see this is not the case.

PROPOSITION 3.5. *For all $g, n, m, t_1, \dots, t_n$, we have $\alpha_{g,m}(t_1, \dots, t_n) = -\frac{1}{2}$.*

Proof. To prove this proposition we shall find a subvariety $W \subset \mathscr{S}_g(t_1, \dots, t_n)$ which is symplectomorphic to a moduli space of the type $\mathscr{S}_1(t_n)$; in other words to a moduli space occurring for a surface of genus $g = 1$ with one marked point. We shall examine the intersection of $D_m(a_i)$ with W, and compare the result to a computation of the variation of the symplectic form using Proposition 2.2 and an explicit diffeomorphism of W with $\mathbb{CP}^1$ obtained by the methods of [5]. This comparison will enable us to compute $\alpha_{g,m}(t_1, \dots, t_n)$.

For convenience we take $m = n$. We define the variety W by looking at points of $\mathscr{S}_g(t_1, \dots, t_n)$ corresponding to representations ρ of $\pi_1(\Sigma^g - \{p_1, \dots, p_n\})$ with fixed values on $a_1, \dots, a_{g-1}, b_1, \dots, b_{g-1}, c_1, \dots, c_{n-1}$, and satisfying $\rho(\prod_{i=1}^{g-1}[a_i, b_i]) = \rho(\prod_{j=1}^{n-1} c_j)$. Geometrically, we cut the surface Σ^g into a surface of genus $g = 1$ with one marked point, whose fundamental group is generated by a_g, b_g, and c_n, and into a subsurface Σ' of genus $g - 1$ with $n - 1$ marked points, whose fundamental group is generated by $a_1, \dots, a_{g-1}, b_1, \dots, b_{g-1}, c_1, \dots, c_{n-1}$; let $i: \Sigma' - \{p_1, \dots, p_{n-1}\} \to \Sigma^g - \{p_1, \dots, p_n\}$ denote the inclusion. We then consider representations of $\pi_1(\Sigma^g - \{p_1, \dots, p_n\})$ with some *fixed* value on the latter surface (see Fig. 5). To be concrete, we choose representation $\bar{\rho} \in R_g(t_1, \dots, t_n)$ such that $\bar{\rho}|_{i_*\pi_1(\Sigma' - \{p_1, \dots, p_{n-1}\})}$ is irreducible, and that $\rho(\prod_{i=1}^{g-1}[a_i, b_i]) = \bar{\rho}(\prod_{j=1}^{n-1} c_j)$.† We now consider the space of conjugacy classes of representations $\rho \in R_g(t_1, \dots, t_n)$ such that $\operatorname{tr} \rho(x) = \operatorname{tr} \bar{\rho}(x)$ whenever $x \in i_*\pi_1(\Sigma' - \{p_1, \dots, p_{n-1}\})$. Now when a surface is decomposed into two subsurfaces, the corresponding moduli spaces satisfy a gluing relation; see [2, 4, 5]. In our case where we are considering representations with *fixed* values on one of the subsurfaces, the moduli space of such representations is given by $SU(2) \times \mathscr{S}_1(t_n, 0) = SU(2) \times \mathscr{S}_1(t_n)$. Thus we can find a subvariety $W = 1 \times \mathscr{S}_1(t_n)$ of this space symplectomorphic to $\mathscr{S}_1(t_n)$. To find the constant $\alpha_{g,m}(t_1, \dots, t_n)$, it suffices to consider $r_n|_W$ and compare it with $D_n(a_g) \cap W$.

†No such representations exist if $g = n = 1$; our argument can be easily modified to accomodate that case.

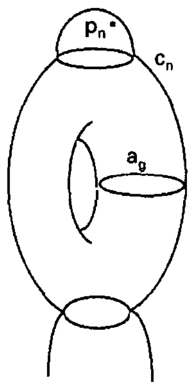

Fig. 5. The subvariety W corresponds to representations whose values are fixed away from a subsurface of genus one shown.

We compute $r_n|_W$ by Proposition 2.2, which shows that the Chern class r_n is given by the variation of the symplectic form $\omega^g_{t_1,\dots,t_n}$ as t_n varies. This result then holds also for the subvariety W; as t_n varies the symplectic form on W varies, and its derivative is $r_n|_W$. On the other hand, the work of [2, 5] shows that $W \simeq \mathscr{S}_1(t_n)$ is symplectomorphic to $\mathbb{CP}^1$, equipped with the symplectic form $\frac{1}{2}(1-t_n)\,\omega_{\mathbb{CP}^1}$, where $\omega_{\mathbb{CP}^1}$ is the usual symplectic form on $\mathbb{CP}^1$. The identification is obtained by using the moment map $\mu\colon W \to [\frac{1}{2}t_n, \frac{1}{2}(2-t_n)]$ given by sending a conjugacy class $[\rho] \in W$ to $\mu([\rho]) = (1/\pi \cos^{-1}(\frac{1}{2}\operatorname{tr}(\rho(a_g))))$. Then the image of the moment map is $\mu(W) = [\frac{1}{2}t_n, \frac{1}{2}(2-t_n)]$, and the endpoints of the interval correspond to conjugacy classes of representations ρ where $\rho([a_g, c_n]) = 1$; in other words to the points of $W \cap D_n(a_g)$. On the other hand, the variation of the symplectic form on $\mathscr{S}_1(t_n)$ is given by the cocycle dual to $-\frac{1}{2}$ the sum of the cycles represented by the endpoints of the image of the moment map, that is, by $-\frac{1}{2}$ the sum of the north and south poles on $\mathbb{CP}^1$.

3.2. *Cycles of the second type*

We now turn our attention to the cycle $D_m(X)$ where $X \in \{c_1, \dots, \hat{c}_m, \dots, c_n\}$. The first point to notice is the following:

Proposition 3.6. $H_{6g+2n-8}(D_i(c_j)) = \mathbb{Z} \oplus \mathbb{Z}$.

Proof. Points in $D_i(c_j)$ correspond to representations $\rho \in \operatorname{Hom}(\pi_1(\Sigma^g - \{p_1, \dots, p_n\}), SU(2))$ with $[\rho(c_i), \rho(c_j)] = 1$. Then $\operatorname{tr}\rho(c_i)\rho(c_j)$ can take two values; it is either $2\cos\pi(t_i + t_j)$ or else $2\cos\pi(|t_i - t_j|)$. Let us denote the corresponding subsets of $D_i(c_j)$ by $D_i(c_j)^+$ and $D_i(c_j)^-$. The representations lying in either $D_i(c_j)^+$ or $D_i(c_j)^-$ are reducible on the three holed sphere subspace of Σ^g bounded by curves homotopic to c_i, c_j and $c_i * c_j$ (see Fig. 3). Such representations are rigid; thus $D_i(c_j)^+$ and $D_i(c_j)^-$ are symplectomorphic to moduli spaces corresponding to the surface obtained from Σ^g by removing this three-holed sphere, and assigning the appropriate trace to the values of the representations on $c_i * c_j$. In other words, we have symplectomorphisms

$$\begin{aligned} s_+ &: \mathscr{S}_g(t_1, \dots, \hat{t}_i, \dots, \hat{t}_j, \dots, t_n, t_i + t_j) \to D_i(c_j)^+ \\ s_- &: \mathscr{S}_g(t_1, \dots, \hat{t}_i, \dots, \hat{t}_j, \dots, t_n, |t_i - t_j|) \to D_i(c_j)^-. \end{aligned} \tag{5}$$

Note incidentally that $\{t_1, \dots, \hat{t}_i, \dots, \hat{t}_j, \dots, t_n, t_i + t_j\}$ and $\{t_1, \dots, \hat{t}_i, \dots, \hat{t}_j, \dots, t_n, |t_i - t_j|\}$ are admissible if $\{t_1, \dots, t_n\}$ is.

Thus we have identified $D_i(c_j)^{\pm}$ with connected manifolds, so that they themselves must be connected.

Just as in the case of $D_m(a_i)$ we may orient each of these components using the symplectic form, by requiring $\int_{D_i(c_j)\pm} \omega^{3g+n-4} > 0$. Thus we obtain two generators $[D_i(c_j)^+]$ and $[D_i(c_j)^-]$ for $H_{6g+2n-8}(D_i(c_j))$. Now the computation of the dual homology class to r_m reduces to the computation of two constants. Let $\iota: D_i(c_j) \to \mathscr{S}_g(t_1, \ldots, t_n)$, $\iota_{\pm}: D_i(c_j)^{\pm} \to \mathscr{S}_g(t_1, \ldots, t_n)$ denote the inclusions.

PROPOSITION 3.7. *The homology class dual to r_i lies in $\iota_* H_{6g+2n-8}(D_i(c_j))$, and may be written as $\alpha_{i,j,n,g}(t_1, \ldots, t_n)[D_i(c_j)^+] + \beta_{i,j,n,g}(t_1, \ldots, t_n)[D_i(c_j)^-]$.*

As before, the constants $\alpha_{i,j,n,g}(t_1, \ldots, t_n)$ and $\beta_{i,j,n,g}(t_1, \ldots, t_n)$ may depend *a priori* on i, j, n, g or $t_1, \ldots, t_n$. In fact they only depend on the values of t_i and t_j. The final result is the following proposition.

PROPOSITION 3.8. *Let $0 < t_i + t_j < 1$. The constants $\alpha_{i,j,n,g}(t_1, \ldots, t_n)$ and $\beta_{i,j,n,g}(t_1, \ldots, t_n)$ are given by*

$$\begin{aligned} \alpha_{i,j,n,g}(t_1, \ldots, t_n) &= \tfrac{1}{2} \\ \beta_{i,j,n,g}(t_1, \ldots, t_n) &= \tfrac{1}{2}\,\mathrm{sign}(t_j - t_i). \end{aligned} \tag{6}$$

Proof. We proceed as in the proof of Proposition 3.5. In order to compute the constants $\alpha_{i,j,n,g}(t_1, \ldots, t_n)$ and $\beta_{i,j,n,g}(t_1, \ldots, t_n)$ we again make use a subvariety Y of $\mathscr{S}_g(t_1, \ldots, t_n)$ which looks like a moduli space corresponding to a surface of genus $g = 1$, where we may perform the computation using Proposition 2.2 and the theory of toric varieties. In this case Y will be symplectomorphic to the moduli space $\mathscr{S}_1(t_i, t_j)$.

We define Y in analogy with the definition of W given in Section 3.1; for convenience we take $i = n-1$ and $j = n$. We consider the moduli of representations ρ of $\pi_1(\Sigma^g - \{p_1, \ldots, p_n\})$ which take some fixed value on $a_1, \ldots, a_{g-1}, b_1, \ldots, b_{g-1}, c_1, \ldots, c_{n-2}$, and with $\rho(\prod_{i=1}^{g-1} [a_i, b_i]) = \rho(\prod_{j=1}^{n-2} c_j)$. Geometrically, we cut Σ^g into a surface of genus $g = 1$ with two marked points p_{n-1}, p_n, and a surface of genus $g - 1$ with $n - 2$ marked points, and consider representations of $\pi_1(\Sigma^g - \{p_1, \ldots, p_n\})$ with fixed values on the fundamental group of the latter subsurface (see Fig. 6). As in the proof of Proposition 3.5, the moduli space of such representations is given by $SU(2) \times \mathscr{S}_1(t_{n-1}, t_n, 0) = SU(2) \times \mathscr{S}(t_{n-1}, t_n)$. We may therefore choose a subvariety Y sumplectomorphic to $\mathscr{S}(t_{n-1}, t_n)$. To find the constants $\alpha_{i,j,n,g}(t_1, \ldots, t_n)$ and $\beta_{i,j,n,g}(t_1, \ldots, t_n)$ it suffices to compare $V_m^g(t_1, \ldots, t_n)|_Y$ with $Y \cap D_i(c_j)^{\pm}$.

As before, we may compute $c_1(V_m^g(t_1, \ldots, t_n)|_Y)$ using variation of the symplectic form by Proposition 2.2, the methods of [5] which allow us to identify Y with a toric variety, and the theory of toric varieties. The identification of Y with a toric variety proceeds as follows. We use the standard generators a_g, b_g, c_{n-1}, c_n of $\pi_1(\Sigma^g - \{p_1, \ldots, p_n\})$, and define two moment maps $\mu_1, \mu_2: Y \to \mathbb{R}^2$ by sending a conjugacy class $[\rho] \in Y$ of representations of $\pi_1(\Sigma^g - \{p_1, \ldots, p_n\})$ to

$$\begin{aligned} \mu_1([\rho]) &= \tfrac{1}{\pi}\cos^{-1}(\tfrac{1}{2}\,\mathrm{tr}\,\rho(a_g)) \\ \mu_2([\rho]) &= \tfrac{1}{\pi}\cos^{-1}(\tfrac{1}{2}\,\mathrm{tr}\,\rho(c_{n-1} * c_n)). \end{aligned} \tag{7}$$

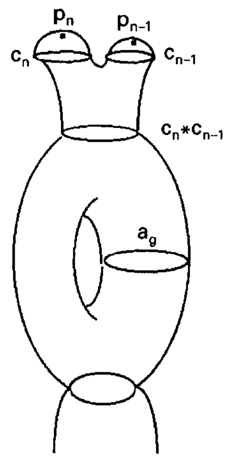

Fig. 6. The subvariety Y is formed from representations whose values are fixed outside a subsurface of genus one shown.

The results of [5] show that μ_1 and μ_2 combine to give the moment map $\mu: Y \to \mathbb{R}^2$ for an $S^1 \times S^1$-action on Y, whose image is the convex polyhedron in $\mathbb{R}^2$ given by the inequalities

$$\begin{gathered} 0 \leqslant x_2 \leqslant 2x_1 \\ 0 \leqslant 2x_1 + x_2 \leqslant 2 \\ |t_{n-1} - t_n| \leqslant x_2 \leqslant t_{n-1} + t_n \\ x_2 + t_{n-1} + t_n \leqslant 2. \end{gathered} \tag{8}$$

Now we use the following elementary fact about toric varieties. Suppose we are given a smooth toric variety M^{2n} equipped with a family of symplectic forms Ω_t. Then the image of the moment map $\mu:(M, \Omega_t \to \mathbb{R}^n$ gives a family of convex polyhedra $\Delta_t \subset \mathbb{R}^n$ given by some walls $B_i = \{x \in \mathbb{R}^n\colon \langle x, v_i\rangle = \lambda_i(t)\}$. Suppose that as t varies, the polyhedra Δ_t are combinatorially identical and differ only by a linear dependence of some $l_m(t)$ on t, and that as t is increased, the wall B_m of Δ_t moves *outward*. Then $d/dt\, \Omega_t$ is given by the cohomology class dual to the homology class represented by the cycle $[\mu^{-1}(B_m)]$, where the orientation of $\mu^{-1}(B_m)$ is given by requiring $\int_{\mu^{-1}(B_m)} (\Omega_t)^{n-1} > 0$.

Applying this fact to the case of the toric variety Y, we see that as t_n is varied the symplectic form $\omega^g_{t_1, \dots, t_n}$ changes by the dual of the homology class $\frac{1}{2}[V^+] + \frac{1}{2}[V^-]$, where V^+ is given by

$$V^+ = [\mu^{-1}(\{x_2 = t_{n-1} + t_n\})] = [D_i(c_j)^+ \cap Y]$$

while

$$V^- = \operatorname{sign}(t_{n-1} - t_n)\ [\mu^{-1}(\{x_2 = |t_{n-1} - t_n|\})] = [D_i(c_j)^- \cap Y].$$

The factor of $\frac{1}{2}$ is due to the fact that the circle action in question is not effective.

In order to prove Theorem 3 we shall also need the following result, which can be proved by comparing $D^g_m(c_k)(t_1, \dots, t_n) \cap D^g_n(c_{n-1})(t_1, \dots, t_n)$ with $\iota_+ \circ s_+(D^g_m(c_k)(t_1, \dots, t_{n-2}, (t_{n-1} + t_n))$ and $\iota_- \circ s_-(D^g_m(c_k)(t_1, \dots, t_{n-2}, |t_{n-1} - t_n|)$; alternatively we may use the explicit formula of eq. (3) for the Chern class r_i:

Proposition 3.9. *Take $i = n-1$, $j = n$, and denote the cohomology classes $r^g_m(t_1, \dots, t_{n-2}, t_{n-1} + t_n) = c_1((V^g_m)(t_1, \dots, t_{n-2}, t_{n-1} + t_n))$ and $r^g_m(t_1, \dots, t_{n-2}, |t_{n-1} - t_n|) =$*

$c_1(V^g_m(t_1, \dots, t_{n-2}, |t_{n-1} - t_n|))$ by r_m^+, r_m^-, respectively; we continue to write $r_m = r^g_m(t_1, \dots, t_n)$. Then if $t_n > t_{n-1}$,

$$
\begin{aligned}
(\iota_\pm \circ s_\pm)^* r_m &= r_m^\pm \quad \text{for } m \leqslant n-2 \\
(\iota_\pm \circ s_\pm)^* r_{n-1} &= \pm\, r_{n-1}^\pm \qquad\qquad (9)\\
(\iota_\pm \circ s_\pm)^* r_n &= r_{n-1}^\pm .
\end{aligned}
$$

4. RECURSION RELATIONS AND THE CONJECTURES OF NEWSTEAD AND WITTEN

In this section we apply the results of Section 3 to prove Theorems 1–3 of the Introduction. Slight generalizations of these are stated in Theorems 4.1, 4.2 and 4.5.

4.1. The Newstead Conjecture

We first turn our attention to the proof of Theorem 1. This is the simplest result in the sense that its proof depends only on the fact, stated in Proposition 3.3, that the homology class dual to r_m is supported in $D_m(X)$ for $X \in \{a_1, \dots, a_g, b_1, \dots, b_g\}$; we do not require the computation of the dual homology class contained in Propositions 3.5 and 3.8. Recall that we are working with $\mathscr{S}_g(t_1, \dots, t_n)$ where $(t_1, \dots, t_n)$ is *admissible.*

THEOREM 4.1. *The cohomology class $(r_{m_1})^{k_1} \cdots (r_{m_n})^{k_n}$ vanishes whenever $\sum_{i=1}^n k_i \geqslant 2g + n - 1$.*

Proof. To avoid notational complications we work with the cohomology class $(r_n)^{k_n}(r_{n-1})^{k_{n-1}} \cdots (r_{n-1})^{k_{n-1}}$, and assume that $\sum_{i=1}^n k_i = 2g + n - 1$. Let us write out the standard generators $\{a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_n\}$ of $\pi_1(\Sigma^g - \{p_1, \dots, p_n\})$ as $\{x_1, \dots, x_{2g+n}\}$, where

$$
x_i = \begin{cases} a_i & \text{if } 1 \leqslant i \leqslant g \\ b_{i-g} & \text{if } g+1 \leqslant i \leqslant 2g \\ c_{i-2g} & \text{if } 2g+1 \leqslant i \leqslant 2g+n. \end{cases} \qquad (10)
$$

Then by Proposition 3.3, the homology class dual to $(r_n)^{k_n}(r_{n-1})^{k_{n-1}} \cdots (r_{n-1})^{k_{n-l}}$ must be supported on the submanifold $D \subset \mathscr{S}_g(t_1, \dots, t_n)$ given by

$$
\begin{aligned}
D = \bigcap_{j=0}^{l-1} D_{n-j}(c_{n-j-1}) \cap \bigcap_{i=1}^{k_n - 1} D_n(x_i) \cap \bigcap_{i=k_n}^{k_{n-1}+k_n-2} D_{n-1}(x_i) \cap \cdots \\
\cap \bigcap_{i=k_n+k_{n-1}+\cdots+k_{n-(l-1)}-(l-1)}^{2g+n-l-1} D_{n-l}(x_i). \qquad (11)
\end{aligned}
$$

We claim $D = \emptyset$.

To see this, note that since we have assumed $\sum_{i=1}^n k_i = 2g + n - 1$, points in D are equivalence classes $[\rho]$ of representations ρ of $\pi_1(\Sigma^g - \{p_1, \dots, p_n\})$ where $[\rho(c_i), \rho(c_j)] = 1$ for all i and j; and therefore also $[\rho(c_i), \rho(x_j)] = 1$ for all i and j. No such representations correspond to points in $\mathscr{S}_g(t_1, \dots, t_n)$ where the set $(t_1, \dots, t_n)$ is admissible. For such a representation would be conjugate to a representation $\sigma: \pi_1(\Sigma^g - \{p_1, \dots, p_n\}) \to T$ satisfying $\prod_{i=1}^n (\sigma(c_j)) = \prod_{i=1}^{2g} \sigma([a_i, b_i]) = 1$, whence $\sum_i \varepsilon_i t_i \in 2\mathbb{Z}$ for some n-tuple $(\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$.

4.2. The Donaldson recursion relation

We now proceed to the proof of Theorem 2. For this we shall need the full calculation of the homology class dual to r_m contained in Proposition 3.5. We actually prove the following generalization.

THEOREM 4.2. *Suppose $k_1, \ldots, k_n \in \mathbb{Z}$, and suppose $k_n = 2r < 2g$ is even. Write $k = \sum_i k_i$, $r_m^g = r_m^g(t_1, \ldots, t_n)$, $r_m^{g-r} = r_m^{g-r}(t_1, \ldots, t_n)$. Then we have the following recursion relation*:

$$\int_{\mathscr{S}_g(t_1,\ldots,t_n)} (r_1^g)^{k_1} \cdots (r_n^g)^{k_n} (\omega^g_{t_1,\ldots,t_n})^{3g+n-3-k}$$
$$= 2^{-r} r! \binom{3g+n-k-3}{r} \int_{\mathscr{S}_{g-r}(t_1,\ldots,t_n)} (r_1^{g-r})^{k_1} \cdots (r_{n-1}^{g-r})^{k_{n-1}} (\omega^{g-r}_{t_1,\ldots,t_n})^{3(g-r)+n-3-(k-2r)}. \quad (12)$$

To prove this theorem we make use of Proposition 3.5, which tells us that

$$\int_{\mathscr{S}_g(t_1,\ldots,t_n)} (r_1^g(t_1,\ldots,t_n))^{k_1} \cdots (r_n^g(t_1,\ldots,t_n))^{k_n} (\omega^g_{t_1,\ldots,t_n})^{3g+2n-3-k}$$
$$= 2^{-2r} \int_D \iota^*((r_1^g(t_1,\ldots,t_n))^{k_1} \cdots (r_n^g(t_1,\ldots,t_n))^{k_n} (\omega^g_{t_1,\ldots,t_n})^{3g+2n-3-k}). \quad (13)$$

where $D = \bigcap_{i=g-r+1}^{g} (D_n(a_i) \cap D_n(b_i))$,[‡] and where $\iota : D \to \mathscr{S}_g(t_1, \ldots, t_n)$ denotes the inclusion. To make further progress we must study the submanifold D.

PROPOSITION 4.3. *There exists a symplectomorphism $s : (\mathscr{S}_{g-r}(t_1, \ldots, t_n), \omega^{g-r}_{t_1,\ldots,t_n}) \times ((S^1 \times S^1)^r, 2\omega_{(S^1 \times S^1)^r}) \to (D, \iota^* \omega^g_{t_1,\ldots,t_n})$, where $\omega_{(S^1\times S^1)^r}$ denotes the usual symplectic form on $(S^1 \times S^1)^r$.*

Proof. Points in D are conjugacy classes $[\rho]$ of representations ρ in $V_n^g(t_1, \ldots, t_n)$ where $\rho(a_i), \rho(b_i) \in T$ for $i > g - r$. Let $F = \langle a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_n \rangle$ denote the free group on the standard generators. The conditions for a homomorphism $\sigma : F \to SU(2)$ to descend to a representation in $V_n^g(t_1, \ldots, t_n)$ corresponding to a point in D are then

$$\sigma(a_i), \sigma(b_i) \in T \quad \text{for } i > g - r \quad (14a)$$

$$\sigma(c_n) \in T \quad (14b)$$

$$\sigma\left(\prod_{i=1}^{g-r} [a_i, b_i]\right) = \sigma\left(\prod_{j=1}^{n} (c_j)\right). \quad (14c)$$

Thus any such representation corresponds uniquely to a pair consisting of a representation in $V_n^{g-r}(t_1, \ldots, t_n)$ (conditions (14a) and (14b) and a point of $(S^1 \times S^1)^r$ given by the values of σ on a_i, b_i where $i > g - r$. Geometrically such representations are reducible to T on a subsurface of genus r of Σ^g; see Fig. 7. The computation of the symplectic form can be performed using the methods of [1, 5]; the factor of 2 is the 2 of $SU(2)$.

To compare the forms $(\iota \circ s)^*(r_m^g(t_1, \ldots, t_n))$ and $r_m^{g-r}(t_1, \ldots, t_n)$, we need the following lemma, which like its counterpart, Proposition 3.9, may be proved using the explicit formula (3) for the Chern forms f_m:

[‡]Note the submanifolds $D_n(a_i)$ and $D_n(b_i)$ have transverse intersections.

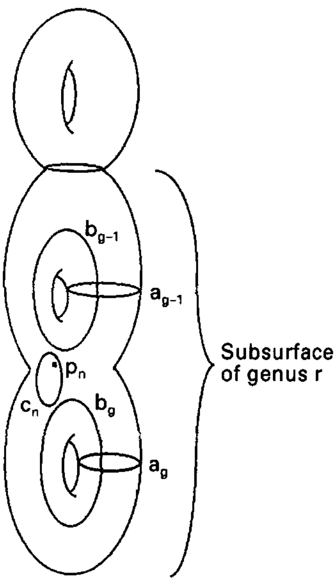

Fig. 7. The subvariety D consists of representations which are reducible on the subsurface shown.

LEMMA 4.4. *The Chern classes* $r_m^g(t_1, \dots, t_n)$ *and* $r_m^{g-r}(t_1, \dots, t_n)$ *are related by*

$$(\iota \circ s)^*(r_m^g(t_1, \dots, t_n)) = r_m^{g-r}(t_1, \dots, t_n). \tag{15}$$

The proof of Theorem 4.2 now follows by combining Proposition 4.3 and Lemma 4.4.

Remark. In view of Proposition 2.2, the recursion relation of Theorem 4.2 may be recast as follows. Let $V(s, t) = \sum_{g=1}^{\infty} \int_{\mathscr{S}(t)} e^{\omega} s^{-g} (g-1)!$. Then

$$\frac{d}{ds} V(s,t) = -\frac{1}{2} \frac{d^2}{dt^2} V(s,t). \tag{16}$$

4.3. Recursion relations in n

Finally, we come to Theorem 3. Combining Propositions 3.6, 3.7, and 3.9, we have the following result:

THEOREM 4.5. *Let* $x \in \mathbb{R}$, *and suppose* $t_n + t_{n-1} < 1$. *Write* $r_m = r_m^g(t_1, \dots, t_n)$, $r_m^+ = r_m^g(t_1, \dots, t_{n-2}, t_{n-1} + t_n)$, $r_m^- = r_m^g(t_1, \dots, t_{n-2}, |t_{n-1} - t_n|)$; *similarly write* $\omega = \omega^g_{t_1, \dots, t_n}$, $\omega^+ = \omega^g_{t_1, \dots, t_{n-2}, t_{n-1} + t_n}$, $\omega^- = \omega^g_{t_1, \dots, t_{n-2}, |t_{n-1} - t_n|}$. *Then*

$$\begin{aligned}\int_{\mathscr{S}_g(t_1, \dots, t_n)} (r_1)^{k_1} \cdots (r_n)^{k_n} e^{x\omega} &= \frac{1}{2} \int_{\mathscr{S}_g(t_1, \dots, t_{n-2}, t_{n-1}+t_n)} (r_1^+)^{k_1} \cdots (r_{n-2}^+)^{k_{n-2}} (r_{n-1}^+)^{k_{n-2}+k_n-1} e^{x\omega^+} \\ &\quad - \frac{(-1)^{k_{n-1}}}{2} \int_{\mathscr{S}_g(t_1, \dots, t_{n-2}, |t_{n-1}-t_n|)} (r_1^-)^{k_1} \cdots (r_{n-2}^-)^{k_{n-2}} (r_{n-1}^-)^{k_{n-1}+k_n-1} e^{x\omega^-}.\end{aligned} \tag{17}$$

3. HIGHER-RANK MODULI SPACES

In this section we shall use methods analogous to the ones explored in Sections 2 and 3 to prove some analogs of the Newstead Conjecture for moduli spaces corresponding to representations of $\pi_1(\Sigma^g - \{p_1, \dots, p_n\})$ in $SU(k)$ where $k > 2$. For convenience, we concentrate on the case where $n = 1$.

Given a k-tuple $\mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$, let

$$X(\mathbf{t}) = \begin{pmatrix} e^{i\pi t_1} & & & & \\ & e^{i\pi t_2} & & & \\ & & e^{i\pi t_3} & & \\ & & & \ddots & \\ & & & & e^{i\pi t_k} \end{pmatrix} \in U(k).$$

Consider a two-manifold Σ^g with *one* marked point p_1, as in Section 2. Suppose that $t_1 + \cdots + t_k = 0$, so that $X(\mathbf{t}) \in SU(k)$. We define the representation variety $R_g^k(\mathbf{t})$ as before by $R_g^k(\mathbf{t}) = \{\rho \in \mathrm{Hom}(\pi_1(\Sigma^g - \{p_1\}), SU(k)) : \rho(c_1) = X(\mathbf{t})\}$. On $R_g^k(\mathbf{t})$ we have k commuting S^1 actions; the action of $(e^{\pi i s_1}, \dots, e^{\pi i s_k}) \in (S^1)^k$ is given by the conjugation action of $X(\mathbf{s})$ on the representations in $R_g^k(\mathbf{t})$, where $\mathbf{s} = (s_1, \dots, s_k)$. Let us denote the subgroup of $(S^1)^k$ given by the ith copy of S^1 by T_i.

Let us now assume that the set $\{t_1, \dots, \hat{t}_i, \dots, t_k\}$ is admissible for every i. The moduli space $\mathscr{S}_g^k(\mathbf{t})$ is then defined as the quotient $R_g^k(\mathbf{t})/(T_1 \times \cdots \times T_k)$.[§] Similarly we may define the circle bundles $V_{i,j}^{k,g}(\mathbf{t})$ for $i, j = 1, \dots, k, i \neq j$, by $V_{i,j}^{k,g}(\mathbf{t}) = R_g^k(\mathbf{t})/(T_1 \times \cdots \times \hat{T}_i \times \cdots \times \hat{T}_j \times \cdots \times T_k)$. Let us denote the bundle projections by $\pi_{i,j} : V_{i,j}^{k,g}(\mathbf{t}) \to \mathscr{S}_g^k(\mathbf{t})$, and also write as before $r_{i,j}^{k,g} = c_1(V_{i,j}^{k,g}(\mathbf{t}))$.

To find homology classes dual to the $r_{i,j}^{k,g}$ we may proceed precisely as in Section 3. We choose a standard generator $X \in \{a_1, \dots, a_g, b_1, \dots, b_g\}$ and write $D_{i,j}(X)$ for the subset of $\mathscr{S}_g^k(\mathbf{t})$ corresponding to representations $\rho \in R_g^k(\mathbf{t})$ where $(\rho(a_i))_{i,j} = 0$; note this condition is preserved by the conjugation action of $(S^1)^k$, so that $D_{i,j}(X)$ is well defined. The following result is the analog of Proposition 3.3.

Proposition 5.1. *The circle bundle* $V_{i,j}^{k,g}(\mathbf{t})|_{\mathscr{S}_g^k(\mathbf{t}) - D_{i,j}(X)}$ *has a section for any choice of generator* X.

Using Proposition 5.1 we may now generalize the Newstead Conjecture to the following result about the classes $r_{i,j}^{k,g}$.

Theorem 5.2. (i) *Fix* $j \in \{1, \dots, k\}$. *Then*

$$\left(\prod_{i \neq j} (r_{i,j}^{k,g}) \right)^{2g} = 0. \tag{18}$$

(ii) *Any monomial in the Chern classes* $r_{i,j}^{k,g}$ *of degree greater than* $(2g)\, k(k-1)/2$ *vanishes.*

Proof. To prove (i) we note that by Proposition 5.1, the homology class dual to $(\prod_{i \neq j}(r_{i,j}^{k,g}))^{2g}$ can be chosen to be supported in $D = \bigcap_{l=1}^{g} \bigcap_{i \neq j} (D_{i,j}(a_l) \cap D_{i,j}(b_l))$. We claim $D = \emptyset$. For representations $\rho \in R_g^k(\mathbf{t})$ corresponding to points in D must have, for each l, $(\rho(a_l))_{i,j} = 0$ for all $i \neq j$; since $\rho(a_l)$ is unitary, we must also have $(\rho(a_l))_{j,i} = 0$ for all $i \neq j$. Similarly for all l, $(\rho(b_l))_{i,j} = (\rho(b_l))_{j,i} = 0$ for all $i \neq j$. The representation ρ is then *reducible*,

[§] Note that the diagonal subgroup in $(S^1)^k$ acts trivially on $R_k^g(\mathbf{t})$.

and in particular $(\rho(\prod_{l=1}^{g}[a_l, b_l]))_{j,j} = 1$, which is impossible; we have, by construction, that $(\rho(\prod_{l=1}^{g}[a_l, b_l]))_{j,j} = e^{i\pi t_i}$, and $e^{i\pi t_j} \neq 1$ by admissibility of $\{t_1, \ldots, t_n\}$.

The proof of (ii) is similar and will be described elsewhere.

CONJECTURE 5.3. *Any monomial in the Chern classes $r_{i,j}^{k,g}$ of degree greater than $(2g-1)[k(k-1)/2]$ vanishes.*

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