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A Remark on the Intersection Arrays of Distance-Regular Graphs

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It is shown that the number of columns of type $(1, 1, k-2)$ in the intersection array of a distance-regular graph with valency k and girth >3 is at most four.

1. INTRODUCTION

Let $G=(V, E)$ be a connected graph with vertex set V and edge set E, and let ∂ denote the usual metric on V. For two vertices u, v, let $p^i_j(u, v)$ denote the number of vertices x which satisfies $\partial(x, u)=i$ and $\partial(x, v)=j$. If $p^i_j(u, v)$ depends only on the distance between u and v, rather than the individual vertices, then G is said to be *distance-regular*. In this case, we write $p^i_{jm}=p^i_j(u, v)$, where $m=\partial(u, v)$. Let $d=d(G)$ be the diameter of G and k be the valency of G, and let

$$\begin{Bmatrix} * & c_1 & \cdots & c_{d-1} & c_d \\ 0 & a_1 & \cdots & a_{d-1} & a_d \\ k & b_1 & \cdots & b_{d-1} & * \end{Bmatrix}$$

be the *intersection array* of G, where $a_i=p^i_{1i}$, $b_i=p^{i+1}_{1i}$, $c_i=p^{i-1}_{1i}$. By way of recourse to the inequalities $c_i \leqslant c_{i+1}$ and $b_i \geqslant b_{i+1}$, we may write the intersection array of G in the form

$$\begin{Bmatrix} * & 1 & \cdots & 1 & 1 & \cdots & 1 & \cdots \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots \\ k & k-1 & \cdots & k-1 & k-2 & \cdots & k-2 & \cdots \end{Bmatrix}.$$

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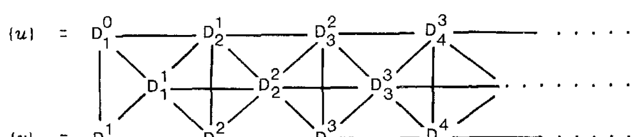
Let r and s be the number of columns of type $(1, 0, k-1)$, $(1, 1, k-2)$, respectively. We obtained the following theorem.

THEOREM. *If $r > 0$ then $s < 5$.*

Recently Biggs, Boshier, and Shawe-Taylor completed the classification of distance-regular graphs of valency three (see [3]). The key of their proof is to show that $s < 4$ if $r > 0$ in the case of $k = 3$. Our result is a partial extension of this fact. In Section 2, we shall introduce the intersection diagram of a distance-regular graph which we have used in our previous paper [2]. The intersection diagram seems to be a powerful tool in the theory of distance-regular graphs. For example, we have simple proofs of [3] and [4] using the intersection diagrams. In Section 3, we shall prove the above theorem using the intersection diagrams.

2. The Intersection Diagrams

Let $G = (V, E)$ be a distance-regular graph. For a vertex u, let $\Gamma_i(u)$ denote the set of all vertices x that satisfies $\partial(x, u) = i$. For any edge (u, v) we put $D^i_j(u, v) = \Gamma_i(u) \cap \Gamma_j(v)$. Remark that $D^i_j(u, v)$ is empty if $|i-j| > 1$, and that there is no edge between $D^i_j(u, v)$ and $D^{i'}_{j'}(u, v)$ if $|i-i'| > 1$ or $|j-j'| > 1$. The family $\{D^i_j(u, v)\}_{i,j}$ is called the *intersection diagram* of G with respect to the edge (u, v). It is helpful to draw the intersection diagram in the form



where $D^i_j(u, v)$ is denoted simply by D^i_j. In this intersection diagram, a line between D^i_j and $D^{i'}_{j'}$ indicates possibility of existence of edges between them. In the following, $E(A, B)$ denotes the set of all edges between subsets A, B of V, and $e(A, B)$ denotes the size of $E(A, B)$. Let (u, v) be a fixed edge and $\{D^i_j\}$ be the intersection diagram of G with respect to the edge (u, v).

LEMMA 1. *Let $x \in D^{i+1}_i$, then*

(1) $e(x, D^i_{i+1}) + e(x, D^{i+1}_{i+1}) + e(x, D^{i+2}_{i+1}) = b_i$,

(2) $e(x, D^{i+2}_{i+1}) = b_{i+1}$,

(3) $e(x, D^i_{i-1}) = c_i$,

(4) $e(x, D^i_{i-1}) + e(x, D^i_i) + e(x, D^i_{i+1}) = c_{i+1}$.

Proof. (1) Since $\partial(v, x) = i$, we have

$$b_i = |\Gamma_1(x) \cap \Gamma_{i+1}(v)|.$$

Here

$$\Gamma_{i+1}(v) = D^i_{i+1} \cup D^{i+1}_{i+1} \cup D^{i+2}_{i+1}.$$

So,

$$\begin{aligned} b_i &= |(\Gamma_1(x) \cap D^i_{i+1}) \cup (\Gamma_1(x) \cap D^{i+1}_{i+1}) \cup (\Gamma_1(x) \cap D^{i+2}_{i+1})| \\ &= e(x, D^i_{i+1}) + e(x, D^{i+1}_{i+1}) + e(x, D^{i+2}_{i+1}). \end{aligned}$$

(2) Since $\partial(u, x) = i+1$, we have

$$b_{i+1} = |\Gamma_1(x) \cap \Gamma_{i+2}(u)|.$$

But

$$\Gamma_1(x) \cap \Gamma_{i+2}(u) \subset D^{i+2}_{i+1}.$$

Therefore we have

$$\Gamma_1(x) \cap \Gamma_{i+2}(u) = \Gamma_1(x) \cap D^{i+2}_{i+1}.$$

Hence

$$b_{i+1} = e(x, D^{i+2}_{i+1}).$$

(3) and (4) Similar to (1) and (2).

LEMMA 2. *Let* $x \in D^i_i$, $i > 0$. *Then*

(1) $e(x, D^i_{i+1}) + e(x, D^{i+1}_{i+1}) = e(x, D^{i+1}_i) + e(x, D^{i+1}_{i+1}) = b_i$.

(2) $e(x, D^{i-1}_i) + e(x, D^{i-1}_{i-1}) = e(x, D^i_{i-1}) + e(x, D^{i-1}_{i-1}) = c_i$.

Proof. As in the proof of Lemma 1.

LEMMA 3. (1) $b_i = b_{i+1}$ *if and only if*

$$e(D^{i+1}_i, D^{i+1}_{i+1}) = e(D^{i+1}_i, D^i_{i+1}) = 0.$$

(2) $c_i = c_{i+1}$ *if and only if*

$$e(D^{i+1}_i, D^i_i) = e(D^{i+1}_i, D^i_{i+1}) = 0.$$

Proof. These are direct consequences of Lemma 1.

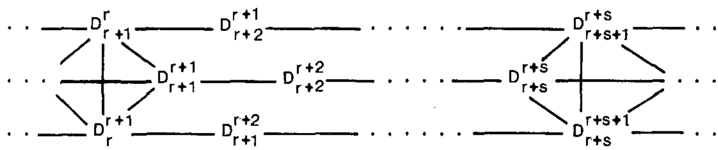
LEMMA 4. *Let* $x \in D_i^i$, $i > 0$. *Then*

(1) $e(x, D_{i+1}^i) = e(x, D_i^{i+1})$.

(2) $e(x, D_i^{i-1}) = e(x, D_{i-1}^i)$.

Proof. Immediate from Lemma 2.

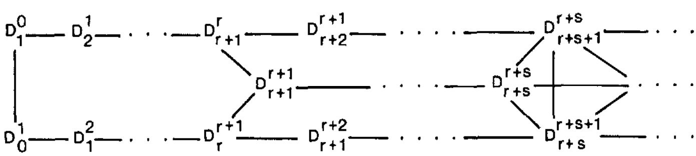
LEMMA 5. *Let* r, s *be positive integers. If* $b_{r+1} = b_{r+2} = \cdots = b_{r+s}$ *and* $c_{r+1} = c_{r+2} = \cdots = c_{r+s}$, *then the intersection diagram of* G *takes the form*



Proof. Use Lemma 3 repeatedly.

3. PROOF OF THEOREM

Let $G = (V, E)$ be a distance-regular graph with valency $k \geq 3$. Number of columns of type $(1, 0, k-1)$, $(1, 1, k-2)$ in the intersection array will be denoted by r and s, respectively. We assume $r > 0$. Then the intersection diagram of G takes the form



by the Lemmas in the previous section.

By way of contradiction, we assume $s > 4$. Note that $r + 1 \geq s$ by Ivanov's theorem (see [4]).

First, we determine the numbers $e(x, D_j^i)$ for various x.

PROPOSITION. (1) *If* $x \in D_i^{i+1}$ *for* $0 < i < r$, *then*

$$e(x, D_{i-1}^i) = 1 \qquad \textit{and} \qquad e(x, D_{i+1}^{i+2}) = k - 1.$$

(2) *If* $x \in D_r^{r+1}$ *then*

$$e(x, D_{r-1}^r) = e(x, D_{r+1}^{r+1}) = 1 \qquad \textit{and} \qquad e(x, D_{r+1}^{r+2}) = k - 2.$$

(3) *If* $x \in D_i^{i+1}$ *for* $r < i < r + s$ *then*

$$e(x, D_{i-1}^i) = e(x, D_i^{i+1}) = 1 \qquad \textit{and} \qquad e(x, D_{i+1}^{i+2}) = k - 2.$$

(4) *If* $x \in D_{r+1}^{r+1}$ *then*

$$e(x, D_r^{r+1}) = e(x, D_{r+1}^r) = 1 \quad \text{and} \quad e(x, D_{r+2}^{r+2}) = k-2.$$

(5) *If* $x \in D_i^i$ *for* $r+1 < i < r+s$ *then*

$$e(x, D_{i-1}^{i-1}) = e(x, D_i^i) = 1 \quad \text{and} \quad e(x, D_{i+1}^{i+1}) = k-2.$$

Proof. We shall only prove (4), the other cases follow along similar lines. Let $x \in D_{r+1}^{r+1}$. By Lemma 2,

$$e(x, D_{r+2}^{r+1}) + e(x, D_{r+2}^{r+2}) = b_{r+1} = k-2.$$

Since there is no edge between D_{r+1}^{r+1} and D_{r+2}^{r+1}, we have

$$e(x, D_{r+2}^{r+1}) = 0.$$

Therefore

$$e(x, D_{r+2}^{r+2}) = k-2.$$

Again by Lemma 2,

$$e(x, D_{r+1}^r) + e(x, D_r^r) = c_r = 1.$$

But now D_r^r is empty. Thus,

$$e(x, D_{r+1}^r) = 1.$$

Similarly we get also

$$e(x, D_r^{r+1}) = 1.$$

For a cycle

$$C\colon \quad x_0, x_1, \dots, x_{m-1}$$

in G, we consider the profile of C which has been defined in [3]. We give a slightly different definition. Let $\{D_j^i\}$ be the intersection diagram of G with respect to the edge (x_0, x_1). Then each x_t $(0 \leqslant t < m)$ is contained in some D_j^i. Put $D(t) = D_j^i$. Then we get a series

$$D(0), D(1), \dots, D(m-1)$$

which will be called the *profile* of the cycle C with respect to (x_0, x_1).

For example, take an edge (x_0, x_1) of G and consider the intersection diagram $\{D_j^i\}$ with respect to (x_0, x_1). Take an edge (x_{r+2}, x_{r+3}) in D_{r+1}^{r+2}, take x_{r+4} in $\Gamma_1(x_{r+3}) \cap D_r^{r+1}$ and take x_{r+5} in $\Gamma_1(x_{r+4}) \cap D_{r+1}^{r+1}$. Connect x_1 and x_{r+2} by a $(r+1)$-path

$$x_1, x_2, \dots, x_{r+2}$$

and connect x_{r+5} and x_0 by a $(r+1)$-path

$$x_{r+5}, \dots, x_{2r+5}, x_0.$$

Then we get a $(2r+6)$-cycle

$$C: \quad x_0, x_1, \dots, x_{2r+5}$$

and the profile of C with respect to (x_0, x_1) is

$$D_1^0, D_0^1, \dots, D_r^{r+1}, D_{r+1}^{r+2}, D_{r+1}^{r+2}, D_r^{r+1}, D_{r+1}^{r+1}, D_{r+1}^{r}, \dots, D_2^1.$$

Now we determine the profiles of C in the above example with respect to (x_0, x_1), (x_1, x_2), (x_2, x_3),.... By the form of the intersection diagram and by the proposition, the profile of C with respect to (x_1, x_2) is

$$D_1^0, D_0^1, \dots, D_r^{r+1}, D_{r+1}^{r+1}, D_{r+1}^{r}, D_{r+2}^{r+1}, D_{r+2}^{r+1}, D_{r+1}^{r}, \dots, D_2^1,$$

where $\{D_j^i\}$ denotes the intersection diagram with respect to (x_1, x_2). The profile of C with respect to (x_2, x_3) is

$$D_1^0, D_0^1, \dots, D_r^{r+1}, D_{r+1}^{r+1}, D_{r+2}^{r+2}, D_{r+2}^{r+2}, D_{r+1}^{r+1}, D_{r+1}^{r}, \dots, D_2^1,$$

and the profile with respect to (x_3, x_4) is

$$D_1^0, D_0^1, D_r^{r+1}, D_{r+1}^{r+2}, D_{r+1}^{r+2}, D_r^{r+1}, D_{r+1}^{r+1}, D_{r+1}^{r}, \dots, D_2^1.$$

But the profile with respect to (x_3, x_4) is same as the profile with respect to (x_0, x_1). This means the length of the cycle must be a multiple of 3. Hence we have $2r+6 \equiv 0 \pmod 3$, $r \equiv 0 \pmod 3$.

To get another condition on r, we take a $(2r+13)$-cycle

$$C': \quad y_0, y_1, \dots, y_{2r+12},$$

whose profile with respect to (y_0, y_1) is

$$D_1^0, D_0^1, \dots, D_r^{r+1}, D_{r+1}^{r+2}, D_{r+2}^{r+3}, D_{r+2}^{r+3}, D_{r+1}^{r+2}, D_r^{r+1},$$
$$D_{r+1}^{r+1}, D_{r+2}^{r+2}, D_{r+3}^{r+3}, D_{r+3}^{r+3}, D_{r+2}^{r+2}, D_{r+1}^{r+1}, D_{r+1}^{r}, \dots, D_2^1.$$

We calculate the profile of C' with respect to (y_1, y_2), (y_2, y_3),.... The profiles with respect to (y_1, y_2) and (y_2, y_3) are determined uniquely. But the profile with respect to (y_3, y_4) has two possibilities, and we must calculate the profiles with respect to (y_3, y_4), (y_4, y_5),... in each case separately. Fortunately, the profiles with respect to (y_7, y_8) coincide in each case. Thus, the profiles with respect to (y_7, y_8) and (y_8, y_9) are uniquely determined. Again the profiles with respect (y_9, y_{10}),..., (y_{12}, y_{13}) have two possibilities. But the profiles with respect to (y_{13}, y_{14}) are coincident, and the profiles with respect to (y_{13}, y_{14}), (y_{14}, y_{15}), and (y_{15}, y_{16}) are uniquely determined. The profile with respect to (y_{15}, y_{16}) coincides with the profile with respect to (y_0, y_1). Therefore $2r + 13 \equiv 0 \pmod{15}$, $r \equiv 1 \pmod 3$. This is a contradiction.

Remark. Two cycles in the above proof are the same as those used in [3]. But the profiles of the $(2r+13)$-cycle are not uniquely determined in our case, i.e., that do not have a *good profile* in terms of [3].

References

1. E. Bannai and T. Ito, "Algebraic Combinatorics I," Benjamin, California, 1984.
2. T. Hilano and K. Nomura, Distance degree regular graphs, *J. Combin. Theory Ser. B* **37** (1984), 96–100.
3. N. L. Biggs, A. G. Boshier, and J. Shawe-Taylor, Cubic distance-regular graphs, to appear.
4. A. A. Ivanov, Bounding the diameter of a distance-regular graphs, *Soviet Math. Dokl.* **28** (1983), 149–152.
5. K. Nomura, An inequality between intersection numbers of a distance-regular graph, *J. Combin. Theory Ser. B*, in press.