The saturation of convergence on the interval \([0, 1]\) for the \(q\)-Bernstein polynomials in the case \(q > 1\) \(\ast\)

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Abstract

In the note, we consider saturation of convergence on the interval \([0, 1]\) for the \(q\)-Bernstein polynomials of a continuous function \(f\) for arbitrary fixed \(q > 1\). We show that the rate of uniform convergence on \([0, 1]\) is \(o(q^{-n})\) if and only if \(f\) is linear. The result is sharp in the following sense: it ceases to be true if we replace "\(o\)" by "\(O\).

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1. Introduction

Let \(q > 0\). For any non-negative integer \(k\), the \(q\)-integer \([k]_q\) is defined by

\[ [k]_q := 1 + q + \cdots + q^{k-1} \quad (k = 1, 2, \ldots), \quad [0]_q := 0; \]

and the \(q\)-factorial \([k]_q\) by

\[ [k]_q! := [1]_q[2]_q \cdots [k]_q \quad (k = 1, 2, \ldots), \quad [0]_q! := 1. \]

For integers \(k, n\) with \(0 \leq k \leq n\), the \(q\)-binomial coefficient is defined by (see [2, p. 12])

\[ \binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}. \]

In [6], Phillips proposed the \(q\)-Bernstein polynomials: for each positive integer \(n\), and \(f \in C[0, 1]\), the \(q\)-Bernstein polynomial of \(f\) is

\[ B_{n,q}(f, x) := \sum_{k=0}^{n} f \left( \frac{[k]_q}{[n]_q} \binom{n}{k}_q x^k \prod_{i=0}^{n-k-1} (1 - q^i x) \right). \]

Note that for \(q = 1\), \(B_{n,q}(f, x)\) is the classical Bernstein polynomial \(B_n(f, x)\):

\[ B_n(f, x) := \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k}(1-x)^{n-k}. \]
In recent years, the $q$-Bernstein polynomials have been investigated intensively and a comprehensive review of the results on $q$-Bernstein polynomials along with extensive bibliography on the subject is given in [4]. From these researches we know that for $q \neq 1$, the convergence properties of the $q$-Bernstein polynomials differ essentially from those of the classical ones. In the case $q > 1$, the $q$-Bernstein polynomials are no longer positive operators. The lack of positivity makes the investigation of convergence in the case $q > 1$ essentially more difficult. However, for a function analytic in a disc $U_R := \{ z \in \mathbb{C}; \; |z| < R \}$ $(R > q)$, it was proved in [3] that the rate of convergence of $\{B_{n,q}(f, z)\}$ to $f(z)$ has the order $q^{-n}$ (versus $1/n$ for the classical Bernstein polynomials). In the note, we consider saturation of convergence of the $q$-Bernstein polynomials for arbitrary fixed $q > 1$. Denote by $C[0, 1]$ (or $C^0[0, 1]$, $1 \leq n \leq \infty$) the space of all continuous (corresponding, $n$ times continuously differentiable) functions on $[0, 1]$ equipped with the uniform norm $\| \cdot \|$. $A(n) = o(B(n))$ represents $\lim_{n \to \infty} A(n)/B(n) = 0$. For the saturation of convergence in the complex domain of the $q$-Bernstein polynomials in the case $q > 1$, we have the following two theorems:

**Theorem A.** (See [8].) Let $R > q > 1$. If a function $f$ is analytic in the disc $U_R = \{ z \in \mathbb{C}; \; |z| < R \}$, then $\|B_{n,q}(f, z) - f(z)\| = o(q^{-n})$ as $n \to \infty$ for infinite number of points having an accumulation point on $U_{R/q}$ if and only if $f$ is linear.

**Theorem B.** (See [5].) Suppose that a function $f$ is analytic in $U_1$ and continuous on $\{ z \in \mathbb{C}; \; |z| \leq 1 \}$. If

$$\max_{|z| \leq 1} |B_{n,q}(f, z) - f(z)| = o(q^{-n}) \quad \text{as} \quad n \to \infty,$$

then $f$ is a linear function.

We remark that the condition of analyticity of $f$ in Theorems A and B cannot be canceled, since in the proofs of Theorems A and B we need to expand $f(z)$ into the power series. In the note we use a completely different method and obtain the following saturation of convergence on the interval $[0, 1]$ for the $q$-Bernstein polynomials for fixed $q > 1$.

**Theorem 1.** Let $q > 1$ and $f \in C[0, 1]$. Then

$$\|B_{n,q}(f) - f\| = \sup_{x \in [0, 1]} |B_{n,q}(f, x) - f(x)| = o(q^{-n}), \quad \text{as} \quad n \to \infty, \quad (1.1)$$

if and only if $f$ is a linear function.

**Remark 1.** The above result is sharp in the following sense: the notation “$o$” cannot be replaced by the notation “$O$”. Indeed, $\sup_{x \in [0, 1]} |B_{n,q}(t^2, x) - x^2| = O(q^{-n})$ as $n \to \infty$, however, $f(x) = x^2$ is not a linear function.

The next theorem shows that if we add the condition that $f$ is convex on $[0, 1]$, then the condition in Theorem 1 can be weakened.

**Theorem 2.** Let $q > 1$, $f \in C[0, 1]$, and $f$ be convex on $[0, 1]$. If $f$ has the derivatives at $\frac{1}{q^m}$, $m = 1, 2, \ldots$, and satisfies

$$\left|B_{n,q}\left(f, \frac{1}{q^m}\right) - f\left(\frac{1}{q^m}\right)\right| = o(q^{-n}) \quad \text{as} \quad n \to \infty, \quad m = 1, 2, \ldots, \quad (1.2)$$

then $f$ is linear on $[0, 1]$.

**Remark 2.** In the case $q \in (0, 1)$, the saturation of convergence for the $q$-Bernstein polynomials was obtained by Wang in [7]. Although properties of convergence of the $q$-Bernstein polynomials are completely different for the cases $q < 1$ and $q > 1$, there is some similarity concerning saturation. See Theorem 4 in [7] for comparison.

2. Proofs of Theorems 1–2

Let $q > 1$ and $f \in C[0, 1]$. In [3], among others, Ostrovskas shows for each $m = 0, 1, 2, \ldots$,

$$\lim_{n \to \infty} B_{n,q}\left(f, \frac{1}{q^m}\right) = f\left(\frac{1}{q^m}\right).$$

Based on this, we show the following deeper result. It can be viewed as a discrete analogue of Voronovskaya’s Theorem for $q$-Bernstein polynomials for fixed $q > 1$.

**Lemma 1.** Let $q > 1$. If $f \in C^1[0, 1]$, then for each $m = 1, 2, \ldots$,

$$\lim_{n \to \infty} q^n \left(B_{n,q}\left(f, \frac{1}{q^m}\right) - f\left(\frac{1}{q^m}\right)\right) = \left(1 - \frac{1}{q^m}\right) \left(\frac{f\left(\frac{1}{q^m}\right) - f(1)}{1 - \frac{1}{q^m}} - f'\left(\frac{1}{q^m}\right)\right). \quad (2.1)$$
Proof. Let the \( q \)-Bernstein base polynomials \( p_{n,k}(q; x) \) be defined by

\[
p_{n,k}(q; x) := \binom{n}{k} q^{k} \prod_{s=0}^{n-k-1} (1 - q^s x).
\]

Then

\[
p_{n,n-k}(q; x) \left( q; \frac{1}{q^m} \right) = 0 \quad \text{for } m < k \leq n,
\]

and for \( 0 \leq k \leq m \),

\[
p_{n,n-k}(q; x) \left( q; \frac{1}{q^m} \right) = \binom{n}{k} q^{k} \prod_{s=0}^{n-k-1} \left( 1 - \frac{q^{k+1}}{q^m} \right) = \left( q^n - 1 \right) \cdots \left( q^{n-k} - 1 \right) \frac{1}{q^m} \prod_{s=0}^{m-k-1} \left( 1 - \frac{1}{q^s} \right) = q^{n(k-m)} \prod_{s=0}^{m-k-1} \left( 1 - \frac{1}{q^s} \right).
\]

It follows from the definition of \( B_{n,q}(f, x) \) and (2.2) that

\[
B_{n,q} \left( f, \frac{1}{q^m} \right) = \sum_{k=0}^{n} f \left( \frac{[n-k]_q}{[n]_q} \right) p_{n,n-k}(q; \frac{1}{q^m}) = \sum_{k=0}^{m} f \left( \frac{[n-k]_q}{[n]_q} \right) p_{n,n-k}(q; \frac{1}{q^m}).
\]

Hence,

\[
I := \lim_{n \to \infty} q^n \left( B_{n,q} \left( f, \frac{1}{q^m} \right) - f \left( \frac{1}{q^m} \right) \right) = \lim_{n \to \infty} q^n \sum_{k=0}^{n-2} f \left( \frac{[n-k]_q}{[n]_q} \right) p_{n,n-k}(q; \frac{1}{q^m}) \lim_{n \to \infty} q^n \sum_{k=0}^{m} f \left( \frac{[n-m+k]_q}{[n]_q} \right) p_{n,n-m+k}(q; \frac{1}{q^m}) =: I_1 + I_2 + I_3.
\]

Since \( f \in C[0, 1] \) and for \( 0 \leq k \leq m \),

\[
\lim_{n \to \infty} \frac{[n-k]_q}{[n]_q} = \frac{1}{q^k},
\]

we get

\[
\lim_{n \to \infty} f \left( \frac{[n-k]_q}{[n]_q} \right) = f \left( \frac{1}{q^k} \right), \quad 0 \leq k \leq m.
\]

By (2.3) and (2.5) we get for \( 0 \leq k \leq m - 2 \),

\[
\lim_{n \to \infty} q^n f \left( \frac{[n-k]_q}{[n]_q} \right) p_{n,n-k}(q; \frac{1}{q^m}) = \lim_{n \to \infty} \frac{m!}{[m]_q} f \left( \frac{1}{q^m} \right) = 0,
\]

and therefore,

\[
I_1 = 0.
\]

Let us compute \( I_2 \). Since

\[
\lim_{n \to \infty} q^n p_{n,n-m+1}(q; \frac{1}{q^m}) = \lim_{n \to \infty} [m]_q \left( 1 - \frac{1}{q^m} \right) \cdots \left( 1 - \frac{1}{q^{n-m+2}} \right) = [m]_q,
\]

by (2.5) we get

\[
I_2 = \lim_{n \to \infty} q^n f \left( \frac{[n-m+1]_q}{[n]_q} \right) p_{n,n-m+1}(q; \frac{1}{q^m}) = [m]_q f \left( \frac{1}{q^{m-1}} \right).
\]
Finally, we compute $I_3$. From (2.3) we have
\[
I_3 = \lim_{n \to \infty} q^n \left( f \left( \frac{[n-m]_k}{[n]_q} \right) \left( 1 - \frac{1}{q^n} \right) \cdots \left( 1 - \frac{1}{q^{n+m-1}} \right) - f \left( \frac{1}{q^m} \right) \right)
\]
\[
= \lim_{n \to \infty} q^n \left( f \left( \frac{[n-m]_k}{[n]_q} \right) - f \left( \frac{1}{q^m} \right) \right) \left( 1 - \frac{1}{q^n} \right) \cdots \left( 1 - \frac{1}{q^{n+m-1}} \right)
\]
\[
+ \lim_{n \to \infty} f \left( \frac{1}{q^m} \right) q^n \left( \left( 1 - \frac{1}{q^n} \right) \cdots \left( 1 - \frac{1}{q^{n+m-1}} \right) \right) - 1
\]
\[
=: I_4 + I_5. \tag{2.8}
\]
Note that $f \in C^1[0, 1]$. Then
\[
I_4 = \lim_{n \to \infty} q^n \left( f \left( \frac{[n-m]_k}{[n]_q} \right) - f \left( \frac{1}{q^m} \right) \right) \left( 1 - \frac{1}{q^n} \right) \cdots \left( 1 - \frac{1}{q^{n+m-1}} \right)
\]
\[
= \lim_{n \to \infty} f \left( \frac{[n-m]_k}{[n]_q} \right) - f \left( \frac{1}{q^m} \right) \frac{q^{n-m} - q^n}{q^n - 1} = -f' \left( \frac{1}{q^m} \right) \left( 1 - \frac{1}{q^m} \right). \tag{2.9}
\]
On the other hand, we have
\[
\lim_{n \to \infty} q^n \left( \left( 1 - \frac{1}{q^n} \right) \cdots \left( 1 - \frac{1}{q^{n+m-1}} \right) \right) - 1 = -(1 + q + \cdots + q^{m-1}) = -[m]_q,
\]
which, together with (2.8) and (2.9), implies
\[
I_3 = -f' \left( \frac{1}{q^m} \right) \left( 1 - \frac{1}{q^m} \right) - [m]_q f \left( \frac{1}{q^m} \right). \tag{2.10}
\]
Combining with (2.4), (2.6), (2.7), and (2.10), we get
\[
I = [m]_q \left( f \left( \frac{1}{q^{m-1}} \right) - f \left( \frac{1}{q^m} \right) \right) - \left( 1 - \frac{1}{q^m} \right) f' \left( \frac{1}{q^m} \right)
\]
\[
= \left( 1 - \frac{1}{q^m} \right) \left( f \left( \frac{1}{q^{m-1}} \right) - f \left( \frac{1}{q^m} \right) \right) \left( \frac{1}{q^{m-1}} - \frac{1}{q^m} \right) f' \left( \frac{1}{q^m} \right).
\]
The proof of Lemma 1 is complete. \qed

**Remark 3.** From the proof of Lemma 1, we know that the condition $f \in C^1[0, 1]$ can be weakened. In fact, if a continuous function $f(x)$ on $[0, 1]$ has the left derivatives at the points $\frac{1}{q^m}$, $m = 1, 2, \ldots$, then (2.1) holds with the derivatives replaced by the left derivatives.

**Proof of Theorem 1.** We note that $B_{n,q}(f) = f$ whenever $f$ is a linear function, so it suffices to prove that $f$ is linear if
\[
\|B_{n,q}(f) - f\| = \sup_{x \in [0,1]} |B_{n,q}(f,x) - f(x)| = o(q^{-n}) \quad \text{as} \quad n \to \infty.
\]
Denote by $\Pi_n$ the space of all algebraic polynomials of degree at most $n$, and by $E_n(f)$ the best approximation of $f(x)$ by $\Pi_n$ in $C[0,1]$, i.e.,
\[
E_n(f) = \inf_{g \in \Pi_n} \|f - g\|.
\]
Obviously,
\[
E_n(f) \leq \|f - B_{n,q}(f)\| = o(q^{-n}).
\]
By the theorem of Bernstein about characterization of the best approximation of analytic functions by polynomials (see [1]), we know that $f$ is analytic in some open region containing $[0, 1]$ in the complex plane. Then $f \in C^\infty[0, 1]$. By Lemma 1 and (1.1), we know for each $m = 1, 2, \ldots$,
\[
f' \left( \frac{1}{q^m} \right) = \frac{f\left( \frac{1}{q^{m-1}} \right) - f \left( \frac{1}{q^m} \right)}{\frac{1}{q^{m-1}} - \frac{1}{q^m}} = f'(\xi_m), \quad \xi_m \in \left( \frac{1}{q^m}, \frac{1}{q^{m-1}} \right).
\]
Hence, for each \( m = 1, 2, \ldots \), \( f''(\eta_m) = 0 \) for some \( \eta_m \in (\frac{1}{q^m}, \frac{1}{q^{m-1}}) \). Since \( f \) (and hence \( f'' \)) is analytic in some open region containing [0, 1], by the Unicity Theorem for analytic functions we get \( f'' = 0 \). Thus, \( f \) is linear. Theorem 1 is proved. \( \square \)

In order to prove Theorem 2, we need the following lemma:

**Lemma 2.** Suppose that \( f \in C[a, b] \) and \( f \) is convex on \([a, b] \). If

\[
\frac{f(b) - f(a)}{b - a} = f'_+(a),
\]

then \( f \) is linear on \([a, b] \).

**Proof.** Let \( g(x) = f(x) - (f(a) + \frac{f(b) - f(a)}{b - a}(x - a)) \) for \( x \in [a, b] \). Then \( g(a) = g(b) = 0 \), \( g'_+(a) = 0 \), and \( g(x) \) is convex and continuous on \([a, b] \). Hence, \( g(x) \leq 0 \) for \( x \in [a, b] \), and \( g \) has the left and right derivatives \( g_- \) and \( g'_+ \) on \([a, b] \). Assume that \( g \) attains the minimum at \( \xi \in (a, b) \). Obviously, \( g'_{-}(\xi) \leq 0 \). Note that for any \( x \in (a, \xi) \), we have

\[
0 = g'_+(a) \leq g'_{-}(x) \leq g'_{+}(x) \leq g'_{-}(\xi) \leq 0,
\]

which means \( g'(x) = 0 \) for \( x \in (a, \xi) \) and therefore, \( g(x) = g(a) = 0 \) for \( x \in [a, \xi] \). Since \( g(\xi) = 0 \) is the minimum of \( g \) on \([a, b] \), we get that \( g(x) \geq 0 \) for \( x \in [a, b] \). It follows that \( g = 0 \) and \( f \) is linear on \([a, b] \). Lemma 2 is proved. \( \square \)

**Proof of Theorem 2.** By Remark 3 and (1.2), we know that for each \( m = 1, 2, \ldots \),

\[
\frac{f\left(\frac{1}{q^m}\right) - f\left(\frac{1}{q^{m+1}}\right)}{\frac{1}{q^m} - \frac{1}{q^{m+1}}} = f'_\left(\frac{1}{q^m}\right) = f'_\left(\frac{1}{q^{m+1}}\right).
\]

It follows from Lemma 2 that for each \( m = 1, 2, \ldots \), \( f \) is linear on \([\frac{1}{q^m}, \frac{1}{q^{m+1}}] \). Note that for each \( m = 1, 2, \ldots \), \( f \) is piecewise linear on \([\frac{1}{q^m}, 1]\) and has the derivatives at the knots \( \frac{1}{q^k}, k = 1, 2, \ldots, m - 1 \). Hence, \( f \) is linear on \([\frac{1}{q^{m+1}}, 1]\), and by continuity \( f \) is linear on \([0, 1]\). The proof of Theorem 2 is complete. \( \square \)

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**References**