Existence theory and $Q$-matrix characterization for the generalized linear complementarity problem: Revisited

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Abstract

We provide conditions under which a vertical block matrix is a $Q$-matrix if one or all representative sub-matrices are $Q$-matrices and vice versa. It is also shown, by means of counterexamples, that Eq. (3) of [A.A. Ebiefung, Existence theory and $Q$-matrix characterization for the generalized linear complementarity problem, Linear Algebra Appl. 223/224 (1995) 155–169] is incorrect.

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1. Introduction

A vertical block matrix $N$ of dimension $m \times n$, $m \geq n$, is said to be of type $(m_1, \ldots, m_n)$ if it is partitioned, row-wise, into $n$ blocks so that the $j$th block, $N_j$, is of dimension $m_j \times n$. Given $N$ and a vector $q \in \mathbb{R}^m$ the Cottle–Dantzig generalized linear complementarity problem is to find $w \in \mathbb{R}^m$ and $z \in \mathbb{R}^n$ such that

\[ w = Nz + q, \quad w \geq 0, \quad z \geq 0, \]

\[ z_j \prod_{i=1}^{m_j} (N_j z + q_j)^i = 0 \quad (j = 1, \ldots, n). \]
We assume that the vector \( q \) is partitioned conformably with \( N \). Denote the above problem by \( \text{GLCP}(q, N) \).

Cottle and Dantzig [1] showed that if \( N \) is a strictly positive vertical block matrix or a \( P \)-matrix, then the \( \text{GLCP}(q, N) \) has a solution. The fact that the \( \text{GLCP}(q, N) \) has a unique solution when \( N \) is a \( P \)-matrix was established by Szanc [15]. By means of a system of linear inequalities in \( N \), Ebiefung and Kostreva [6] characterized existence and nonexistence of solutions, and presented a procedure for solving the \( \text{GLCP}(q, N) \). That the problem can be solved by linear programs was given by Mangasarian [12]. For other results and applications, see the papers [2,5–16] and references cited therein.

In this paper, we develop conditions for existence of solutions. We also point out errors that are contained in [4] by giving counter examples.

The rest of the paper is organized as follows. In Section 2, we present notation and definitions needed for the rest of the paper. In Section 3, we give counter examples to the previous theorems and provide new existence results. The last section, Section 4, is devoted to concluding remarks.

2. Definitions and notation

**Definition 1.** By an \( m \times n, m \geq n \), vertical block matrix \( N \) of type \((m_1, \ldots, m_n)\), we mean

\[
N = \begin{bmatrix}
N^1 \\
\vdots \\
N^n
\end{bmatrix},
\]

where the \( j \)th block is \( m_j \times n \) and \( m = \sum_{j=1}^{n} m_j \). The vectors \( w \in \mathbb{R}^m \) and \( q \in \mathbb{R}^m \) are also partitioned conformably with the entries in the blocks of \( N \):

\[
w = \begin{bmatrix}
w^1 \\
\vdots \\
w^n
\end{bmatrix}, \quad q = \begin{bmatrix}
q^1 \\
\vdots \\
q^n
\end{bmatrix},
\]

where \( q^j \) and \( w^j \) are \( m_j \times 1 \) column vectors.

**Definition 2.** Let \( N \) be a vertical block matrix of type \((m_1, \ldots, m_n)\). An \( n \times n \) matrix \( M \) is called a representative sub-matrix of \( N \) if its \( j \)th row is from the \( j \)th block, \( N^j \), of \( N \). A vertical block matrix of type \((m_1, \ldots, m_n)\) has \( \prod_{j=1}^{n} m_j \) representative sub-matrices.

**Definition 3.** A vertical block matrix \( N \) of type \((m_1, \ldots, m_n)\) is a \( Q \)-matrix if and only if the \( \text{GLCP}(q, N) \) has a solution for all \( q \in \mathbb{R}^m \).

**Definition 4.** Let \( N \) be a vertical block matrix of type \((m_1, \ldots, m_n)\). \( N \) is said to possess property \( \Phi \) or have the \( \Phi \)-Property if and only if all its representative sub-matrices are \( \Phi \)-matrices, where \( \Phi \) is some matrix class.

3. Existence of solutions

In Theorems 1 and 2 of [4] the author characterized the set of \( Q \)-matrices in terms of representative sub-matrices. These theorems are stated here for reference purposes.
Theorem 1. Let $N$ be a vertical block matrix of type $(m_1, \ldots, m_n)$ and $q$ in $\mathbb{R}^m$. The GLCP($q$, $N$) has a solution if and only if there is a representative sub-matrix $M$ and a vector $\overline{q}$, formed from $q$ by taking entries corresponding to the rows in $M$, so that LCP($\overline{q}$, $M$) has a solution $(\overline{w}, \overline{z})$ that satisfies $N\overline{z} + \overline{q} \geq 0$.

Theorem 2. Let $N$ be a vertical block matrix of type $(m_1, \ldots, m_n)$. Then $N$ is a $Q$-matrix if and only if for every vector $q \in \mathbb{R}^m$, there exists a representative sub-matrix $M$ and a vector $q \in \mathbb{R}^n$, whose entries correspond to the rows of $M$, such that LCP($q$, $M$) has a solution $(w, z)$ that satisfies $Nz + q \geq 0$.

In [4] the author had the following notation and claim: Suppose $N$ is a $Q$-matrix. For each $q \in \mathbb{R}^m$, and $j = 1, \ldots, n$, let

$$f_j = \min_{1 \leq i \leq m_j} \min_{z} \{ N^j_i z + q^j_i : Nz + q \geq 0, z \geq 0 \}.$$  

Such $f_j$ exists since $N$ is a $Q$-matrix. For a given $1 \leq j \leq n$, let $(i_j, 1 \leq i_j \leq m_j)$, be an index of $i$ such that

$$f_j = N^j_{i_j} z + q^j_{i_j} = \min_{1 \leq i \leq m_j} \min_{z} \{ N^j_i z + q^j_i : Nz + q \geq 0, z \geq 0 \}.$$

Define an $n \times n$ matrix $\overline{M}$ and a vector $\overline{q} \in \mathbb{R}^n$ by

$$\overline{M}_j = N^j_{i_j}, \quad \overline{q}_j = q^j_{i_j}, \quad j = 1, \ldots, n. \tag{2}$$

Then $\overline{M}$ is a representative sub-matrix of $N$. Suppose that $\overline{M}$ satisfies the inequality

$$0 \leq \overline{M}_j z + \overline{q}_j \leq (N^j z + q^j)_i,$$

where $i = 1, \ldots, m_j$, $j = 1, \ldots, n$, and $\overline{z} \in \mathbb{R}^n_+$. In [4] the author claimed that Eq. (3) is a true statement. The following example shows that Eq. (3) is not true in general.

Example 1. Let $N$ be a vertical block matrix of type $(2, 2)$ and $q \in \mathbb{R}^4$,

$$N = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 0 & 3 \\ 0 & 4 \end{bmatrix}, \quad q = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}.$$  

The corresponding representative sub-matrices and vectors are, respectively,

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix},$$

and

$$q_1 = q_2 = q_3 = q_4 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$  

Moreover,

$$f_1 = \min \{ \min \{ z_1 - 1, 2z_1 - z_2 - 1 : Nz + q \geq 0, z \geq 0 \} : z \geq 0 \} = \min \{ 0, 0 \} = 0.$$

$$f_2 = \min \{ \min \{ 3z_2 - 1, 4z_2 - 1 : Nz + q \geq 0, z \geq 0 \} : z \geq 0 \} = \min \{ 0, 333 \} = 0.$$
The representative sub-matrices and vectors selected according to Eq. (2) are

\[ M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 2 & -1 \\ 0 & -3 \end{bmatrix}, \quad q_1 = q_3 = \begin{bmatrix} -1 \\ -3 \end{bmatrix}. \]

The LCP\((q_1, M_1)\) has the solution \((z_1, z_2) = (1, 1/3)\) with \(Nz + q = (0, 2/3, 0, 1/3)\), and the LCP\((q_3, M_3)\) has the solution \((z_1, z_2) = (2/3, 1/3)\) with \(Nz + q = (-1/3, 0, 0, 1/3)\). Although LCP\((q_1, M_1)\) solves GLCP\((q, N)\), LCP\((q_3, M_3)\) does not. Thus \(M_1\) satisfies Eq. (3) and \(M_3\) does not. Consequently, Eq. (3) in [4] is not true in general.

**Definition 5.** Let \(N\) be a vertical block matrix of type \((m_1, \ldots, m_n)\) and \(q \in \mathbb{R}^m\) partitioned conformably with the blocks of \(N\). A representative sub-matrix \(M\) and a vector \(q \in \mathbb{R}^n\) are said to be compatible if for each \(j = 1, \ldots, n\), there exists an index \(i_j, 1 \leq i_j \leq m_j\), such that

\[ M_j = N_{i_j}^j, \quad q_j = q_{i_j}^j. \]

We shall call the pair \((\bar{q}, \bar{M})\) a compatible pair.

**Assumption 1.** Let \(N\) be a vertical block matrix of type \((m_1, \ldots, m_n)\) and \(q \in \mathbb{R}^m\) partitioned conformably with the blocks of \(N\). Let \(\bar{M}\) be a representative sub-matrix of \(N\) and \(\bar{q} \in \mathbb{R}^n\) a compatible vector. Assume that there exists a nonnegative vector \(\bar{z} \in \mathbb{R}^n\) such that \((\bar{q}, \bar{M})\) satisfies the inequality

\[ 0 \leq \bar{M}_j \bar{z} + \bar{q}_j \leq (N_j^i \bar{z} + q_j^i)_i, \]

where \(i = 1, \ldots, m_j, j = 1, \ldots, n\).

**Theorem 3.** Let \(\bar{M}\) be a representative sub-matrix of \(N\). Suppose that \((\bar{q}, \bar{M})\) satisfies Assumption 1 with respect to \(q \in \mathbb{R}^m\) and the nonnegative vector \(\bar{z}\). Then \(\bar{z}\) is a solution of GLCP\((q, N)\) if and only if it is a solution of LCP\((\bar{q}, \bar{M})\).

**Proof.** Suppose that there is a nonnegative vector \(\bar{z} \in \mathbb{R}^n\) such that \((\bar{q}, \bar{M})\) satisfies Assumption 1 with respect to \(\bar{z}\). Then \(\bar{z}\) is feasible to LCP\((\bar{q}, \bar{M})\). If \(\bar{z}\) is a solution of GLCP\((q, N)\), the feasibility condition, \(N\bar{z} + q \geq 0\), and the complementary conditions

\[ \bar{z}_j \prod_{i=1}^{m_j} (N_j^i \bar{z} + q_j^i)_i = 0 \quad (j = 1, \ldots, n) \]

imply that

\[ \bar{z}_j \left( \min_{1 \leq i \leq m_j} \{ N_i^j \bar{z} + q_j^i \} \right) = 0. \]

Thus there is an index \(k \in \{1, \ldots, m_j\}\) such that

\[ \min_{1 \leq i \leq m_j} \{ N_i^j \bar{z} + q_j^i \} = (N_k^j \bar{z} + q_k^j) \]

and

\[ \bar{z}_j (N_k^j \bar{z} + q_k^j) = 0 \quad (j = 1, \ldots, n). \]

If \((N_k^j \bar{z} + q_k^j) = 0\), then for the \(j\)th row of \(\bar{M}\),

\[ 0 \leq \bar{M}_j \bar{z} + \bar{q}_j \leq 0 \]
by Assumption 1.

Thus if \((N^l_j z^l_j + q^l_j) = 0\) or \(z^l_j = 0\), then

\[ z^l_j (M^l_j z^l_j + q^l_j) = 0 \quad (j = 1, \ldots, n). \]

Hence, \(z\) solves the \(LCP(\bar{q}, \bar{M})\).

Conversely, suppose that \((\bar{q}, \bar{M})\) satisfies Assumption 1 and that \(z\) solves \(LCP(\bar{q}, \bar{M})\). Then \(z\) is feasible to \(GLCP(q, N)\). The complementary conditions of \(LCP(\bar{q}, \bar{M})\) imply that

\[ z^l_j (M^l_j z^l_j + q^l_j) = 0 \quad (j = 1, \ldots, n). \]

By the definition of \(\bar{M}\) and \(q^l_j\),

\[ M^l_j z^l_j + q^l_j = \min_{1 \leq i \leq m_j} \{ N^l_i z^l_i + q^l_i \}, \]

where \(j = 1, \ldots, n\). Consequently,

\[ z^l_j \prod_{i=1}^{m_j} (N^l_i z^l_i + q^l_i) = 0 \quad (j = 1, \ldots, n). \]

That is, \(z\) solves the \(GLCP(q, N)\). This completes the proof. \(\square\)

In Theorem 5 of [4] it was claimed that \(N\) is a \(Q\)-matrix if and only if every representative sub-matrix is a \(Q\)-matrix. The proof of this theorem was based on the incorrect Eq. (3) of that paper. The following example shows that the theorem is also incorrect.

**Example 2.** Let \(N\) be a vertical block matrix of type \((2, 2)\) and \(q \in \mathbb{R}^4\),

\[ N = \begin{bmatrix} 1 & -6 \\ -3 & 6 \\ 2 & -2 \\ 3 & -2 \end{bmatrix}, \quad q = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 4 \end{bmatrix}. \]

The corresponding representative sub-matrices and vectors are:

\[ M_1 = \begin{bmatrix} 1 & -6 \\ 2 & -2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & -6 \\ 3 & -2 \end{bmatrix}, \quad M_3 = \begin{bmatrix} -3 & 6 \\ 2 & -2 \end{bmatrix}, \quad M_4 = \begin{bmatrix} -3 & 6 \\ 3 & -2 \end{bmatrix}. \]

\[ q_1 = q_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \quad q_3 = q_4 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}. \]

The representative sub-matrices are \(Q\)-matrices, which can be verified by using complementary cones. The \(LCP(q_1, M_1)\) and \(LCP(q_2, M_2)\) have the solution \((w_1, w_2, z_1, z_2) = (5, 4, 0, 0)\) with \(Nz + q = (5, -2, 4, 4)\). Moreover, the \(LCP(q_3, M_3)\) and \(LCP(q_4, M_4)\) have the solution \((w_1, w_2, z_1, z_2) = (10, 0, 0, 2)\) with \(Nz + q = (-7, 10, 0, 0)\). By Theorems 1 and 2, the \(GLCP(q, N)\) has no solution. Thus \(N\) is not a \(Q\)-matrix.

**Remark.** In the context of the generalized order complementarity problem, GOCP, it was shown in [9] by examples that a \(Q\)-matrix may not possess the \(Q\)-property and vice versa. Since the GOCP can be recast as a GLCP, these results also apply to the GLCP.
By Example 2, if all representative sub-matrices are \( Q \)-matrices, \( N \) may not be a \( Q \)-matrix. However, it is still interesting to know conditions under which a vertical block matrix is a \( Q \)-matrix when one or all representative sub-matrices are \( Q \)-matrices. In what follows, we provide these conditions.

**Theorem 4.** Let \( N \) be a vertical block matrix of type \((m_1, \ldots, m_n)\) and \( \overline{M} \) a \( Q \)-matrix and a representative sub-matrix of \( N \). Suppose that for every \( q \in R^m \) there is a solution \( \bar{z} \) of \( LCP(\overline{q}, \overline{M}) \) such that the compatible pair \((\bar{q}, \overline{M})\) satisfies Assumption 1 with respect to \( \bar{z} \).

Then \( N \) is a vertical block \( Q \)-matrix. Moreover, \( \bar{z} \) solves \( GLCP(q, N) \).

**Proof.** Suppose that \( \overline{M} \) is a \( Q \)-matrix. For each \( \bar{q} \in R^n \) define a vector \( q \in R^m \) by

\[
q_i^j = \begin{cases} \bar{q}_j , & \text{if } \overline{M}_j = N_{i,j}^j , \ i \in \{1, \ldots, m_j \} , \\ c \in R , & \text{otherwise} , \end{cases}
\]

where \( j = 1, \ldots, n \). Then \((\bar{q}, \overline{M})\) is a compatible pair. Let \( \bar{z} \geq 0 \) be a solution of \( LCP(\bar{q}, \overline{M}) \). Since \((\bar{q}, \overline{M})\) satisfies Assumption 1 for \( q \in R^m \) with respect to \( \bar{z} \), we have that

\[
0 \leq \overline{M}_j \bar{z} + \bar{q}_j \leq (N_i^j \bar{z} + q_i^j) ,
\]

where \( i = 1, \ldots, m_j , \ j = 1, \ldots, n \). By Theorem 3, \( \bar{z} \) is a solution to both \( LCP(\bar{q}, \overline{M}) \) and \( GLCP(q, N) \). That \( N \) is a \( Q \)-matrix follows since the choices of \( c \in R \) and \( \bar{q} \in R^n \) are arbitrary. This completes the proof.

**Corollary 1.** Let \( N \) be a vertical block matrix of type \((m_1, \ldots, m_n)\). Suppose that each representative sub-matrix of \( N \) is a \( Q \)-matrix. Assume that if \( \overline{M} \) is a representative sub-matrix of \( N \) and \( q \in R^m \), then \( \bar{z} \) solves \( LCP(\bar{q}, \overline{M}) \) and \( GLCP(q, N) \). That \( N \) is a \( Q \)-matrix follows since the choices of \( c \in R \) and \( \bar{q} \in R^n \) are arbitrary. This completes the proof.

**Theorem 5.** Let \( N \) be a vertical block \( Q \)-matrix of type \((m_1, \ldots, m_n)\) and \( \overline{M} \) a representative sub-matrix of \( N \). Suppose that for every \( q \in R^m \) there is a solution \( \bar{z} \) of \( GLCP(q, N) \) such that the compatible pair \((\bar{q}, \overline{M})\) satisfies Assumption 1 with respect to \( \bar{z} \). Then \( \overline{M} \) is a \( Q \)-matrix. Moreover, \( \bar{z} \) solves \( LCP(\bar{q}, \overline{M}) \).

**Proof.** Suppose that \( N \) is a \( Q \)-matrix. For \( q \in R^m \) let \( \bar{z} \) be a solution of \( GLCP(q, N) \). Let \( \overline{M} \) be a representative sub-matrix of \( N \). Define a vector \( \bar{q} \) by

\[
\bar{q}_j = q_i^j \quad \text{iff} \quad \overline{M}_j = N_{i,j}^j ,
\]

\( i \in \{1, \ldots, m_j \} , \ j = 1, \ldots, n \). Then \((\bar{q}, \overline{M})\) is a compatible pair.

By hypothesis, \((\bar{q}, \overline{M})\) satisfies Assumption 1 with respect to \( \bar{z} \). By Theorem 3, \( \bar{z} \) solves the \( LCP(\bar{q}, \overline{M}) \). Since the choice of \( q \in R^m \) is arbitrary, \( \overline{M} \) is a \( Q \)-matrix. This completes the proof.

**Corollary 2.** Let \( N \) be a vertical block \( Q \)-matrix of type \((m_1, \ldots, m_n)\) and \( q \in R^m \). Let \( \bar{z} \) be a solution of \( GLCP(q, N) \). Assume that if \( \overline{M} \) is a representative sub-matrix of \( N \), then \((\bar{q}, \overline{M})\) satisfies Assumption 1 with respect to \( \bar{z} \). Then every representative sub-matrix of \( N \) is a \( Q \)-matrix.
We have used Assumption 1 to provide conditions under which $N$ is a $Q$-matrix if one or each representative sub-matrix is a $Q$-matrix and vice versa. Is Assumption 1 reasonable and attainable? The answer is demonstrated in Theorem 6.

**Theorem 6.** If $\tilde{z} > 0$ solves GLCP$(q, N)$, then there exists a representative sub-matrix that satisfies Assumption 1 with respect to $\tilde{z}$.

**Proof.** Suppose that $\tilde{z} > 0$ solves the GLCP$(q, N)$. Its complementary conditions imply that for some $k \in \{1, \ldots, m_j\}$,

$$\tilde{z}_j (N_k^j \tilde{z} + q_k^j) = 0 \quad (j = 1, \ldots, n).$$

Since $\tilde{z} > 0$, we have that $(N_k^j \tilde{z} + q_k^j) = 0$. Consequently, and since $N\tilde{z} + q \geq 0$,

$$N_k^j \tilde{z} + q_k^j = 0 = \min \{N_i^j \tilde{z} + q_i^j : i = 1, \ldots, m_j\} \quad (j = 1, \ldots, n).$$

Define a matrix $\tilde{M}$ and a vector $\tilde{q}$ by

$$\tilde{M}_j = N_k^j, \quad \tilde{q}_j = q_k^j, \quad j = 1, \ldots, n.$$

Then

$$0 = \tilde{M}_j \tilde{z} + \tilde{q}_j \leq (N^j \tilde{z} + q^j)_i,$$

where $i = 1, \ldots, m_j$, $j = 1, \ldots, n$. The matrix $\tilde{M}$ and the vector $\tilde{q}$, defined by

$$\tilde{M} = \begin{bmatrix} \tilde{M}_1 \\ \vdots \\ \tilde{M}_n \end{bmatrix}, \quad \tilde{q} = \begin{bmatrix} \tilde{q}_1 \\ \vdots \\ \tilde{q}_n \end{bmatrix},$$

satisfy the conditions of Assumption 1. This completes the proof. \( \square \)

**Example 3.** Let $N$ be a vertical block matrix of type $(2, 2)$ and $q \in \mathbb{R}^4$,

$$N = \begin{bmatrix} 2 & 0 \\ 2 & 1 \\ 0 & 4 \\ 1 & 4 \end{bmatrix}, \quad q = \begin{bmatrix} -2 \\ 1 \\ -1 \\ -2 \end{bmatrix}.$$  

The vector $(z_1, z_2) = (1, 1/4)$ solves GLCP$(q, N)$ since $NZ + q = (0, 13/4, 0, 0)$. Notice that

$$N_1^1 z + q_1^1 = 0, \quad N_2^1 z + q_2^1 = \frac{13}{4},$$

and that

$$N_1^1 z + q_1^1 = 0 \leq 0 = N_1^1 z + q_1^1,$$

$$N_1^1 z + q_1^1 = 0 \leq \frac{13}{4} = N_2^1 z + q_2^1.$$

The same analysis applies to $N_2^2 z + q_2^2 = 0$ and $N_2^2 z + q_2^2 = 0$. Thus the representative sub-matrices,

$$M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix},$$

satisfy the conditions of Theorem 6.
Suppose that $GLCP(q, N)$ is solvable with solution $\bar{z}$. If there exists a representative sub-matrix $\bar{M}$ such that $(\bar{q}, \bar{M})$ satisfies Assumption 1 with respect to $\bar{z}$, then $\bar{z}$ solves both $LCP(\bar{q}, \bar{M})$ and $GLCP(q, N)$ by Theorem 3. Thus when Assumption 1 is satisfied, even for only one representative sub-matrix, it may be sufficient to solve the associated LCP rather than the GLCP. This may be advantageous, since $N$ is $m \times n$ and $\bar{M}$ is $n \times n$, and $m \gg n$ in many instances, provided that finding $\bar{M}$ does not involve the worst-case scenario, which is exponential [3]. We illustrate this observation in Example 4.

Example 4. Let $N$ be a vertical block matrix of type $(2, 2)$ and $q \in \mathbb{R}^4$, where

$$N = \begin{bmatrix} 2 & 3 \\ 2 & 1 \\ -2 & 4 \\ -3 & 4 \end{bmatrix}, \quad q = \begin{bmatrix} -7 \\ -5 \\ 0 \\ 2 \end{bmatrix}.$$ 

The representative sub-matrices in Theorem 3 can be selected as follows. Define $f_1$ and $f_2$ as follows:

$$f_1 = \min \{ \min \{2z_1 + 3z_2 - 7, 2z_1 + z_2 - 5: Nz + q \geq 0, z \geq 0\} \} = \min\{0, 0\} = 0.$$ 

$$f_2 = \min \{ \min \{-2z_1 + 4z_2, -3z_1 + 4z_2 + 2: Nz + q \geq 0, z \geq 0\} \} = \min\{0, 0\} = 0.$$ 

For this particular $N$ and $q$, each representative sub-matrix satisfies Assumption 1 with respect to the vector $\bar{z} = (2, 1)$, a common solution to all the minimization problems. Notice that $N\bar{z} + q = (0, 0, 0, 0)$ and so $\bar{z} = (2, 1)$ is a complementary solution. It is easy to verify that $\bar{z} = (2, 1)$ solves each $LCP(5\bar{q}_k, \bar{M}_k), k = 1, 2, 3, 4$.

4. Conclusion

We provide conditions under which a vertical block matrix is a $Q$-matrix when one or each representative sub-matrix is a $Q$-matrix and vice versa. We also give counterexamples to show that Eq. (3) in [4] is incorrect without further assumptions. Theorems 3, 4, 5, 6, and algorithm 1 in [4] are also incorrect since their proofs used Eq. (3) in [4].

The results show that if $GLCP(q, N)$ is solvable, then an appropriately selected representative sub-matrix and a corresponding $q$-vector can solve it. As demonstrated by Example 4, linear programs can be used to select such a representative sub-matrix. Solving the GLCP through LCP is advantageous since there are no efficient methods for solving the GLCP for a general vertical block matrix. However, finding a representative sub-matrix for the LCP may be exponential in the worst-case scenario. How to verify Assumption 1 for an arbitrary $q$ is an open question, and is not discussed in this paper.

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