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Oscillatory criteria for Third-Order difference equation with impulses

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ABSTRACT

In this paper, we investigate the oscillation of Third-order difference equation with impulses. Some sufficient conditions for the oscillatory behavior of the solutions of Third-order impulsive difference equations are obtained.

1. Introduction

Consider the impulsive difference equation

$$\begin{cases} \triangle^3 x(n) + p(n)x(n-\tau) = 0, & n \neq n_k, k = 1, 2, 3 \ldots \\ x(n_k) = a_k x(n_k - 1), & k = 1, 2, 3 \ldots \\ \triangle x(n_k) = b_k \triangle x(n_k - 1), & k = 1, 2, 3 \ldots \\ \triangle^2 x(n_k) = c_k \triangle^2 x(n_k - 1), & k = 1, 2, 3 \ldots \end{cases} \tag{1}$$

where $a_k > 0, b_k > 0, c_k > 0, p(n) \geq 0, p(n) \not\equiv 0, 0 < n_0 < n_1 < n_2 < \cdots < n_k < \cdots$ and $\lim_{k\to\infty} n_k = \infty, \tau \in N, \triangle x(n) = x(n+1) - x(n)$.

It is well known that equations with impulses have been considered by many authors. The theory of impulsive differential/difference equations is emerging as an important area of investigation, since it is much richer than the corresponding theory of differential/difference equations without impulse effects. Moreover, such equations may exhibit several real-world phenomena, such as rhythmical beating, merging of solutions, and noncontinuity of solutions.

In recent years, there has been increasing interest in the oscillation/nonoscillation of impulsive differential/difference equations, and numerous papers have been published on this class of equations and good results were obtained (see [1–8, 10–15] etc. and the references therein). But there are fewer papers on impulsive difference equations [5–7].

For example, in [5], Mingshu Peng researched the equation

$$\triangle(r_{n-1}|\triangle(x_{n-1} - x_{n-\tau-1})|^{\alpha-1}\triangle(x_{n-1} - x_{n-\tau-1})) + f(n, x_n, x_{n-l}) = 0,$$
$$r_{n_k}|\triangle(x_{n_k} - x_{n_k-\tau})|^{\alpha-1}\triangle(x_{n_k} - x_{n_k-\tau}) = M_k(r_{n_k-1}|\triangle(x_{n_k-1} - x_{n_k-\tau-1})|^{\alpha-1}\triangle(x_{n_k-1} - x_{n_k-\tau-1})).$$

He obtained sufficient conditions for oscillation of all solutions of the equation.

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In [3], Wu Xiu-Li et al. discussed the equation

$$[r(t)x'(t)]' + p(t)x'(t) + Q(t, x(t)) = 0, \quad t \geq t_0, t \neq t_k, k = 1, 2, \ldots,$$
$$x(t_k^+) = g_k(x(t_k)), \qquad x'(t_k^+) = h_k(x'(t_k)),$$
$$x(t_0^+) = x_0, \qquad x'(t_0^+) = x_0'.$$

They also investigated the oscillation of the above equation.

In [10], Wan and Mao paid attention to the following system

$$\begin{cases} x'''(t) + p(t)x(t) = 0, \quad t \geq t_0, t \neq t_k, k = 1, 2, 3 \ldots \\ x(t_k^+) = a_k x(t_k), \qquad x'(t_k^+) = b_k x'(t_k), \qquad x''(t_k^+) = c_k x''(t_k), \\ x(t_0^+) = x_0, \qquad x'(t_0^+) = x_0', \qquad x''(t_0^+) = x_0''. \end{cases}$$

The sufficient conditions are obtained for all solutions either oscillating or asymptotically tending to zero.

In this paper, we study Eq. (1) and we get some sufficient conditions for the oscillation of solutions of Eq. (1).

Definition 1. By a solution of (1) we mean a real-valued sequence $\{x_n\}$ defined on $\{n_0 - \tau, n_0 - \tau + 1, n_0 - \tau + 2, \ldots\}$ which satisfies (1) for $n \geq n_0$.

Definition 2. A solution of Eq. (1) is said to be nonoscillatory if the solution is eventually positive or eventually negative; otherwise, the solution is said to be oscillatory.

This paper is organized as follows. In Section 2, we shall offer some lemmas and theorems. To illustrate our results, some examples are included in Section 3.

2. Main results

In order to prove our theorems, we need the following lemmas.

Lemma 1. *Assume that $x(n)$ is a solution of (1), and the following conditions are satisfied:*
H1*:* $(n_1 - n_0) + b_1(n_2 - n_1) + b_1b_2(n_3 - n_2) + \cdots + b_1b_2b_3\cdots b_m(n_{m+1} - n_m) + \cdots = \infty$,
H2*:* $(n_1 - n_0) + c_1(n_2 - n_1) + c_1c_2(n_3 - n_2) + \cdots + c_1c_2c_3\cdots c_m(n_{m+1} - n_m) + \cdots = \infty$, *for some $i \in \{1, 2\}$, there exists $N \geq n_0$ such that $\triangle^{i+1} x(n) \geq 0 (\leq 0)$, $\triangle^i x(n) > 0 (< 0)$ for $n \geq N$. Then $\triangle^{i-1} x(n) > 0 (< 0)$ holds for sufficiently large n.*

Proof. We only prove the conclusion under the assumption that $\triangle^{i+1} x(n) \geq 0$, $\triangle^i x(n) > 0$. Without loss of the generality, suppose $N = n_0$. From $\triangle^{i+1} x(n) \geq 0$, we know that $\triangle^i x(n)$ is monotonically nondecreasing in $[n_k, n_{k+1})$, $k = 0, 1, 2, \ldots$. Hence

$$\triangle^i x(n) \geq \triangle^i x(n_k), \quad n \in [n_k, n_{k+1}).$$

Summing the above inequality from n_k to $n_{k+1} - 1$, we have

$$\triangle^{i-1} x(n_{k+1}) \geq \triangle^{i-1} x(n_k) + \triangle^i x(n_k)(n_{k+1} - n_k). \tag{2}$$

So

$$\triangle^{i-1} x(n_2) \geq \triangle^{i-1} x(n_1) + \triangle^i x(n_1)(n_2 - n_1),$$

thus

$$\begin{aligned} \triangle^{i-1} x(n_3) &\geq \triangle^{i-1} x(n_2) + \triangle^i x(n_2)(n_3 - n_2) \\ &\geq \triangle^{i-1} x(n_1) + \triangle^i x(n_1)(n_2 - n_1) + d_2 \triangle^i x(n_2 - 1)(n_3 - n_2) \\ &\geq \triangle^{i-1} x(n_1) + \triangle^i x(n_1)(n_2 - n_1) + d_2 \triangle^i x(n_1)(n_3 - n_2) \end{aligned}$$

where

$$d_k = \begin{cases} c_k, & i = 2, \\ b_k, & i = 1. \end{cases}$$

By induction, we get

$$\triangle^{i-1} x(n_k) \geq \triangle^{i-1} x(n_1) + \triangle^i x(n_1)[(n_2 - n_1) + d_2(n_3 - n_2) + \cdots + d_2d_3\cdots d_{k-1}(n_k - n_{k-1})].$$

From (H1) or (H2), we know that there exists l such that $\triangle^{i-1} x(n_k) > 0$ for $k \geq l$. Since $\triangle^i x(n) > 0$, we get

$$\triangle^{i-1} x(n) > \triangle^{i-1} x(n_k) > 0, \quad n \in [n_k, n_{k+1}), n_k \geq n_l.$$

We complete the proof. □

Lemma 2. *Let $x(n)$ be a solution of Eq. (1) and,* (H1) *and* (H2) *hold, for some $i \in \{1, 2, 3\}$, there exists a constant $N (N \geq n_0)$, such that $x(n) > 0$, $\triangle^i x(n) \leq 0$, $\triangle^i x(n) \not\equiv 0$ in any interval $[n, \infty)$. Then $\triangle^{i-1} x(n) > 0$ holds for sufficiently large n.*

Proof. Without loss of generality, we assume $N = n_0$. We first prove that for any $n_k \ge n_0$, $\triangle^{i-1} x(n_k) > 0$ holds. Otherwise, we can choose $n_j > n_0$, such that $\triangle^{i-1} x(n_j) \le 0$. From $\triangle^{i} x(n) \le 0$, we get that $\triangle^{i-1} x(n)$ is nonincreasing in any $[n_k, n_{k+1})$. By the condition, there exists $n_l \ge n_j$ such that $\triangle^i x(n) \not\equiv 0$, $n_l \le n < n_l + 1$. We assume $l = j$. Hence

$$\triangle^{i-1} x(n_{j+1}) = d_{j+1} \triangle^{i-1} x(n_{j+1} - 1) < d_{j+1} \triangle^{i-1} x(n_j) \le 0, \tag{3}$$

where

$$d_{j+1} = \begin{cases} c_{j+1}, & i = 3, \\ b_{j+1}, & i = 2 \\ a_{j+1}, & i = 1. \end{cases}$$

So

$$\triangle^{i-1} x(n) \le \triangle^{i-1} x(n_{j+1}) < 0, \quad n \in [n_{j+1}, n_{j+2}).$$

By induction, $\triangle^{i-1} x(n) < 0$ holds for $n \in [n_{j+q}, n_{j+q+1})$, where $q \in N$. Then $\triangle^i x(n) \le 0$, $\triangle^{i-1} x(n) < 0$, $n \in [n_{j+1}, \infty)$. By Lemma 1, we get that $\triangle^{i-2} x(n) < 0$ holds for sufficiently large n. Using Lemma 1 repeatedly, we get $x(n) < 0$, this contradicts $x(n) > 0$. So $\triangle^{i-1} x(n_k) > 0$ holds for any n_k. Since $d_{j+1} > 0$, $\triangle^{i-1} x(n)$ is nonincreasing in $[n_k, n_{k+1})$, we have that $\triangle^{i-1} x(n) > 0$ holds eventually. The proof is complete. □

From Lemmas 1 and 2, we obtain the following lemma.

Lemma 3. *Let $x(n) \ge 0$ be a solution of Eq. (1),* (H1) *and* (H2) *hold, then for sufficiently large n either* (i) *or* (ii) *holds, where* (i) $\triangle^2 x(n) > 0$, $\triangle x(n) > 0$; (ii) $\triangle^2 x(n) > 0$, $\triangle x(n) < 0$.

Lemma 4. *Assume that $x(n) \ge 0$ (≤ 0) for $n \ge N \ge n_0$. Moreover, assume that $x(n)$ is monotonically nonincreasing (monotonically nondecreasing) in $[n_k, n_{k+1})(n_k \ge n^*)$ for sufficiently large n^*, $\sum_{k=1}^{\infty} [x(n_k) - x(n_k - 1)]$ converges. Then there exists a finite limit*

$$\lim_{n \to \infty} x(n) = \alpha \ge 0 (\le 0).$$

Proof. Let $g_l = \sum_{k=1}^{l} [x(n_k) - x(n_k - 1)]$, $\lim_{l \to \infty} g_l = c$. Define a function

$$y(n) = -g_k + x(n), \quad n \in [n_k, n_{k+1}), k \in N. \tag{4}$$

Next we will prove that $y(n)$ is nonincreasing and bounded on $[n_0, \infty)$. From the definition of $y(n)$, we get

$$\begin{aligned} y(n_{k+1} - 1) &= -g_k + x(n_{k+1} - 1) = -g_k - [x(n_{k+1}) - x(n_{k+1} - 1)] + x(n_{k+1}) \\ &= -g_{k+1} + x(n_{k+1}) = y(n_{k+1}). \end{aligned}$$

For any $n_0 < a < b < \infty$, if there is k such that $a, b \in [n_k, n_{k+1})$, then

$$y(a) = -g_k + x(a) \ge -g_k + x(b) = y(b).$$

If there exist $m, k \in N$ such that $0 < m < k$ and $a \in [n_m, n_{m+1})$, $b \in [n_k, n_{k+1})$, from the nonincreasing of $x(n)$ on $[n_k, n_{k+1})$, $y(n_{k+1} - 1) = y(n_{k+1})$, we obtain

$$\begin{aligned} y(a) &= -g_m + x(a) \ge -g_m + x(n_{m+1} - 1) = y(n_{m+1} - 1) \\ &\ge y(n_k) = -g_k + x(n_k) \ge -g_k + x(b) = y(b). \end{aligned}$$

So $y(n)$ is nonincreasing on $[n_0, \infty)$, and $y(n)$ has a lower bound. Therefore, $\lim_{n\to\infty} y(n) = A$, $\lim_{n \to \infty} x(n) = A + c \ge 0$. □

Theorem 1. *Assume that* (H1) *and* (H2) *hold, $\sum_{k=1}^{\infty} |a_k - 1|$ converges, $\sum^{\infty} n^{(2)} p(n) = \infty$, then every bounded solution of Eq. (1) either oscillates or tends asymptotically to zero with fixed sign.*

Proof. Let $x(n) > 0$ be a bounded solution of Eq. (1). By Lemma 3, either (i) $\triangle^2 x(n) > 0$, $\triangle x(n) > 0$; or (ii) $\triangle^2 x(n) > 0$, $\triangle x(n) < 0$ holds for $n \ge N_0$.

We first prove that (i) does not hold. Otherwise, $\triangle x(n_k) = \gamma > 0$ holds for some $n_k \ge N_0$. From $\triangle^2 x(n) > 0$, we know that $\triangle x(n)$ is monotonically increasing for $n \in [n_{k+i-1}, n_{k+i})(i \in N)$. For $n \in [n_k, n_{k+1})$,

$$\triangle x(n) > \triangle x(n_k) = \gamma > 0,$$

in particular,

$$\triangle x(n_{k+1} - 1) > \gamma > 0.$$

For $n \in [n_{k+1}, n_{k+2})$,

$$\triangle x(n) > \triangle x(n_{k+1}) = b_{k+1}\triangle x(n_{k+1} - 1) > b_{k+1}\gamma > 0.$$

In particular, $\triangle x(n_{k+2} - 1) > b_{k+1}\gamma > 0$.

By induction for $n \in [n_{k+m}, n_{k+m+1})$ $(m \geq 2)$, we get

$$\triangle x(n) > b_{k+m}b_{k+m-1}b_{k+m-2}\cdots b_{k+1}\gamma > 0,$$

so

$$\triangle x(n_{k+m+1}) > b_{k+m+1}b_{k+m}\cdots b_{k+1}\gamma > 0.$$

Summing the inequality $\triangle x(n) > \gamma$ from n_k to $n_{k+1} - 1$,

$$x(n_{k+1}) \geq x(n_k) + \gamma(n_{k+1} - n_k).$$

Summing the inequality $\triangle x(n) > b_{k+1}\gamma$ from n_{k+1} to $n_{k+2} - 1$,

$$\begin{aligned} x(n_{k+2}) &\geq x(n_{k+1}) + b_{k+1}\gamma(n_{k+2} - n_{k+1}) \\ &\geq x(n_k) + \gamma(n_{k+1} - n_k) + b_{k+1}\gamma(n_{k+2} - n_{k+1}). \end{aligned}$$

By induction, for any $m \geq 2$,

$$x(n_{k+m}) \geq x(n_k) + \gamma(n_{k+1} - n_k) + b_{k+1}\gamma(n_{k+2} - n_{k+1}) + \cdots + b_{k+m-1}b_{k+m-2}\cdots b_{k+1}\gamma(n_{k+m} - n_{k+m-1}).$$

Considering the condition (H1), we know that the inequality above leads to a contradiction. Therefore, $\triangle x(n) < 0$, that is case (ii) holds.

Now we consider case (ii).

$\triangle x(n) < 0$ shows that $x(n)$ is strictly monotonically decreasing. From the facts that $\sum_{k=1}^{\infty} |a_k - 1|$ converges and $x(n)$ is bounded, we get that $\sum_{k=1}^{\infty} |a_k - 1| x(n_k - 1)$ converges. Consequently,

$$\sum_{k=1}^{\infty} |a_k - 1| x(n_k - 1) = \sum_{k=1}^{\infty} |x(n_k) - x(n_k - 1)|$$

converges. So $\sum_{k=1}^{\infty}[x(n_k) - x(n_k - 1)]$ converges. From Lemma 4, $\lim_{n\to\infty} x(n) = \alpha$, where $0 \leq \alpha < \infty$. We shall prove $\alpha = 0$. Otherwise, $\alpha > 0$, then there exists $n_1 \geq n_0$ such that $x(n - \tau) > \frac{\alpha}{2}$ for $n \geq n_1$. From (1),

$$\triangle^3 x(n) = -p(n)x(n - \tau) < -\frac{\alpha}{2}p(n).$$

The above inequality times $n^{(2)}$ is

$$n^{(2)} \triangle^3 x(n) < -\frac{\alpha}{2} n^{(2)} p(n).$$

Summing the above inequality from n_s to $n_{s+m} - 1$,

$$\sum_{n_s}^{n_{s+m}-1} n^{(2)} \triangle^3 x(n) < -\frac{\alpha}{2} \sum_{n_s}^{n_{s+m}-1} n^{(2)} p(n). \tag{5}$$

That is

$$\begin{aligned} -\frac{\alpha}{2} \sum_{n_s}^{n_{s+m}-1} n^{(2)} p(n) &> \sum_{n_s}^{n_{s+1}-1} n^{(2)} \triangle^3 x(n) + \sum_{n_{s+1}}^{n_{s+2}-1} n^{(2)} \triangle^3 x(n) + \cdots + \sum_{n_{s+m-1}}^{n_{s+m}-1} n^{(2)} \triangle^3 x(n) \\ &= n_{s+m}^{(2)} \triangle^2 x(n_{s+m}) - 2n_{s+m}\triangle x(n_{s+m} + 1) + 2x(n_{s+m} + 2) \\ &\quad - n_s^{(2)} \triangle^2 x(n_s) + 2n_s\triangle x(n_s + 1) - 2x(n_s + 2). \end{aligned}$$

According to $\triangle x(n) < 0$, $\triangle^2 x(n) > 0$, we get

$$-\frac{\alpha}{2} \sum_{n_s}^{n_{s+m}-1} n^{(2)} p(n) \geq 2x(n_{s+m} + 2) - n_s^{(2)} \triangle^2 x(n_s) + 2n_s\triangle x(n_s + 1) - 2x(n_s + 2).$$

Let $m \to \infty$. Since $\sum^{\infty} n^{(2)} p(n) = \infty$, this contradicts the fact that $x(n)$ is bounded. So $\alpha = 0$. The proof is complete. □

Theorem 2. *Assume that* (H1) *and* (H2) *hold,* $\sum_{n=1}^{\infty} |a_n - 1|$ *converges,* $\sum^{\infty} p(n) = \infty$, $\frac{c_k}{a_{k-1}} \leq 1$, $n_k - n_{k-1} \geq \tau + 1$, *then every solution of Eq.* (1) *either oscillates or tends asymptotically to zero with fixed sign.*

Proof. We suppose that $x(n) > 0$ is a solution of Eq. (1). From Lemma 3, for $n \geq N_0$, either
(i) $\triangle^2 x(n) > 0$, $\triangle x(n) > 0$, or (ii) $\triangle^2 x(n) > 0$, $\triangle x(n) < 0$ holds.

We first prove that case (i) does not hold. Otherwise, there exists $n_{k-2} > n_0$, such that $\triangle x(n) > 0$ for $n \in [n_{k-2}, \infty)$. Let $u(n) = \frac{\triangle^2 x(n)}{x(n-\tau-1)}$, then $u(n) > 0$.

$$\triangle u(n) < \frac{\triangle^3 x(n)}{x(n-\tau)} = -p(n) \leq 0,$$

$$\begin{aligned} u(n_k) &= \frac{\triangle^2 x(n_k)}{x(n_k-\tau-1)} = \frac{c_k \triangle^2 x(n_k-1)}{x(n_k-\tau-1)} < \frac{c_k \triangle^2 x(n_{k-1})}{x(n_{k-1})} \\ &= \frac{c_k \triangle^2 x(n_{k-1})}{a_{k-1} x(n_{k-1}-1)} < \frac{c_k}{a_{k-1}} \frac{\triangle^2 x(n_{k-1})}{x(n_{k-1}-\tau-1)} = \frac{c_k}{a_{k-1}} u(n_{k-1}). \end{aligned}$$

Hence $u(n)$ is nonincreasing and nonnegative in $[n_{k+i-1}, n_{k+i})(i \in N)$.

Summing $\triangle u(n) \leq -p(n)$ from n_k to $n_{k+1}-1$, we get

$$u(n_{k+1}) \leq u(n_k) - \sum_{n_k}^{n_{k+1}-1} p(n).$$

Similarly to this, we have

$$\begin{aligned} u(n_{k+2}) &\leq u(n_{k+1}) - \sum_{n_{k+1}}^{n_{k+2}-1} p(n) \\ &\leq u(n_k) - \sum_{n_k}^{n_{k+1}-1} p(n) - \sum_{n_{k+1}}^{n_{k+2}-1} p(n) \leq u(n_k) - \sum_{n_k}^{n_{k+2}-1} p(n). \end{aligned}$$

By induction, we know

$$u(n_{k+m}) \leq u(n_k) - \sum_{n_k}^{n_{k+m}-1} p(n).$$

Let $m \to \infty$, we get a contradiction. So only case (ii) holds.

Now we consider case (ii). From $\triangle x(n) < 0$, we get that $x(n)$ is strictly decreasing in $[n_{k+i-1}, n_{k+i})(i \in N)$.

$$\begin{aligned} &x(n_{k+1}) = a_{k+1} x(n_{k+1}-1) < a_{k+1} x(n_k), \\ &x(n_{k+2}) = a_{k+2} x(n_{k+2}-1) < a_{k+2} a_{k+1} x(n_k), \\ &\vdots \\ &x(n_{k+i}) = a_{k+i} x(n_{k+i}-1) < a_{k+i} a_{k+i-1} \cdots a_{k+1} x(n_k), \\ &\vdots \end{aligned}$$

Since $\sum_{n=1}^{\infty} |a_n - 1| < \infty$ implies $\prod_{n=1}^{\infty} a_n < \infty$, $\triangle x(n) < 0$, we get that $x(n)$ is bounded on $[n_0, \infty)$. So $\sum_{i=1}^{\infty} |a_i - 1| x(n_i - 1)$ converges, further we get that $\sum_{i=1}^{\infty} [x(n_i) - x(n_i - 1)]$ converges. By Lemma 4, we know $\lim_{n\to\infty} x(n) = \alpha$, , $0 \leq \alpha < \infty$. We shall prove $\alpha = 0$. Otherwise, $\alpha > 0$, then there exists $N_1 \geq N_0$ such that $x(n) \geq \frac{\alpha}{2}$ for $n \geq N_1$. From Eq. (1),

$$\triangle^3 x(n) = -p(n) x(n-\tau) \leq -\frac{\alpha}{2} p(n), \quad n \geq N_1.$$

Summing the above inequality from n_s to $n_{s+m}-1$,

$$-\frac{\alpha}{2} \sum_{n_s}^{n_{s+m}-1} p(n) \geq \sum_{n_s}^{n_{s+m}-1} \triangle^3 x(n) = \triangle^2 x(n_{s+m}) - \triangle^2 x(n_s).$$

Let $m \to \infty$, we get a contradiction. So $\alpha = 0$. The proof is complete. □

Theorem 3. *Assume that* (H1) *and* (H2) *hold,*

$$\limsup_{n\to\infty} \sum_{s=n-\tau}^{n-1} (s-n)^{(2)} p(s) > 2, \tag{6}$$

or

$$\limsup_{n\to\infty}\sum_{s=n-\tau}^{n-1}(s-n+\tau)^{(2)}p(s) > 2, \tag{7}$$

$\sum_{k=1}^{\infty}|a_k-1|<\infty$, *then every bounded solution of Eq.* (1) *oscillates.*

Proof. Let $x(n)$ be a positive bounded solution of Eq. (1). From the proof of Theorem 1, we get $\triangle^3 x(n) < 0$, $\triangle^2 x(n) > 0$, $\triangle x(n) < 0$.

Consider the case that (6) holds. Using the discrete Taylor formula, we have that for k, n large and $k \le n$

$$x(k) > \frac{(k-n)^{(2)}}{2}\triangle^2 x(n).$$

Hence

$$x(k-\tau) > \frac{(k-n)^{(2)}}{2}\triangle^2 x(n-\tau).$$

Substituting this into Eq. (1) we have

$$\triangle^3 x(k) + p(k)\frac{(k-n)^{(2)}}{2}\triangle^2 x(n-\tau) < 0. \tag{8}$$

Summing (8) in k from $n-\tau$ to $n-1$ we have

$$\triangle^2 x(n) - \triangle^2 x(n-\tau) + \frac{\triangle^2 x(n-\tau)}{2}\sum_{k=n-\tau}^{n-1}p(k)(k-n)^{(2)} < 0.$$

We have

$$\limsup_{n\to\infty}\sum_{s=n-\tau}^{n-1}(s-n)^{(2)}p(s) < 2,$$

which contradicts (6). The proof corresponding to (7) is similar, so we omit it here. □

3. Examples

Example 1.

$$\begin{cases} \triangle^3 x(n) + \dfrac{1}{n^{(3)}}x(n-1) = 0, & n \ge 1, n \ne 2^m, m = 1, 2, \ldots, \\ x(2^m) = \left(1+\dfrac{1}{2^m}\right)x(2^m-1), \\ \triangle x(2^m) = \left(1+\dfrac{1}{3^m}\right)\triangle x(2^m-1), \\ \triangle^2 x(2^m) = \left(1+\dfrac{1}{5^m}\right)\triangle^2 x(2^m-1). \end{cases} \tag{9}$$

(H1) and (H2) are satisfied, and $\sum_{n=1}^{\infty}|a_n-1| = \sum_{n=1}^{\infty}\frac{1}{2^n}$ converges, $\sum_{n=1}^{\infty}n^{(2)}p(n) = \sum_{n=1}^{\infty}n^{(2)}\frac{1}{n^{(3)}} = \infty$. So every bounded solution of Eq. (9) either oscillates or tends asymptotically to zero with fixed sign.

Remark. In [9], Li studied the oscillation of higher-order neutral difference equation. The result obtained here includes and improves the sufficient condition of Theorem 1 in [9].

Example 2.

$$\begin{cases} \triangle^3 x(n) + \dfrac{1}{n}x(n-1) = 0, & n \ge 7, n \ne 3m, m \ge 3, \\ x(3m) = \left(1-\dfrac{1}{m^2}\right)x(3m-1), \\ \triangle x(3m) = \left(1-\dfrac{1}{m}\right)\triangle x(3m-1), \\ \triangle^2 x(3m) = \left(1-\dfrac{1}{m}\right)\triangle^2 x(3m-1). \end{cases} \tag{10}$$

(H1) and (H2) are satisfied, and $\sum_{n=1}^{\infty}|a_n-1|=\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty}p(n)=\sum_{n=1}^{\infty}\frac{1}{n}=\infty$, $n_k-n_{k-1}=3>1+1=2$,

$$\frac{c_k}{a_{k-1}}=\frac{1-\frac{1}{k}}{1-\frac{1}{(k-1)^2}}\le 1.$$

So every solution of Eq. (10) either oscillates or tends asymptotically to zero with fixed sign.

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