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Monotonicity Properties of Musielak–Orlicz Spaces and Dominated Best Approximation in Banach Lattices

H. Hudzik* and W. Kurc

Faculty of Mathematics and Computer Science, University of Poznań, Poznań 60-769, Poland

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Criteria for strict monotonicity, lower local uniform monotonicity, upper local uniform monotonicity and uniform monotonicity of a Musielak–Orlicz space endowed with the Amemiya norm and its subspace of order continuous elements

monotonicity of these spaces equipped with the Luxemburg norm are also given. Some applications to dominated best approximation are presented.

1. PRELIMINARIES

In the following, X always denotes a Banach lattice with a lattice norm $\|\cdot\|$. Following [6] recall that X is said to be uniformly monotone (UM) (UMB in [5]) if for every $\varepsilon > 0$ there exists $\beta(\varepsilon) > 0$ such that $\|f + g\| > 1 + \beta(\varepsilon)$ whenever $f, g \in X^+$ (the positive cone in X), $\|f\| = 1$ and $\|g\| \geqslant \varepsilon$. It is known (see [23]) that X is UM if and only if for every $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that $\|f - g\| \geqslant 1 - \eta(\varepsilon)$ whenever $f \geqslant g$, $f, g \in X^+$, $\|f\| = 1$ and $\|g\| \geqslant \varepsilon$. X is said to be strictly monotone (STM) if $\|f - g\| < \|f\|$ whenever $f \geqslant g \geqslant 0$ and $g \neq 0$. From the characterizations of local uniform monotonicity for E^Φ and h^Φ given in this paper it follows that this property must in general be split into the ULUM and LLUM (LUM in [23]) properties which are defined below. X is said to be upper locally uniformly monotone (ULUM) if for any $f \in X^+$ with $\|f\| = 1$ and any $\varepsilon > 0$ there is $\delta = \delta(f, \varepsilon) > 0$ such that $\|f + g\| > 1 + \delta(f, \varepsilon)$ whenever $g \in X^+$ and $\|g\| \geqslant \varepsilon$. On the other hand, if for any $f \in X^+$ with $\|f\| = 1$ and any $\varepsilon > 0$ there is $\delta = \delta(f, \varepsilon) > 0$ such that $\|f - g\| \leqslant 1 - \delta(f, \varepsilon)$ whenever $g \in X^+$, $g \leqslant f$ and $\|g\| \geqslant \varepsilon$ then X is said to be lower locally uniformly monotone (LLUM). The LLUM property is considered in [6–8] and in [23], whereas the ULUM property is considered in [26]. The UM and STM properties in a normed lattice X are

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nothing but uniform rotundity (UR) and rotundity (R) (see [31]) restricted to comparable elements in the positive cone X^+, respectively (see [23], Propositions 1.2 and 1.3 and [16]). Therefore, they involve both the geometric and the order structure of X. Consequently, these properties play in the dominated best approximation a similar role as UR and R in the best approximation for Banach spaces ([23]). In the last section, we will also deal with properties CWLLUM and H^+STM weaker than LLUM (see [23]). It will be proved that these properties coincide in Banach lattices. H^+STM and CWLLUM play a crucial role in a characterization of the dominated best approximation in Banach lattices (see Theorems 4.1 and 4.2). It is worth noticing that property H^+STM can be viewed as a lattice version of the Kade–Klee property with the rotundity (HR). Let X^* denote the dual space of X. Recall, a Banach lattice is said to be CWLLUM (CWLUM in [23]) if for any nonnegative $x^* \in X^*$ with $\|x^*\| = 1$, any nonnegative $x \in X$ with $\|x\| = 1$ and any sequence (x_n) in X satisfying $0 \leqslant x_n \leqslant x$ for all n, the condition $x^*(x - x_n) \to \|x\|$ implies $\|x_n\| \to 0$. It is clear that LLUM implies CWLLUM. Also, we say that X has the H^+ property ([23]) if $\|x - x_n\| \to 0$ whenever $0 \leqslant x_n \leqslant x$ and $x_n \to x$ weakly.

A subset K of a Banach lattice X is said to be a sublattice of X if it is closed with respect to the finite suprema and infima. Let us point out that K need not be a linear subspace and that any order interval $[x, y]$ is a typical example of such a K. We write $K \leqslant f$ whenever given any sublattice K and f in X there holds $f \geqslant g$ for all $g \in K$. Similarly $f \leqslant K$ is defined. If $K \leqslant f$ or $f \leqslant K$ is satisfied let

$$P_K(f) = \{h \in K : \|f - h\| = \inf_{g \in K} \|f - g\|\}.$$

Since $dist(f, K) = \inf_{g \in K \geqslant f} \|g - f\| = \inf_{v \in V \leqslant f} \|f - v\|$, where $V = 2f - K$, one can restrict to the case $K \leqslant f$ only. Therefore, we always refer to such problems as to the dominated best approximation problems ([23]). The dominated best approximation problem is solvable, if $P_K(f) \neq \emptyset$. The problem of dominated best approximation is said to be uniquely solvable if $\mathrm{Card}(P_K(f)) = 1$. The problem is said to be stable if for every minimizing sequence $\{h_n\}$ in K, i.e., a sequence in K such that $\|f - h_n\| \to \inf_{h \in K} \|f - h\|$, there holds $d(h_n, P_K(f)) \to 0$ as $n \to \infty$. Finally, the problem is said to be strongly solvable if it is uniquely solvable and stable. In the last section we give a natural application of the LLUM, ULUM, CWLLUM and H^+STM properties to the problem of dominated best approximation. In particular, a characterization of the unique solvability in terms of STM property together with the order continuity ($0 \leqslant x_n \downarrow 0$ implies $\|x_n\| \to 0$), and in terms of the CWLLUM property is given. These results extend the corresponding results from [23 pp. 181–182] to the case of Banach lattices.

In the sequel let (T, Σ, μ) be either a nonatomic or a purely atomic (counting) measure space. We always assume that (T, Σ, μ) is nontrivial, complete and σ-finite. Let Φ: $T \times \mathsf{R} \to [0, +\infty]$ be a function such that $\Phi(t, \cdot)$ is even, convex (nontrivial), vanishing and continuous at zero and left continuous on R_+ for μ-a.e. $t \in T$ with $\Phi(\cdot, u)$ Σ-measurable for each $u \in \mathsf{R}$. In the case of the counting measure ($T = \mathsf{N}$ and $\mu(\{n\}) = 1$ for any $n \in \mathsf{N}$) we will write $\Phi_n(u)$ instead of $\Phi(n, u)$. In this case, Φ will be identified with (Φ_n). Also, we will write $0 < \Phi$ if $0 < \Phi(t, u)$ for μ-a.e. $t \in T$ and all $u > 0$. Similarly $\Phi < \infty$ is defined. It will be assumed in the most of Section 2 below that for μ-a.e. $t \in T$, $\Phi(t, u)/u \to \infty$ (resp. $\Phi_n(u)/u \to \infty$ for all $n \in \mathsf{N}$) as $u \to \infty$. This condition will be called the (∞)-condition. A function Φ satisfying the above conditions and the (∞)-condition is nothing but a Musielak–Orlicz function. We denote by $L^0 = L^0(\mu)$ (resp. l^0) the space of all Σ-measurable functions from T into R (resp. the space of all real sequences). We consider the functional $I_\Phi(\cdot)$ on L^0 (resp. l^0) defined by

$$I_\Phi(f) = \int_T \Phi(t, f(t))\, d\mu \qquad (\forall f \in L^0)$$

(resp. $I_\Phi(f) = \sum_{n=1}^{\infty} \Phi_n(x_n)$ for $f = (x_n) \in l^0$). Then $I_\Phi(\cdot)$ is a convex (even) modular on L^0 (resp. l^0), cf. [27, 28] and [32].

The Musielak–Orlicz space L^Φ (resp. l^Φ) consists of all $f \in L^0$ (resp. $f \in l^0$) such that $I_\phi(\lambda f) < \infty$ for some $\lambda > 0$ (depending on f). The subspace E^Φ of order continuous elements in L^Φ is defined by $E^\Phi = \{f \in L^0 : I_\Phi(\lambda f) < \infty \text{ for all } \lambda > 0\}$. The corresponding subspace for l^Φ is defined by $h^\Phi = \{f = (x_n) \in l^0 : \forall(\lambda > 0)\ \exists(m \in \mathsf{N}) \text{ s.t. } \sum_{n=m}^{\infty} \Phi_n(\lambda x_n) < \infty\}$. In these four spaces we consider the Luxemburg norm

$$\|f\|_\Phi = \inf\left\{\lambda > 0 : I_\Phi\left(\frac{f}{\lambda}\right) \leqslant 1\right\}$$

and the Orlicz norm

$$\|f\|_\Phi^0 = \sup\{|\langle f, g\rangle| : I_{\Phi^*}(g) \leqslant 1\},$$

where $\langle f, g\rangle = \int_T f(t)\, g(t)\, d\mu$ ($\langle f, g\rangle = \sum_{n=1}^{\infty} f_n g_n$) for $f, g \in L^Phi$ (resp. $f, g \in l^\Phi$) and Φ^* denotes the Young conjugate of Φ. Both norms are lattice monotone norms and they are equivalent:

$$\|f\|_\Phi \leqslant \|f\|_\Phi^0 \leqslant 2\, \|f\|_\Phi,$$

where $f \in L^\Phi$ (resp. $f \in l^\Phi$). Since the Banach spaces $(L^\Phi, \|\cdot\|_\Phi)$ and $(L^\Phi, \|\cdot\|_\Phi^0)$ are not isometric in general, the geometry of the spaces must be studied separately. In what follows, we will be concerned with the

Amemiya norm (see Section 2) rather than the Orlicz norm. This is because the Amemiya norm is easier to study and coincides with the Orlicz norm in the most important cases (see Section 2).

In the case of a nonatomic measure space we say that Φ satisfies the Δ_2-condition ($\Phi \in \Delta_2$) if there exists a set T_0 with $\mu(T_0) = 0$ and a nonnegative function $h \in L_1(\mu)$ such that

$$\Phi(t, 2u) \leqslant 2\Phi(t, u) + h(t)$$

for every $t \in T \setminus T_0$, $u \in \mathrm{R}$ (see [9, 27]).

In the case of the counting measure μ we say that Φ satisfies the δ_2^0-condition ($\Phi \in \delta_2^0$) if there exist positive numbers a and K, a natural number m and a sequence (c_n) in $[0, +\infty]$ such that $\sum_{n=m}^{\infty} c_n < \infty$ and

$$\Phi_n(2u) \leqslant K\Phi_n(u) + c_n$$

for all $n \in \mathrm{N}$ and $u \in \mathrm{R}$ such that $\Phi_n(u) \leqslant a$ (see [2, 17]). If the δ_2^0-condition for Φ holds with $m = 1$, i.e., (c_n) can be chosen in $[0, +\infty)$, we say that Φ satisfies the δ_2-condition ($\Phi \in \delta_2$). If all functions Φ_n are finite, then $\Phi \in \delta_2^0$ is equivalent to $\Phi \in \delta_2$.

It is known that $E^\Phi = L^\Phi$ iff $\Phi \in \Delta_2$ and $h^\Phi = l^\Phi$ iff $\Phi \in \delta_2^0$. Moreover, for a sequence (x_n) in L^Φ (resp. l^Φ) $\|x_n\|_\Phi \to 0$ is equivalent to $I_\Phi(x_n) \to 0$ if and only if $\Phi \in \Delta_2$ (resp. $\Phi \in \delta_2^0$) and $\Phi > 0$ (see [14]).

The main aim of the paper is to characterize the UM, LLUM, ULUM and STM properties in the spaces L^Φ, E^Φ, l^Φ and h^Φ equipped with the Amemiya norm (which will be defined below) in terms of the function Φ. Next, these properties are applied to solve some dominated best approximation problems. For the Luxemburg norm this was done in [22] and [23] by the second named author. In this paper we fill these results in the case of the ULUM property for the Luxemburg norm and develop our study for the LLUM and ULUM properties for the Amemiya norm in the context of strong solvability of the upper dominated best approximation problems. Recall that the UM property has applications to ergodic theory as well (see [1]).

It is worth noticing that the results concerning the Amemiya norm differ substantially from that for the Luxemburg norm. For example, for the Luxemburg norm (see [23]) L^Φ is STM iff it is UM (equivalently LLUM) and this is equivalent to $\Phi > 0$ and $\Phi \in \Delta_2$, whereas for the Amemiya norm: L^Φ is STM iff $\Phi > 0$ and L^Φ is LLUM iff it is ULUM (equivalently, iff $\Phi > 0$ and $\Phi \in \Delta_2$). In the case of the spaces E^Φ and h^Φ, equipped with the Amemiya norm, the differences become greater.

The results we prove below concern four Orlicz-type spaces defined above for a continuous as well as a counting measure space (T, Σ, μ). Although our results are presented only in the case of a nonatomic or purely atomic measure space they can be easily extended to any σ-finite measure space. If S denotes the set of all atoms in Σ, then (T, Σ, μ) is a direct sum of two measure spaces $(S, \Sigma \cap S, \mu)$ and $(S, \Sigma \cap (T \backslash S), \mu)$ which are purely atomic and nonatomic, respectively. It is easy to see that L^{Φ} has a monotonicity property (A) if and only if both $L_{\Phi}(S, \Sigma \cap S, \mu)$ and $L_{\Phi}(S, \Sigma \cap (T \backslash S), \mu)$ have property (A).

In the sequel we will need the following lemma which was proved in [13] for the Luxemburg norm only, but in view of the equivalence of the Luxemburg and Amemiya norms $\|\cdot\|^{A}_{\Phi}$ (cf. Section 2 below) this result is valid for the Amemiya norm too.

LEMMA 1.1. *Let $\Phi \in \Delta_2$ (resp. $\Phi \in \delta_2^0$) and $\Phi > 0$. Then for every $\varepsilon > 0$ there exists $\sigma(\varepsilon) > 0$ such that $I_{\Phi}(f) \geqslant \sigma(\varepsilon)$ whenever $f \in L^{\Phi}$ (resp. $F \in l^{\Phi}$) and $\|f\|^{A}_{\Phi} \geqslant \varepsilon$.*

2. CHARACTERIZATIONS OF THE MONOTONICITY PROPERTIES

The Amemiya norm $\|\cdot\|^{A}_{\Phi}$ is defined by the formula

$$\|f\|^{A}_{\Phi} = \inf_{k>0} \frac{1}{k}(1 + I_{\Phi}(kf)),$$

where $f \in L^{\Phi}$ (resp. $F \in l^{\Phi}$) (see [21, 24, 29] for the Orlicz spaces and [28] for the general case). It is well known that $\|f\|^{0}_{\Phi} \leqslant \|f\|^{A}_{\Phi}$ for any $f \in L^{\Phi}$ (resp. l^{Φ}) and that if Φ is finitely valued and the (∞)-condition is satisfied then $\|f\|^{A}_{\Phi} = \|f\|^{0}_{\Phi}$ for all $f \in L^{\Phi}$ (resp. l^{Φ}), see [10] and in the case of Orlicz space [29].

Remark. Assuming that for μ-a.e. $t \in T$ there holds $\Phi(t, u)/u \to \infty$ as $u \to \infty$ (resp. for all $n \in \mathrm{N}$ there holds $\Phi_n(u)/u \to \infty$ as $u \to \infty$) the infimum in the Amemiya formula is attained for a certain $k = k(f) > 0$:

$$\|f\|^{A}_{\Phi} = \frac{1}{k}(1 + I\Phi(kf))$$

(see [11] for Orlicz spaces and [24] for the general case).

This formula is of great importance when the geometry of L^{Φ} (resp. l^{Φ}) under the Orlicz norm is studied because it does not refer explicitly to the conjugate Φ^{*} of Φ.

Let 1_A stand for the characteristic function of a given set A. We start with the following proposition.

PROPOSITION 2.1 *Let Φ be a Musielak–Orlicz function such that $\Phi \notin \Delta_2$ (resp. $\Phi \notin \delta_2^0$) and in the nonatomic case assume that $\Phi < \infty$. Then there exists a set $A \in \Sigma$ (resp. $A \subset \mathsf{N}$) of positive finite measure such that $\mu(T \backslash A) > 0$ and for any $\varepsilon > 0$ there exists a sequence of positive elements (f_n) in E^Φ (resp. h^Φ) with pointwise disjoint supports in $T \backslash A$ such that*

$$I_\Phi\left(\sum_{n=1}^{\infty} f_n\right) \leqslant \min(1, \varepsilon), \tag{1}$$

$$\frac{1}{1+\varepsilon} \leqslant \|f_k\|_\Phi \leqslant \left\|\sum_{n=1}^{\infty} f_n\right\| \leqslant 1 \qquad (\forall k \in \mathsf{N}), \tag{2}$$

$$\frac{1}{1+\varepsilon} \leqslant \|f_k\|_\Phi^0 \leqslant \left\|\sum_{n=1}^{\infty} f_n\right\|_\Phi^0 \leqslant \left\|\sum_{n=1}^{\infty} f_n\right\|_\Phi^A \leqslant 1+\varepsilon \qquad (\forall k \in \mathsf{N}). \tag{3}$$

Proof. We will give a proof in the sequence case only. The proof for a nonatomic measure proceeds analogously. Take an arbitrary $\varepsilon \in (0, 1)$. We can construct as in the proof of Theorem 3 in [2] a sequence (f_k) in h^Φ with pairwise disjoint supports in $\mathsf{N} \backslash A$, where A is an arbitrary finite subset of N, such that

$$I_\Phi(f_k) \leqslant 2^{-k}\varepsilon \qquad (\forall k \in \mathsf{N}) \tag{4}$$

and

$$I_\Phi((1+\varepsilon) f_k) > 1 \qquad (\forall k \in \mathsf{N}). \tag{5}$$

Inequality (4) yields inequality (1) immediately. Inequalities (4) and (5) yield

$$\frac{1}{1+\varepsilon} \leqslant \|f_k\|_\Phi \leqslant \|f_k\|_\Phi^0 \qquad \text{and} \qquad \|f_k\|_\Phi \leqslant 1 \qquad (\forall k \in \mathsf{N}). \tag{6}$$

Next, in view of inequality (4),

$$\|f_k\|_\Phi^0 \leqslant \left\|\sum_{n=1}^{\infty} f_n\right\|_\Phi^0 \leqslant \left\|\sum_{n=1}^{\infty} f_n\right\|_\Phi^1 \leqslant 1 + \sum_{n=1}^{\infty} I_\Phi(f_n) \leqslant 1+\varepsilon. \tag{7}$$

By (1) there holds

$$\left\|\sum_{n=1}^{\infty} f_n\right\|_\Phi \leqslant 1. \tag{8}$$

Combining inequalities (6), (7), and (8), we obtain inequalities (2) and (3).

THEOREM 2.2. *Let Φ be a Musielak–Orlicz function satisfying the (∞)-condition and let L^Φ, E^Φ (resp. l^Φ, h^Φ) be equipped with the Amemiya norm. When considering E^Φ assume additionally that $\Phi < \infty$. Then the following assertions are equivalent*:

(i) L^Φ *is STM*,

(ii) E^Φ *is STM*,

(iii) E^Φ *is LLUM*,

(iv) $\Phi > 0$.

The same hold for l^Φ and h^Φ instead of L^Φ and E^Φ, respectively.

Proof. (iv) $\Rightarrow$ (i). Let $\Phi > 0$ and $f, g \in L^\Phi$ be such that $0 \leqslant g \leqslant f$, $\|f\|_\Phi^A = 1$ and $g \neq 0$. Let $k > 0$ be such that

$$\|f\|_\Phi^A = \frac{1}{k}(1 + I_\Phi(kf)). \tag{9}$$

Since Φ (as a convex function vanishing at zero), is a superadditive function on R_+, it follows that

$$I_\Phi(kf) = I_\Phi(k(f-g)+kg) \geqslant I_\Phi(k(f-g)) + I_\Phi(kg)$$

and so

$$I_\Phi(k(f-g)) \leqslant I_\Phi(kf) - I_\Phi(kg). \tag{10}$$

By (9) and (10),

$$\begin{aligned}\|f-g\|_\Phi^A &\leqslant \frac{1}{k}(1 + I_\Phi(k(f-g))) \\ &\leqslant \frac{1}{k}(1 + I_\Phi(kf)) - \frac{1}{k} I_\Phi(kg) \\ &= \|f\|_\Phi^A - \frac{1}{k} I_\Phi(kg) \\ &< \|f\|_\Phi^A\end{aligned}$$

because $(1/k)\, I_\Phi(kg) > 0$ by the assumption that $g \neq 0$ and $\Phi > 0$.

The implication (i) $\Rightarrow$ (ii) is obvious. Now we will prove the implication (ii) $\Rightarrow$ (iv).

Assume that the condition $\Phi > 0$ is not satisfied and define the measurable function $p: T \to \mathsf{R}_+$ by

$$p(t) = \sup\{u \geqslant 0: \Phi(t, u) = 0\} \tag{11}$$

and let $A = \{t \in T: p(t) > 0\}$. Obviously, $\mu(A) > 0$. There exists a sequence (T_n) of measurable pairwise disjoint sets T_n in T of finite positive measure such that $\bigcup_{n=1}^{\infty} T_n = T$ and (see [18])

$$L^\infty \,|T_n \hookrightarrow E^\Phi|\, T_n \qquad (\forall n \in \mathsf{N}).$$

There are $m \in \mathsf{N}$ satisfying $\mu(T_m \cap A) > 0$ and $l > 0$ such that the set

$$C = \{t \in T_m: p(t) \geqslant l\}$$

is of positive measure. We have $l 1_C \in E^\Phi$ and $f = 1_{T_{m+1}} \in E^\Phi$. Let $\|f\|_\Phi^A = (1/k_0)(1 + I_\Phi(k_0 f))$ and

$$g(t) = f(t) + \frac{l}{k_0} 1_C(t).$$

Clearly $f \leqslant g$, $f \neq g$ and

$$\|g\|_\Phi^A = \inf_{k>0} \frac{1}{k} (1 + I_\Phi(kg)) \geqslant \inf_{k>0} \frac{1}{k} (1 + I_\Phi(kf)) = \|f\|_\Phi^1. \tag{12}$$

On the other hand $\Phi(t, l) = 0$ for μ-a.e. $t \in T$, whence $I_\Phi(k_0((1/k_0)\, 1_C)) = 0$ and consequently

$$\begin{aligned} \|f\|_\Phi^A &= \frac{1}{k_0} (1 + I_\Phi \Phi(k_0 f)) \\ &= \frac{1}{k_0} (1 + I_\Phi(k_0 g)) \\ &\geqslant \inf_{k>0} \frac{1}{k} (1 + I_\Phi(kg)) = \|g\|_\Phi^1. \end{aligned} \tag{13}$$

Combining (12) and (13), it follows that $\|g\|_\Phi^A = \|f\|_\Phi^A$, which means that E^Φ is not STM, so the implications (ii) $\Rightarrow$ (iv) is proved.

Now, we will prove the implication (ii) $\Rightarrow$ (iii). Suppose (ii) holds. Then, as we just proved, condition (iv) is satisfied. Let $0 \leqslant g \leqslant f$, f and $g \in E^\Phi$, $\|f\|_\Phi^A = 1$ and $\|g\|_\Phi^A \geqslant \varepsilon$, where $0 < \varepsilon \leqslant 1$. Then in the same way as in the proof of the implication (iv) $\Rightarrow$ (i), taking into account the fact that $(1/k)\, I_\Phi(kg) \geqslant I_\Phi(g)$ for every $k \geqslant 1$, it follows that

$$\|f-g\|_\Phi^1 \leqslant \|f\|_\Phi^A - I_\Phi(g). \tag{14}$$

To finish the proof of the implication it is enough to show that there exists $\delta = \delta(f, \varepsilon) \in (0, 1)$ (δ independent of g) such that $I_\Phi(g) \geqslant \delta$.

If this is not true, then there exists a sequence (f_n) such that $0 \leqslant f_n \leqslant f$, $\|f_n\|_\Phi^A \geqslant \varepsilon$ and $I_\Phi(f_n) \to 0$. This yields that $\Phi(t, f_n(t)) \to 0$ in measure and by the σ-finitness of the measure space and since $\Phi > 0$, that $f_{n_k} \to 0$ μ-a.e. in T. Taking any $\lambda > 0$, it follows that $0 \leqslant \lambda f_{n_k} \leqslant \lambda f$ and $I_\Phi(\lambda f) < \infty$. So, by the Lebesgue dominated convergence theorem, $I_\Phi(\lambda f_{n_k}) \to 0$. Since $\lambda > 0$ is arbitrary we have $\|f_{n_k}\|_\Phi^A \to 0$, a contradiction.

Consequently, there exists a number δ mentioned above such that $I_\Phi(g) \geqslant \delta$ whence, by (14), we get $\|f-g\|_\Phi^A \leqslant 1-\delta$, which finishes the proof.

THEOREM 2.3. *Let Φ be a Musielak–Orlicz function satisfying the (∞)-condition and consider the Amemiya norm. For the space E^Φ (in the case of a nonatomic measure) let us assume additionally that $\Phi < \infty$. Then the following assertions are equivalent*:

(i) *L^Φ is UM,*

(ii) *L^Φ is LLUM,*

(iii) *L^Φ is ULUM,*

(iv) *E^Φ is ULUM,*

(v) (a) $\Phi > 0$ *and* (b) $\Phi \in \Delta_2$.

The same results hold for the counting measure space with the spaces l^Φ and h^Φ instead of L^Φ and E^Φ respectively, and with $\Phi \in \delta_2^0$ instead of $\Phi \in \Delta_2$.

Proof. We restrict ourselves to the case of a nonatomic measure space because the proof for the counting measure is essentially the same. The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious. By Theorem 2.2. and the fact that ULUM implies STM we know that (iv) $\Rightarrow \Phi > 0$. Therefore, to prove that (iv) $\Rightarrow$ (v) we only need to show that (iv) $\Rightarrow \Phi \in \Delta_2$.

Assume for a moment that $\Phi \notin \Delta_2$ and $\Phi < \infty$. By Proposition 2.1, it follows that there exists a set $A \subset T$ of positive finite measure with $\mu(T \backslash A) > 0$ and a sequence (f_n) in E^Φ with pairwise disjoint supports in $T \backslash A$ such that $I_\Phi(f_n) \leqslant 2^{-n}$ and $\|f_n\|_\Phi^A \geqslant \|f_n\|_\Phi \geqslant 1/1+2^{-n} \geqslant 2/3$. Let f be a function in E^Φ such that $\operatorname{supp}(f) \subset A$ and $\|f\|_\Phi^A = 1$. Let $k_0 > 1$ be such that $\|f\|_\Phi^A = (1/k_0)(1 + I_\Phi(k_0 f)$. Defining $g_n = (1/k_0)\, f_n$, we have $I_\Phi(k_0 g_n) \leqslant 2^{-n}$ and $\|g_n\|_\Phi^A \geqslant 2/3k_0$. Hence $\|f+g_n\|_\phi^A \leqslant (1/k_0)(1 + I_\phi(k_0(f+g_n))) = (1/k_0)(1 + I_\Phi(k_0 f)) + (1/k_0)\, I_\Phi(k_0 g_n) \leqslant 1 + 2^{-n}$.

Combining this with $\|g_n\|_\Phi^A \geqslant 2/3k_0$ ($n \in \mathsf{N}$), we conclude that E^Φ is not a ULUM space. This finishes the proof of the implication (iv) ⇒ (v).

Now we will prove that (v) ⇒ (i). Assume that $f, g \in L^\Phi$ (resp. l^Φ), $0 \leqslant g \leqslant f$, $\|f\|_\Phi^A = 1$ and $\|g\|_\Phi^A \geqslant \varepsilon > 0$. By Lemma 1.1, $I_\Phi(g) \geqslant \sigma(\varepsilon) > 0$. Let $k > 0$ be such that

$$1 = \|f\|_\Phi^A = \frac{1}{k}(1 + I_\Phi(kf)). \tag{15}$$

Proceeding as in the proof of the implication (iv) ⇒ (i) in Theorem 2.2 it follows, by $k > 1$, that

$$\begin{aligned} \|f - g\|_\Phi^1 &\leqslant \frac{1}{k}(1 + I_\Phi(kf)) - \frac{I}{k} I_\Phi(kg) \\ &= \|f\|_\Phi^A - \frac{1}{k} I_\Phi(kg) \\ &\leqslant \|f\|_\Phi^A - I_\Phi(g) \\ &\leqslant \|f\|_\Phi^A - \sigma(\varepsilon), \end{aligned}$$

which means that L^Φ is UM. To finish the proof of the theorem it is enough to show that (ii) implies that $\Phi \in \Delta_2$. Assume that $\Phi \notin \Delta_2$ and take an arbitrary $\varepsilon > 0$. Let (f_n) be the sequence from Proposition 2.1. Defining $g_n = f_n / \|f_n\|_\Phi^A$ we get $\|g_n\|_\Phi^A = 1$ and $\|\sum_{n=1}^\infty g_n\|_\Phi^A \leqslant (1+\varepsilon)^2$.

Denote $h = \sum_{n=1}^\infty g_n$ and $h_n = h - g_n$. Since $\sum_{n=1}^k x^*(g_n) \leqslant x^*(h)$ for every positive functional $x^* \in (L^\Phi)^*$ and each $k \in \mathsf{N}$, we conclude that the series $\sum_{n=1}^\infty x^*(g_n)$ is convergent. Since every $x^* \in (L^\Phi)^*$ can be written as a difference of two positive elements from $(L^\Phi)^*$, we conclude that $\sum_{n=1}^\infty x^*(g_n)$ is convergent for every $x^* \in (L^\Phi)^*$. Therefore $x^*(g_n) \to 0$ as $n \to \infty$, whence $x^*(h - g_n) \to x^*(h)$ for every $x^* \in (L^\Phi)^*$. Now $0 \leqslant h - g_n \leqslant h$ so $\limsup_{n\to\infty} \|h - g_n\|_\Phi^A \leqslant \|h\|_\Phi^A$. On the other hand the norm $\|\cdot\|_\Phi^A$ is a lower semicontinuous function in the weak topology, therefore $\|h\|_\Phi^A \leqslant \liminf_{n\to\infty} \|h - g_n\|_\Phi^A$. This yields $\|h - g_n\|_\Phi^A \to \|h\|_\Phi^A$. However, $\|h - (h - g_n)\|_\Phi^A = \|g_n\|_\Phi^A = 1$ for all $n \in \mathsf{N}$, which means that L^Φ is not LLUM whenever $\Phi \notin \Delta_2$, finishing the proof.

Monotonicity properties of a Musielak–Orlicz space and its subspace of order continuous elements equipped with the Luxemburg norm were characterized in [23] for a nonatomic measure and in [22] for the counting measure. Since the ULUM property was omitted we will fill this gap.

THEOREM 2.4. *Let Φ be an arbitrary Musielak–Orlicz function and, in the case of a nonatomic measure, assume that $\Phi < \infty$ when considering E^Φ. Then, for the Luxemburg norm, the following assertions are equivalent*:

(i) L^{Φ} *(resp.* l^{Φ}*) is ULUM,*

(ii) E^{Φ} *(resp.* h^{Φ}*) is ULUM,*

(iii) (a) $\Phi > 0$ *and* (b) $\Phi \in \Delta_2$ *(resp.* $\Phi \in \delta_2^0$*).*

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. In [23] and [22] it has been proved that (iii) implies that L^{Φ} (resp. l^{Φ}) is UM which implies ULUM so that (i) follows. Also, (ii) implies that E^{Φ} is STM and hence that $\Phi > 0$ ([22, 23]). Therefore, to finish the proof we only need to prove that if E^{Φ} (resp. h^{Φ}) is ULUM, then $\Phi \in \Delta_2$ (resp. $\Phi \in \delta_2^0$). We will present the proof for h^{Φ} only. Assuming that $\Phi \notin \delta_2^0$, by Proposition 2.1, there is a sequence (f_n) of positive elements in h^{Φ} with pairwise disjoint supports such that $I_{\Phi}(f_n) \leqslant 2^{-n}$, $\|f_n\|_{\Phi} \geqslant 1/1+\varepsilon$ and $\mathsf{N} \setminus \bigcup_{n=1}^{\infty} \operatorname{supp}(f_n) \neq \emptyset$. Let $f \in h^{\Phi}$, $f \geqslant 0$ be such that $\|f\|_{\Phi} = 1$, $\operatorname{supp}(f) \subset \mathsf{N} \setminus \bigcup_{n=1}^{\infty} \operatorname{supp}(f_n)$. Then $1 \leqslant I_{\Phi}(f+f_n) \leqslant 1 + 2^{-n}$, whence $1 \leqslant \|f + f_n\|_{\Phi} \leqslant 1 + 2^{-n} \to 1 = \|f\|_{\Phi}$. Since $0 \leqslant f \leqslant f + f_n$ and $\|(f+f_n) - f\|_{\Phi} = \|f_n\|_{\Phi} \geqslant 1/1+\varepsilon$ for all $n \in \mathsf{N}$, we conclude that h^{Φ} is not ULUM, which completes the proof.

3. SOME REMARKS

In our criteria for the monotonicity properties of Musielak–Orlicz spaces equipped with the Amemiya norm the (∞)-condition for Φ plays an important role. Under this assumption it was possible to prove the necessity of the condition $\Phi > 0$ for STM of E^{Φ} (resp. h^{Φ}). It is natural to ask for criteria for the monotonicity properties in absence of the (∞)-condition. As it will be seen below the condition $\Phi > 0$ is sufficient for L^{Φ} (resp. l^{Φ}) under the Amemiya norm to be STM but in general it is not necessary. However, for Orlicz sequence spaces as well as for Orlicz function spaces over a nonatomic infinite measure space the condition $\Phi > 0$ is necessary for L^{Φ} (resp. l^{Φ}) with the Amemiya norm to be STM. We will give an example of a STM Orlicz function space corresponding to a finite nonatomic measure space and generated by an Orlicz function not satisfying $\Phi > 0$. Unfortunately, we do not know criteria for the monotonicity properties (even for STM only) for Musielak–Orlicz spaces with an arbitrary Musielak–Orlicz function Φ (i.e. without the additional assumption that Φ satisfies the (∞)-condition) when the space is equipped with the Amemiya norm.

Remark 3.1. Let Φ be a Musielak–Orlicz function such that $\Phi > 0$. Then L^{Φ} (resp. l^{Φ}) equipped with the Amemiya norm is STM.

Proof. We note that the (∞)-condition was not assumed. Let $\Phi > 0$ and let $0 \leqslant g \leqslant f$, $f, g \in L^{\Phi}$, $g \neq 0$ and $\|f\|_{\Phi}^{A} = 1$. Then $\|f\|_{\Phi}^{A} = \inf_{k \geqslant 1} (1/k)(1 + I_{\Phi}(kf))$ and $(1/k)\, I_{\Phi}(kg) \geqslant I_{\Phi}(g)$ for all $k \geqslant 1$. It follows in the same way as in the proof of Theorem 2.2 that

$$\|f-g\|_{\Phi}^{1} \leqslant \frac{1}{k}(1+I_{\Phi}(kf)) - \frac{1}{k} I_{\Phi}(kg)$$

$$\leqslant \frac{1}{k}(1+I_{\Phi}(kf)) - I_{\Phi}(g)$$

for all $k \geqslant 1$, whence

$$\|f-g\|_{\Phi}^{A} \leqslant \|f\|_{\Phi}^{A} - I_{\Phi}(g) < \|f\|_{\Phi}^{A},$$

i.e., L^{Φ} is STM. The proof of STM for l^{Φ} is the same.

Remark 3.2. In the case of an Orlicz sequence space l^{Φ} as well as in the case of an Orlicz function space L^{Φ} over a nonatomic infinite measure space considered with the Amemiya norm the condition $\Phi > 0$ is necessary for STM of L^{Φ} (resp. l^{Φ}) even if Φ does not satisfy the (∞)-condition.

Proof. Assume that $\Phi > 0$ is not satisfied, i.e. $a(\Phi) = \sup\{u \geqslant 0 : \Phi(u) = 0\} > 0$, and define $x = a(\Phi)\, 1_A$, where A is chosen so that $\mu(A) = \mu(T \setminus A) = \infty$ (resp. $\mathrm{Card}(A) = \mathrm{Card}(\mathsf{N} \setminus A) = \infty$). Then $1 + I_{\Phi}(x) = 1$ since $\Phi(a(\Phi)) = 0$. On the other hand $(1/k)(1 + I_{\Phi}(kx)) > 1$ for every $k \in (0, 1)$ and $I_{\Phi}(kx) = \infty = (1/k)(1 + I_{\Phi}(kx))$ for every $k > 1$. Therefore $\|x\|_{\Phi}^{A} = 1 + I_{\Phi}(x) = 1$. In the same way we can prove that for $y = a(\Phi)\, 1_T$ (resp. $y = a(\Phi)\, 1_N$) we also have that $\|y\|_{\Phi}^{A} = 1$. Since $0 \leqslant x \leqslant y$ and $x \neq y$ this means that L^{Φ} (resp. l^{Φ}) is not STM.

The example presented below shows that the condition $\Phi > 0$ need not be necessary in general for L^{Φ} equipped with the Amemiya norm to be STM.

EXAMPLE 3.3. Consider the Lebesgue measure space on $[0, \gamma)$ with $0 < \gamma \leqslant \infty$ and the Orlicz function $\Phi(u) = \max(0, |u| - 1)$. It is known (see [15]) that $L^{\Phi} = L^1 + L^{\infty}$ and $\|x\|_{\Phi}^{0} = \|x\|_{\Phi}^{A} = \inf\{\|w\|_1 + \|z\|_{\infty} : w \in L^1, z \in L^{\infty}, x = w + z\} = \|x\|_{L^1 + L^{\infty}}$, i.e., both the Orlicz norm $\|x\|_{\Phi}^{0}$ and the Amemiya norm $\|x\|_{\Phi}^{A}$ coincide with the natural norm $\|x\|_{L^1 + L^{\infty}}$ considered in $L^1 = L^{\infty}$. On the other hand, we know (see [20]) that $\|x\|_{L^1 + L^{\infty}} = \int_0^1 x^*(t)\, dt$, where x^* is the nonincreasing rearrangement of x. Hence $\|x\|_{\Phi}^{A} = \int_0^1 x^*(t)\, dt$. If $\gamma \leqslant 1$, then $\int_0^1 x^*(t)\, dt = \int_0^{\gamma} x^*(t)\, dt = \int_0^{\gamma} |x(t)|\, dt$, so $L^1 + L^{\infty} = L^1$ and $= \|x\|_{L^1 + L^{\infty}} = \|x\|_1$ for every $x \in L^1 + L^{\infty} = L^{\phi}$ and consequently L^{Φ} with the Amemiya norm is STM (even UM).

Assume now that $\gamma > 1$. It is evident that the characteristic functions $x = 1_{[0, 1)}$, $y = 1_{[0, \gamma)}$ have the norms in $L^1 + L\infty$ equal to one, so for $\gamma > 1$ the space L^{Φ} is not STM.

4. LLUM, ULUM AND THE DOMINATED BEST APPROXIMATION PROBLEMS

Let $K \subset X$ be a closed sublattice and let $f \in X$ be arbitrary but fixed and such that $f \geqslant K$. As it was stated in Section 1 the case $f \leqslant K$ is also included. Before we apply the LLUM and ULUM properties to dominated best approximation we will prove some auxiliary results which are also of independent interest.

We say a Banach lattice X has the H^+ STM property (cf. [23]) if X has the H^+ property (see Section 1) and X is STM. From the definitions it follows immediately that LLUM implies H^+STM. In view of Theorem 4.1 below the H^+STM spaces, CWLLUM spaces and the STM spaces with order continuous norm coincide.

In this section Banach lattices are assumed to be σ-(Dedekind) complete.

THEOREM 4.1. *For any Banach lattice X the following statements are equivalent*:

(i) *X is CWLLUM.*

(ii) *X is H^+STM.*

(iii) *X is STM and order continuous.*

Proof. (i) ⇒ (ii). This follows from the definitions. (ii) ⇒ (iii). It is well known (cf. [25]) that if X is not order continuous there exists a sequence (y_n) of disjoint, positive, elements in X which is equivalent to the unit vector basis (e_n) in c_0 and a vector $y \in X$ such that $y_n \leqslant y$. Let $i(e_n) = y_n$ be the respective isomorphism ([25]). Given $l \in X^*$, we have $l(y_n) = (l \circ i)(e_n)$. Clearly $l \circ i \in c_0^*$ and therefore $(l \circ i)(e_n) \to 0$ since (e_n) is weakly convergent to zero. In view of the weak lower semi-continuity of the norm it follows that $\|y\| \leqslant \liminf_{n \to \infty} \|y - y_n\| \leqslant \limsup_{n \to \infty} \|y - y_n\| \leqslant \|y\|$. Hence $\|y - y_n\| \to \|y\|$, where we can assume without loss of generality that $\|y\| = 1$. Collecting the above facts we conclude that $\|y - y_n\| \to 1$, $y - y_n \to y$ weakly and $0 \leqslant y - y_n \leqslant y$, so in virtue of (i) $\|y - (y - y_n)\| = \|y_n\| \to 0$, a contradiction because on the other hand $\|y_n\| = \|i(e_n)\| \geqslant m\|e_n\|_\infty = m > 0$.

(iii) ⇒ (ii). We apply the well-known fact (cf. [25]) that a Banach lattice X has an order continuous norm $\|\cdot\|$ if and only if there exists an equivalent lattice norm $\|\cdot\|_1$ such that if $x_n \to x$ weakly and $\|x_n\|_1 \to \|x\|_1$ then $\|x - x_n\|_1 \to 0$. Let $0 \leqslant x_n \leqslant x$ and $x_n \to x$ weakly. Then $\|x_n\|_1 \to \|x\|_1$. Indeed, it follow by the weak lower semi-continuity and the monotonicity of the norm $\|\cdot\|_1$ that $\|x\|_1 \leqslant \liminf_{n \to \infty} \|x_n\|_1 \leqslant \limsup_{n \to \infty} \|x_n\|_1 \leqslant \|x\|_1$. Therefore, $\|x - x_n\|_1 \to 0$ and consequently $\|x - x_n\| \to 0$ since the norms under consideration are equivalent.

(iii) ⇒ (i). If not, there exists a non-negative functional x^* with $\|x^*\| = 1$, a non-negative x with $\|x\| = 1$ and a sequence (y_n) satisfying

$0 \leqslant y_n \leqslant x$ such that $x^*(x - y_n) \geqslant 1 - 1/2^n$ and $\|y_n\| \geqslant \alpha > 0$. Define $x_n = \bigvee_{k=n}^{\infty} y_k$ and $x_0 = \bigwedge_{n=1}^{\infty} x_n$. Then $0 \leqslant x^*(y_n) \leqslant 1/2^n$ since $1 - x^*(y_n) \geqslant x^*(x - y_n) \geqslant 1 - 1/2^n$. Moreover,

$$x^*(x_n) = x^*\left(\bigvee_{k=n}^{\infty} y_k\right) \leqslant \sum_{k=n}^{\infty} x^*(y_k) \leqslant \frac{1}{2^{n-1}}. \tag{16}$$

Therefore $1 \geqslant \|x - x_n\| \geqslant x^*(x - x_n) \geqslant x^*(x - y_n) - x^*(x_n) + x^*(y_n) \geqslant 1 - 1/2^n - 1/2^{n-1}$ and so $\|x - x_n\| \to 1$. Next, $x_n \downarrow$, $0 \leqslant y_n \leqslant x_n \leqslant x$ and $0 \leqslant x_0 = \inf_n x_n \leqslant x_n \leqslant x$ since X is σ-(Dedekind) complete by the assumption. Moreover, by the order continuity of X, x_n converges in the norm to x_0. Thus, $1 \geqslant \|x - x_0\| \geqslant \|x - x_n\|$ and $\|x - x_n\| \to 1$. Therefore, $1 = \|x\| = \|x - x_0\|$ and by the STM of X it must be $x_0 = 0$. Finally, by $\inf_n x_n = x_0 = 0$, we obtain that $0 < \alpha \leqslant \|y_n\| \leqslant \|x_n\| \to 0$, a contradiction, and the proof is finished.

The following theorem is an extension of Theorem 3.4 of [23] to the case of general Banach lattices.

THEOREM 4.2. *A banach lattice X is STM and order continuous (equivalently, CWLLUM space) if and only if for each closed sublattice K of X and every $f \in X$ such that $K \leqslant f$, the dominated best approximation problem is uniquely solvable.*

Proof. The proof of the necessity is the same as in the proof of the first implication of Proposition 3.3 in [23] and therefore is omitted. The proof of the sufficiency proceeds in the same way as the proof of the second implication of Proposition 3.3 in [23]. However, to prove the closedness of the sublattice K, we apply the Dini theorem for normed lattices. We use the fact that if (f_α) is downward directed net which is weakly convergent to f then $f = \inf_\alpha f_\alpha$.

Since LLUM implies CWLLUM, from the results above we obtain the following application of the LLUM property in the dominated best approximation.

THEOREM 4.3. *If a Banach lattice X is LLUM then for each closed sublattice K of X and every $f \in X$ such that $K \leqslant f$, the dominated best approximation problem is uniquely solvable.*

Finally, we give an application of the ULUM property in the dominated best approximation.

THEOREM 4.4. *Let X be an order continuous Banach lattice with the ULUM property. Then for every closed sublattice K of X and for every $f \in X$, the dominated best approximation problem is strongly solvable.*

Proof. Assume that X is an order continuous Banach lattice with the ULUM property. The order continuity implies that the problem is solvable and thanks to the ULUM (STM in fact) the problem is uniquely solvable (cf. Theorem 4.2), i.e., there is a unique $y_0 \in K$ such that $\|f - y_0\| = \inf_{y \in K} \|f - y\|$, i.e., $P_K(f) = \{y_0\}$.

Let (y_n) be a minimizing sequence. Since K is a sublattice, $u_n = \bigvee_{k=1}^{n} y_k \in K$. Moreover, since X is σ-complete, $u_n \leqslant u = \bigvee_{n=1}^{\infty} u_n \leqslant f$. Hence $0 \leqslant u - u_n \downarrow 0$ and since X is order continuous $\|u - u_n\| \to 0$ so that $u \in K$, since K is norm closed; Now,

$$\alpha = \inf_{y \in K} \|f - y\| \leftarrow \|f - y_n\| \geqslant \|f - u_n\| \geqslant \|f - u\| \geqslant \|f - y_0\| = \alpha,$$

whence $u \in P_K(f)$. Now the uniqueness of the solvability implies that $u = y_0$. Next,

$$\|(f - y_0) + (y_0 - y_n)\| \to \|f - y_0\| = \alpha,$$

where $y_n \leqslant y_0 = u \leqslant f$. Now, by the ULUM of X this yields $\|y_0 - y_n\| \to 0$, which means that the problem of the dominated b.a. is stable. In virtue of the uniqueness of the solvability, we conclude that the dominated best approximation problem under ULUM is strongly solvable.

As a corollary to Theorems 2.3 and 3.4 in [23], in virtue of the fact that Musielak–Orlicz spaces are σ-(Dedekind) complete Banach lattices for both kind of norms, we obtain the following result.

THEOREM 4.5. *In Musielak–Orlicz spaces L^{Φ} (resp. l^{Φ}) endowed with the Luxemburg or the Amemiya norm (for this norm we assume that the (∞)-condition is satisfied) the problem of the dominated best approximation with respect to closed sublattices is always strongly solvable.*

Remark. When we consider Theorem 4.4 for the Musielak–Orlicz spaces or for their subspaces of order continuous elements for the Luxemburg norm as well for the Amemiya norm we need not assume order continuity. Indeed, it follows automatically from the ULUM property for the Musielak–Orlicz space (Theorems 2.3 and 2.4).

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