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On Subgroups of the Special Linear Group Containing the Special Orthogonal Group

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INTRODUCTION

Let V be an n-dimensional vector space over a field K of characteristic not 2 and as usual let $GL_n(K)$ and $SL_n(K)$ be the general and special linear groups of V. Let Q be a quadratic form of Witt index $v \geqslant 1$ on V whose associated symmetric bilinear form, given by

$$B(\mathbf{x}, \mathbf{y}) = Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y}), \qquad \forall \mathbf{x}, \mathbf{y} \in V,$$

is non-degenerate, and let $O_n(K)$, $SO_n(K)$ and $GO_n(K)$ be the orthogonal, special orthogonal and general orthogonal groups of Q.

In [5], Dye showed that if n is even and if $\tilde{K}$ is a field of characteristic 2, then considered as a subgroup of the symplectic group $Sp_n(\tilde{K})$, $O_n(\tilde{K})$ is maximal in $Sp_n(\tilde{K})$ if and only if $\tilde{K}$ is perfect. In [6], he proved the maximality in $SL_n(K)$ of $SL_n(K) \cap GSp_n(K)$ (for $n \geqslant 4$); he denoted the latter group by $SGSp_n(K)$. In this paper we consider a situation that may be considered analogous to both of these results. Ideally one would like to prove the maximality in $SL_n(K)$ of $SO_n(K)$. However, $SO_n(K)$ is usually properly contained in its normaliser in $SL_n(K)$; the normaliser is $GO_n(K) \cap SL_n(K)$ which we denote by $SGO_n(K)$ (adapting Dye's notation) and call the special general orthogonal group of Q. It will follow from Lemma 1 that $SGO_n(K)$ is the stabilizer in $SL_n(K)$ of the set of singular 1-dimensional subspaces of V. In Sections 2 and 3 we prove the theorems stated below. In Section 4 we give conditions for $GO_n(K)$ to be maximal in $GL_n(K)$ and for $SO_n(K)$ to be maximal in $SL_n(K)$ (Theorems III and V).

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We also consider the projective groups $PSGO_n(K)$ and $PSO_n(K)$ and give conditions for them to be maximal in $PSL_n(K)$.

THEOREM I. *If* $n \geqslant 3$, *then any proper subgroup of* $SL_n(K)$ *containing* $SO_n(K)$ *lies in* $SGO_n(K)$.

A Corollary to this theorem is that $SGO_n(K)$ is maximal in $SL_n(K)$ when $n \geqslant 3$. We are then also able to determine the subgroups of $SL_n(K)$ containing $SO_n(K)$. Unfortunately, the theorem cannot be extended to the case $n=2$ as $SO_2(K)$ stabilizes each of two 1-dimensional subspaces; there thereby arise two reducible subgroups containing $SO_2(K)$ that don't lie in $SGO_2(K)$. Although for finite fields of order >11 it is clear from [2] or [12] that any proper subgroup of $SL_2(K)$ containing $SO_2(K)$ lies either in $SGO_2(K)$ or one of the given reducible subgroups, it is not clear that this can be extended to infinite fields. However, in most cases, we can still prove the maximality of $SGO_2(K)$ in $SL_2(K)$.

THEOREM II. *If* $n=2$, *then* $SGO_n(K)$ *is maximal in* $SL_n(K)$, *except when* $K=GF(q)$ *with* $q \leqslant 11$.

Dye comments in [5] that his result there is unusual in that it is "geometric" but not true for all fields of characteristic 2. In contrast, when the characteristic is not 2 there are only exceptions for very small fields. One reason for this difference is that any element of a field of characteristic not 2 may be expressed as the difference between two squares, whereas in the characteristic 2 case the same may only be said of perfect fields.

Our approach is geometric in nature, although there are differences between the cases $n \geqslant 3$ and $n=2$. We show that any subgroup of $SL_n(K)$ properly containing $SO_n(K)$ but not lying in $SGO_n(K)$ (properly containing $SGO_n(K)$ if $n=2$) contains a generating set of transvections for $SL_n(K)$. In the proof of Theorem II we use the known maximality of $SGO_2(K)$ in $SL_2(K)$ for finite K (see Result 1). The maximality of $SGO_3(K)$ in $SL_3(K)$ is also known for finite K (see Result 2), although the case $K=GF(3)$ is the only one that we assume.

1. FURTHER NOTATION AND PRELIMINARY RESULTS

Our notation mostly follows [4]. We note only that the conjugate of a subspace U will be written U' and that when U is non-isotropic and $E_n(K)$ is a subgroup of $GO_n(K)$, the subgroup of $E_n(K)$ consisting of those elements that fix each vector in U' will be denoted by $E(U)$.

The following results are stated in terms of our notation; we follow standard practice in writing, for example, $SL_n(K)=SL_n(q)$ when $K=GF(q)$.

Result 1 (Dickson [2]). If $K = GF(q)$, then $SGO_2(q)$ is maximal in $SL_2(q)$ except when $q \leqslant 11$.

Dickson actually lists the subgroups of $PSL_2(q)$ (rather than those of $SL_2(q)$) and the exceptional cases are more neatly described in this form; that the exceptional cases may be considered by reference to $PSL_2(q)$ follows from the fact that $SGO_2(K)$ contains the centre of $SL_2(K)$. It may be seen from Dickson's list that

$$PSGO_2(3) < V_4 < PSL_2(3),$$

$$PSGO_2(5) < A_4 < PSL_2(5),$$

$$PSGO_2(7) < S_4 < PSL_2(7),$$

$$PSGO_2(9) < S_4 < PSL_2(9),$$

$$PSGO_2(11) < A_5 < PSL_2(11),$$

where V_4 is the four group and A_4, A_5 and S_4 are alternating and symmetric groups. In each case, the given group is maximal in $PSL_2(q)$ and contains $PSGO_2(q)$ as a maximal subgroup.

Result 2 (Mitchell [9]). If $K = GF(q)$, then $SGO_3(q)$ is maximal in $SL_3(q)$.

A transvection in $SL_n(K)$ is a map of the form

$$: \mathbf{v} \mapsto \mathbf{v} + \rho(\mathbf{v}) \cdot \mathbf{x},$$

where $\mathbf{x}$ is a non-zero vector in V and ρ is a linear form on V with $\rho(\mathbf{x}) = 0$; it is said to be centred on $\mathbf{x}$ and to have axis $\rho^{-1}(0)$. For each pair of subspaces $P \subseteq H$ of dimension 1 and $n-1$, respectively, the subgroup of $SL_n(K)$ generated by all transvections with $\mathbf{x} \in P$ and $\rho^{-1}(0) = H$ will be denoted by $X(P, H)$; this subgroup is sometimes known as a subgroup of root type. If a group generated by transvections contains $X(P, H)$, then P and H are said to be respectively a centre and an axis for that group. As McLaughlin pointed out in [8], the following result is true for any K, even though originally stated only for $GF(2)$.

Result 3 (McLaughlin [8]). If $\hat{F}$ is an irreducible subgroup of $SL_n(K)$ generated by subgroups of root type and if $X(P, H_1), X(P, H_2) \leqslant \hat{F}$ for some P and for distinct axes H_1 and H_2, then $\hat{F} = SL_n(K)$.

The general orthogonal group is defined by $GO_n(K) = \{g \in GL_n(K): Q(g\mathbf{x}) = \lambda_g Q(\mathbf{x}), \forall \mathbf{x} \in V\}$ where $\lambda_g \in K$ is dependent on g and is called the multiplicator of g. The set of all λ_g is a subgroup $M(Q)$ of the multiplicative group K^* of K. The elements in $GO_n(K)$ with multiplicator 1 form $O_n(K)$, and the elements in $O_n(K)$ with determinant 1 form $SO_n(K)$. As $GO_n(K)$ contains the centre of $GL_n(K)$, it follows that $(K^*)^2 \leqslant M(Q)$; if

n is odd, then $M(Q) = (K^*)^2$ (cf. [4, p. 77]). If n is even, then the structure of $M(Q)$ is not known in general, but is known in particular cases: if $K = \mathbb{C}$, then $M(Q) = K^*$; if $K = \mathbb{R}$, then $M(Q) = K^*$ when $\nu = n/2$ and $(K^*)^2$ otherwise; if K is finite, then $M(Q) = K^*$.

LEMMA 1. *$GO_n(K)$ is the stabilizer in $GL_n(K)$ of the set of singular 1-dimensional subspaces of V.*

Proof. We need only show that if $g \in GL_n(K)$ stabilizes the set of singular 1-dimensional subspaces, then $g \in GO_n(K)$. Let $\mathbf{a}, \mathbf{b} \in V$ be singular vectors such that $B(\mathbf{a}, \mathbf{b}) = 1$. As $g(\mathbf{a} + \mathbf{b})$ must be non-singular, $\langle g(\mathbf{a}), g(\mathbf{b}) \rangle$ is hyperbolic and, multiplying g by an appropriate element of $O_n(K)$ if necessary, we may assume that $g(\mathbf{a}) = \mathbf{a}$ and $g(\mathbf{b}) = \lambda \mathbf{b}$ for some $\lambda \in K^*$; thus $Q(g(\mathbf{v})) = \lambda Q(\mathbf{v})$ for all $\mathbf{v} \in \langle \mathbf{a}, \mathbf{b} \rangle$. For $\mathbf{c} \in \langle \mathbf{a}, \mathbf{b} \rangle'$, neither $\langle g(\mathbf{c}), \mathbf{a} \rangle$ nor $\langle g(\mathbf{c}), \mathbf{b} \rangle$ can be hyperbolic, so $g(\mathbf{c}) \in \langle \mathbf{a}, \mathbf{b} \rangle'$. Now $\mathbf{c} + \mathbf{a} - Q(\mathbf{c}) \cdot \mathbf{b}$ is singular, so $Q(g(\mathbf{c})) = \lambda Q(\mathbf{c})$. Hence $g \in GO_n(K)$ with multiplicator λ. ∎

Let us now write $G = SGO_n(K)$ and $G_0 = SO_n(K)$ and let $F \leqslant SL_n(K)$ such that $G_0 < F$ but $F \nleqslant G$ if $n \geqslant 3$ and $G < F$ if $n = 2$; we show that $F = SL_n(K)$. As G does not act transitively on the 1-dimensional subspaces of V, it is clear that $G \neq SL_n(K)$.

2. THE CASE $n \geqslant 3$

We assume throughout this section that $n \geqslant 3$.

PROPOSITION 1. *There exists $f \in F \backslash (F \cap G)$ and a non-zero singular vector $\mathbf{x} \in V$ such that $f(\mathbf{x}) = \mathbf{x}$.*

Proof. We begin by proving the statement of the proposition when $n = 3$ and use it for $n \geqslant 4$. As Witt's theorem (cf. [1, p. 71]) may be amended to show that G_0 acts transitively on the non-zero singular vectors, it will suffice to find f and $\mathbf{x}$ such that $f(\mathbf{x})$ is singular.

Suppose that $n = 3$ and let $h \in F \backslash (F \cap G)$; then h does not normalise G_0. Let ι be the central element of $O_3(K)$ taking $\mathbf{v}$ to $-\mathbf{v}$ for all $\mathbf{v} \in V$, then as $O_3(K)$ is generated by its symmetries (cf. [3]), $\{\iota\sigma : \sigma \text{ a symmetry}\}$ is a generating set for G_0. Thus for some symmetry σ, $\iota h^{-1} \sigma h \notin G_0$; moreover $\iota h^{-1} \sigma h \notin G$ because otherwise the fixed space of $h^{-1} \sigma h$, having dimension 2, would contain a non-singular vector implying that $h^{-1} \sigma h$ and therefore $\iota h^{-1} \sigma h$ has multiplicator 1, i.e., that $\iota h^{-1} \sigma h \in G_0$, a contradiction. Let W be the fixed space of $h^{-1} \sigma h$. If W contains a non-zero singular vector, then we may take $f = \iota h^{-1} \sigma h$. Otherwise W is anisotropic and hence non-isotropic.

Let σ_1 be the symmetry centred on W' and let $h_1 = \sigma_1 h^{-1} \sigma h \in F\backslash(F \cap G)$, then W is the fixed space of h_1 which must therefore be a transvection centred on a vector in W. Let $\mathbf{v} \in W' \backslash \{\mathbf{0}\}$, write $h_1(\mathbf{v}) = \mathbf{v} + \mathbf{w}$ where $\mathbf{w} \in W$ and let $\mathbf{u} \in \langle \mathbf{w} \rangle' \cap W \backslash \{\mathbf{0}\}$; then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is an orthogonal base for V. Let $\mathbf{u}_0 \in \langle \mathbf{u}, \mathbf{v} \rangle \backslash \langle \mathbf{v} \rangle$ such that $\mathbf{u}_0 + \mathbf{w}$ is singular; except when $K = GF(3)$ and $Q(\mathbf{w}) = Q(\mathbf{u}) = -Q(\mathbf{v})$ (in which case, as $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} + 2\mathbf{w} = h_1(\mathbf{v} + \mathbf{w})$ are both singular, we may take $f = h_1$ and $x = \mathbf{v} + \mathbf{w}$) $\mathbf{u}_0$ exists because either $\langle \mathbf{u}, \mathbf{v} \rangle$ is hyperbolic and therefore contains a vector $\hat{\mathbf{u}}$ with $Q(\hat{\mathbf{u}}) = -Q(\mathbf{w})$ or $\langle \mathbf{u}, \mathbf{v} \rangle$ is anisotropic in which case $\nu \geqslant 1$ implies that $\langle \mathbf{u}, \mathbf{v} \rangle$ contains a vector $\hat{\mathbf{u}}$ with $Q(\hat{\mathbf{u}}) = -Q(\mathbf{w})$, and except when $K = GF(3)$ and $\langle \mathbf{u}, \mathbf{v} \rangle$ is hyperbolic (where if $Q(\mathbf{w}) = -Q(\mathbf{u})$ we may take $\mathbf{u}_0 = \mathbf{u}$) the irreducibility of the action of $O(\langle \mathbf{u}, \mathbf{v} \rangle)$ on $\langle \mathbf{u}, \mathbf{v} \rangle$ ensures that there is a vector in $\langle \mathbf{u}, \mathbf{v} \rangle \backslash \langle \hat{\mathbf{u}} \rangle$ with the properties of $\hat{\mathbf{u}}$. Now let σ_0 be the symmetry centred on $\mathbf{u}_0$; then $h_1 \iota \sigma_0 h_1 \iota \sigma_0$ is a non-trivial transvection centred on $\mathbf{w}$ and having fixed space $\langle \mathbf{w}, \mathbf{u}_0 \rangle$. Thus we may take $f = h_1 \iota \sigma_0 h_1 \iota \sigma_0$ and $\mathbf{x} = \mathbf{w} + \mathbf{u}_0$.

Suppose now that $n \geqslant 4$ and let $h \in F\backslash(F \cap G)$; then h does not normalise G_0. As G_0 is generated by involutions with fixed space dimension $n-2$ (cf. [3]), there is such an element g for which $h^{-1} g h \notin G_0$. Any element of G whose fixed space is not totally singular fixes a non-singular vector, i.e., has multiplicator 1 and therefore lies in G_0; as the fixed space of $h^{-1} g h$ has dimension $n-2$, it can only be totally singular if $n - 2 \leqslant \nu \leqslant n/2$, i.e., $n = 4$ and $\nu = 2$, so with the one possible exception, $h^{-1} g h \notin G$. If $n = 4$ and $\nu = 2$, then a refinement of the argument is required: another generating set for G_0 is the set of hyperbolic rotations, i.e., elements whose fixed spaces are the conjugates of hyperbolic 2-dimensional subspaces (cf. [3]), which in this case implies that the fixed spaces are themselves hyperbolic 2-dimensional subspaces. Suppose that $h^{-1} G_0 h \leqslant G$ and that g_1 is a hyperbolic rotation with $h^{-1} g_1 h \notin G_0$, and let P be the fixed space of g_1; then $h^{-1} P$, the fixed space of $h^{-1} g_1 h$, must be totally singular. For any $g_2 \in G_0$, $h^{-1} g_2 h \in G$ implies that $h^{-1} g_2 h \cdot h^{-1} P = h^{-1} g_2 P$ is totally singular; as G_0 acts transitively on the hyperbolic 2-dimensional subspaces (from Witt's theorem) it follows that $h^{-1} P_2$ is totally singular for any hyperbolic 2-dimensional subspace P_2. But any vector lies in a hyperbolic 2-dimensional subspace (cf. [4]) implying that every vector of $h^{-1} V$ is singular, which is absurd. Hence $h^{-1} G_0 h \nleqslant G$ and we may choose an involution g as above with $h^{-1} g h \notin G$.

Let $h_1 = h^{-1} g h$ and let W be the fixed space of h_1; then $\dim W = n - 2$ and h_1 is an involution. If W contains a non-zero singular vector then we may take $\mathbf{x}$ to be such a vector and take $f = h_1$. Otherwise W is anisotropic, hence non-isotropic, and $V = W \oplus W'$. Let $\mathbf{u} \in W$, $\mathbf{v} \in W'$ be non-zero vectors such that $\mathbf{u} + \mathbf{v}$ is singular (such exist since not every singular vector can lie in W'); then $\mathbf{u}$ and $\mathbf{v}$ are non-isotropic, and $h_1(\mathbf{v}) = -\mathbf{v} + \mathbf{w}$ for some $\mathbf{w} \in W$ (as h_1 is an involution). Let $\iota \in G_0$ be the map with fixed space W taking $\mathbf{z}$ to $-\mathbf{z}$ for all $\mathbf{z} \in W'$, let $h_2 = \iota h_1$, let U_1 be a

2-dimensional subspace of W containing $\mathbf{u}$ and $\mathbf{w}$ (U_1 is necessarily non-isotropic) and let $U = U_1 + \langle \mathbf{v} \rangle$; then $\dim U = 3$, U is non-isotropic but not anisotropic, and $h_2 U = U$. Now consider the restriction $\hat{h}_2$ of h_2 to U; $\hat{h}_2$ fixes each vector of U_1 and takes $\mathbf{v}$ to $\mathbf{v} + \mathbf{w}$, and so has determinant 1. Let $SO_3(K)$ and $SGO_3(K)$ be respectively the special orthogonal and special general orthogonal groups of the restriction of Q to U. If $\hat{h}_2 \in SGO_3(K)$ then $\hat{h}_2(\mathbf{x})$ is singular for any singular vector $\mathbf{x} \in U$ so we may take $f = h_2$. Otherwise $SO_3(K) < \langle SO_3(K), \hat{h}_2 \rangle \nleq SGO_3(K)$ and we may apply the case $n = 3$ with $\langle SO_3(K), \hat{h}_2 \rangle$ in place of F, giving an element $\hat{f}$ of $\langle SO_3(K), \hat{h}_2 \rangle$ (with $\hat{f} \notin SGO_3(K)$) that fixes a non-zero singular vector of U. As $SO_3(K)$ may be identified with the subgroup $SO(U)$ of G_0, it follows that $\hat{f}$ is the restriction of some $f \in \langle SO(U), h_2 \rangle$ (with $f \notin G$), i.e., $f \in F \backslash (F \cap G)$ and f fixes a non-zero singular vector, as required.

PROPOSITION 2. *If $K \neq GF(3)$ then there is a transvection in F whose centre is a non-zero singular vector* $\mathbf{x}$ *and whose axis is* $\langle \mathbf{x} \rangle'$.

Proof. Let $\mathbf{x}$ be a non-zero singular vector for which there exists $f \in F \backslash (F \cap G)$ such that $f(\mathbf{x}) = \mathbf{x}$. Let $\tilde{G}_0 = \mathrm{Stab}_{G_0}\langle \mathbf{x} \rangle$, let $\tilde{G} = \mathrm{Stab}_G \langle \mathbf{x} \rangle$ and let $\tilde{F} = \mathrm{Stab}_F \langle \mathbf{x} \rangle$; then $\tilde{F} \nleq \tilde{G}$. We consider the orbits of $\tilde{F}$ acting on the 1-dimensional subspaces of V. The orbits of $\tilde{G}_0$ other than $\{\langle \mathbf{x} \rangle\}$ lie in two classes, $\mathscr{C}_1$ and $\mathscr{C}_2$, consisting respectively of those inside and those outside $\langle \mathbf{x} \rangle'$. By Witt's theorem there is one orbit Ω of singular 1-dimensional subspaces in $\mathscr{C}_2$ and one orbit of non-singular 1-dimensional subspaces for each element of $K^*/(K^*)^2$, i.e., if $\mathbf{u}$ and $\mathbf{v}$ are non-singular vectors outside $\langle \mathbf{x} \rangle'$, then $\langle \mathbf{u} \rangle$ and $\langle \mathbf{v} \rangle$ are in the same orbit of $\tilde{G}_0$ if and only if $Q(\mathbf{u})/Q(\mathbf{v})$ is a square in K; any hyperbolic 2-dimensional subspace containing $\mathbf{x}$ but not lying in $\langle \mathbf{x} \rangle'$ contains a representative of each orbit in $\mathscr{C}_2$. In $\mathscr{C}_1$ we need only note that if $\mathbf{v} \in \langle \mathbf{x} \rangle' \backslash \langle \mathbf{x} \rangle$, then $\langle \mathbf{v} \rangle$ and $\langle \mathbf{v} + \lambda \mathbf{x} \rangle$ are in the same orbit of $\tilde{G}_0$ for all $\lambda \in K$ and that if $\nu \geqslant 2$, then there is one orbit Δ of singular 1-dimensional subspaces, except when $n = 4$ in which case there are two, Δ_1 and Δ_2, corresponding to the totally singular 2-dimensional subspaces of V containing $\mathbf{x}$. We show that under $\tilde{F}$ the orbit Ω is joined to another orbit of $\mathscr{C}_2$.

Suppose first that $\tilde{F}$ does not fix $\langle \mathbf{x} \rangle'$, i.e., for some $h \in \tilde{F}$ and some $\mathbf{v} \in \langle \mathbf{x} \rangle' \backslash \langle \mathbf{x} \rangle$, $h(\mathbf{v}) \notin \langle \mathbf{x} \rangle'$; then $h\langle \mathbf{v}, \mathbf{x} \rangle = \langle h(\mathbf{v}), \mathbf{x} \rangle$ is hyperbolic. Thus the 1-dimensional subspaces $\langle \mathbf{v} + \lambda \mathbf{x} \rangle$ ($\lambda \in K$) lie in the same orbit of $\tilde{G}_0$, and $\{h\langle \mathbf{v} + \lambda \mathbf{x} \rangle : \lambda \in K\}$ is a subset of $\mathscr{C}_2$ containing a representative of each orbit of $\mathscr{C}_2$. Hence under $\tilde{F}$, the orbit of $\tilde{G}_0$ containing $\langle \mathbf{v} \rangle$ is joined to each orbit of $\mathscr{C}_2$, from which it follows that all the orbits in $\mathscr{C}_2$ are joined under $\tilde{F}$.

Suppose now that $\tilde{F}$ fixes $\langle \mathbf{x} \rangle'$; then $\tilde{F}$ fixes $\mathscr{C}_1$ and $\mathscr{C}_2$. If $\nu = 1$ then $\Omega \cup \{\langle \mathbf{x} \rangle\}$ is the set of all singular 1-dimensional subspaces of V and by

Lemma 1 cannot therefore be fixed by $\tilde{F}$, so Ω must be joined to some other orbit in $\mathscr{C}_2$. If $v \geqslant 2$ and $n > 4$ (resp. $v = 2$ and $n = 4$) and Δ (resp. $\Delta_1 \cup \Delta_1$) is fixed by $\tilde{F}$, then as $\Omega \cup \Delta \cup \{\langle \mathbf{x} \rangle\}$ (resp. $\Omega \cup \Delta_1 \cup \Delta_2 \cup \{\langle \mathbf{x} \rangle\}$) is the set of singular 1-dimensional subspaces of V, it follows that $\tilde{F}$ does not fix Ω and so joins Ω to some other orbit in $\mathscr{C}_2$. If $h \in \tilde{F}$ and $\langle \mathbf{v} \rangle \in \Delta$ (resp. $\langle \mathbf{v} \rangle \in \Delta_1 \cup \Delta_2$) such that $h\langle \mathbf{v} \rangle \notin \Delta$ (resp. $h\langle \mathbf{v} \rangle \notin \Delta_1 \cup \Delta_2$), then there is a singular vector $\mathbf{w}$ such that $B(\mathbf{x}, \mathbf{w}) \neq 0$ but $B(\mathbf{v}, \mathbf{w}) = 0$. All but one (i.e., at least three) of the 1-dimensional subspaces of $\langle \mathbf{v}, \mathbf{w} \rangle$ lie in Ω, but as $h(\mathbf{v})$ is non-singular, $h\langle \mathbf{v}, \mathbf{w} \rangle$ has at most two singular 1-dimensional subspaces, so h maps an element of Ω to a non-singular 1-dimensional subspace, i.e., $\tilde{F}$ joins Ω to another orbit in $\mathscr{C}_2$.

Let $f_1 \in \tilde{F}$ and let $\mathbf{y}$ be a singular vector outside $\langle \mathbf{x} \rangle'$ such that $f_1(\mathbf{y})$ is non-singular and outside $\langle \mathbf{x} \rangle'$. Then $\langle \mathbf{x}, \mathbf{y} \rangle$ and $\langle \mathbf{x}, f_1(\mathbf{y}) \rangle = f_1 \langle \mathbf{x}, \mathbf{y} \rangle$ are both hyperbolic, so by Witt's theorem there exists $g_1 \in G_0$ such that $g_1 f_1 \langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$; there further exists $g \in \mathrm{Stab}_{G_0} \langle \mathbf{x}, \mathbf{y} \rangle$ such that $g g_1 f_1(\mathbf{x}) = \mathbf{x}$. We can now write $g g_1 f_1(\mathbf{y}) = \alpha \mathbf{x} + \beta \mathbf{y}$ with $\alpha, \beta \neq 0$ (as $f_1(\mathbf{y})$ is non-singular). Let $\xi \in K \backslash \{0, 1, -1\}$, let $g_2 \in SO(\langle \mathbf{x}, \mathbf{y} \rangle)$ be the map $(\mathbf{x}, \mathbf{y}) \mapsto (\xi \mathbf{x}, \xi^{-1} \mathbf{y})$ and let $f_2 = g_2^{-1} (g g_1 f_1)^{-1} g_2 g g_1 f_1$; then $f_2(\mathbf{x}) = \mathbf{x}$, $f_2(\mathbf{y}) = \mathbf{y} + \xi^{-1}(\xi - \xi^{-1}) \alpha \mathbf{x}$ and for $\mathbf{z} \in \langle \mathbf{x}, \mathbf{y} \rangle'$, $f_2(\mathbf{z}) = \mathbf{z} + \gamma \mathbf{x} + \delta \mathbf{y}$ where $\gamma, \delta \in K$ depend on $\mathbf{z}$. Let $\iota \in SO(\langle \mathbf{x}, \mathbf{y} \rangle)$ be the map $(\mathbf{x}, \mathbf{y}) \mapsto (-\mathbf{x}, -\mathbf{y})$ and let $f_3 = \iota f_2 \iota f_2$; then $f_3(\mathbf{x}) = \mathbf{x}$, $f_3(\mathbf{y}) = \mathbf{y} + 2\xi^{-1}(\xi - \xi^{-1}) \alpha \mathbf{x}$ and for $\mathbf{z} \in \langle \mathbf{x}, \mathbf{y} \rangle'$, $f_3(\mathbf{z}) = \mathbf{z} + \eta \mathbf{x}$ where $\eta \in K$ depends on $\mathbf{z}$. Let $f_4 = \iota f_3 \iota f_3$; then $f_4(\mathbf{x}) = \mathbf{x}$, $f_4(\mathbf{y}) = \mathbf{y} + 4\xi^{-1}(\xi - \xi^{-1}) \alpha \mathbf{x}$ and $f_4(\mathbf{z}) = \mathbf{z}$ for all $\mathbf{z} \in \langle \mathbf{x}, \mathbf{y} \rangle'$. As $4\xi^{-1}(\xi - \xi^{-1}) \alpha \neq 0$, f_4 is a transvection whose centre is $\mathbf{x}$ and whose axis is $\langle \mathbf{x} \rangle'$. ∎

PROPOSITION 3. *If $K \neq GF(3)$, then $F = SL_n(K)$.*

Proof. Let τ be a transvection in F with centre $\mathbf{x}$ (non-zero and singular) and axis $\langle \mathbf{x} \rangle'$, let $\mathbf{y}$ be a singular vector such that $B(\mathbf{x}, \mathbf{y}) = 1$ and write $\tau(\mathbf{y}) = \mathbf{y} + \lambda \mathbf{x}$ where $\lambda \in K \backslash \{0\}$. For $\mu \in K^*$, let τ_μ be the transvection with centre $\mathbf{x}$ and axis $\langle \mathbf{x} \rangle'$ that takes $\mathbf{y}$ to $\mathbf{y} + \mu \lambda \mathbf{x}$, and let $K_1 = \{\mu \in K^*; \tau_\mu \in F\} \cup \{0\}$; then as $\tau_\mu^{-1} = \tau_{-\mu}$ and $\tau_{\mu_1} \tau_{\mu_2} = \tau_{\mu_1 + \mu_2}$, K_1 is an additive subgroup of K. For $\xi \in K^*$, let $g_\xi \in SO(\langle \mathbf{x}, \mathbf{y} \rangle)$ be the map $(\mathbf{x}, \mathbf{y}) \mapsto (\xi \mathbf{x}, \xi^{-1} \mathbf{y})$; then $g_\xi \tau g_\xi^{-1} = \tau_{\xi^2} \in F$, so K_1 contains every square in K. As any element of K may be written as the difference of two squares, it follows that $K_1 = K$. Hence F contains every transvection with centre $\mathbf{x}$ and axis $\langle \mathbf{x} \rangle'$, i.e., $X(\langle \mathbf{x} \rangle, \langle \mathbf{x} \rangle') \leqslant F$.

As G_0 acts transitively on the non-zero singular vectors of V and as $g X(\langle \mathbf{x} \rangle, \langle \mathbf{x} \rangle') g^{-1} = X(\langle g(\mathbf{x}) \rangle, \langle g(\mathbf{x}) \rangle')$, it follows that F contains every transvection whose centre is singular and whose axis is conjugate to the centre. Thus if F_1 is the subgroup of F consisting of all the elements that fix $\langle \mathbf{x}, \mathbf{y} \rangle$ and fix every vector in $\langle \mathbf{x}, \mathbf{y} \rangle'$, then F_1 contains $X(\langle \mathbf{x} \rangle, \langle \mathbf{x} \rangle')$ and

$X(\langle \mathbf{y} \rangle, \langle \mathbf{y} \rangle')$. Thus F_1 acts transitively on the 1-dimensional subspaces of $\langle \mathbf{x}, \mathbf{y} \rangle$ and if $\mathbf{w} \in \langle \mathbf{x}, \mathbf{y} \rangle \backslash \{\mathbf{0}\}$, then F contains $X(\langle \mathbf{w} \rangle, \langle \mathbf{w} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle') = fX(\langle \mathbf{x} \rangle, \langle \mathbf{x} \rangle') f^{-1}$ where $f \in F_1$ such that $f\langle \mathbf{x} \rangle = \langle \mathbf{w} \rangle$. Choose any non-singular vector $\mathbf{v} \in \langle \mathbf{x}, \mathbf{y} \rangle'$, let $\mathbf{u} \in \langle \mathbf{x}, \mathbf{y} \rangle$ such that $Q(\mathbf{u}) = Q(\mathbf{v})$, let $\mathbf{w} \in \langle \mathbf{x}, \mathbf{y} \rangle \cap \langle \mathbf{u} \rangle' \backslash \{\mathbf{0}\}$ and let $g_1 \in SO(\langle \mathbf{u}, \mathbf{v} \rangle)$ be the map $(\mathbf{u}, \mathbf{v}) \mapsto (-\mathbf{v}, \mathbf{u})$; then $g_1 X(\langle \mathbf{w} \rangle, \langle \mathbf{w} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle') g_1^{-1} \leqslant F$ and is a subgroup of root type with centre $\langle \mathbf{w} \rangle$ and axis $g_1(\langle \mathbf{w} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle') \neq \langle \mathbf{w} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle'$. Hence if $\hat{F}$ is the subgroup of F generated by the subgroups of root type of F, then $\langle \mathbf{w} \rangle$ is a centre for $\hat{F}$ with more than one axis.

By Result 3, to show that $\hat{F} = SL_n(K)$ and hence that $F = SL_n(K)$, it remains to show that $\hat{F}$ is irreducible. Let U be a non-zero subspace of V fixed by $\hat{F}$; then as for any non-zero singular vector $\mathbf{a}$, $X(\langle \mathbf{a} \rangle, \langle \mathbf{a} \rangle')$ fixes U if and only if either $\mathbf{a} \in U$ or $U \subseteq \langle \mathbf{a} \rangle'$, every singular vector in V lies in either U or U'. As the singular vectors span V, dimensional considerations imply that U is non-isotropic, so $V = U \oplus U'$; moreover U cannot be anisotropic. As the sum of non-zero singular vectors of U and U' would be a singular vector lying outside both, U' must be anisotropic and so every singular vector lies in U. Hence $U = V$ and $F = SL_n(K)$. ▮

PROPOSITION 4. *If $K = GF(3)$, then $F = SL_n(K)$.*

Proof. We argue by induction on n. If $n = 3$, then, given that $SGO_3(K) = SO_3(K)$ for $K = GF(3)$, $F = SL_n(K)$ by Result 2. This means that if $n \geqslant 4$ and if we could find a non-isotropic $(n-1)$-dimensional subspace U of V and an element of $F \backslash (F \cap G)$ fixing U and every vector in U', then F would contain $SL_{n-1}(K)$ acting as the special linear group on U and as the identity on U'. Thus there would be a centre for F in U with more than one axis containing U'; moreover F would act transitively on the 1-dimensional subspaces of V, so proceeding as in the proof of Proposition 2, Result 3 would imply that $F = SL_n(K)$. Notice that $SGO_n(K) = SO_n(K)$ when n is odd and $SO_n(K)$ is a subgroup of $SGO_n(K)$ of index 2 when n is even.

Suppose that $n \geqslant 4$ and that the statement of the proposition is true for spaces of dimension $< n$. As in the proof of Proposition 2, there exists $f \in F$ such that $f(\mathbf{x}) = \mathbf{x}$ and $f(\mathbf{y}) = \alpha\mathbf{x} + \beta\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are singular vectors such that $B(\mathbf{x}, \mathbf{y}) \neq 0$ and where $\alpha, \beta \in K^*$; we may assume that $B(\mathbf{x}, \mathbf{y}) = 1$. We first show that there exists a non-singular vector $\mathbf{v}$ and an element $h \in F \backslash (F \cap G)$ such that h fixes $\mathbf{v}$ but not $\langle \mathbf{v} \rangle'$. If $\alpha = \beta = -1$, then we may take $\mathbf{v} = \mathbf{x} - \mathbf{y}$ and $h = \iota f$ where $\iota \in SO(\langle \mathbf{x}, \mathbf{y} \rangle)$ takes $\mathbf{x}$ and $\mathbf{y}$ to $-\mathbf{x}$ and $-\mathbf{y}$, respectively; if $\alpha = 1, \beta = -1$, then we may take $\mathbf{v} = \mathbf{x} + \mathbf{y}$ and $h = \iota f$ with ι as above. If $\beta = 1$, if $\rho \in G_0$ is the product of symmetries centred on $\mathbf{x} - \mathbf{y}$ and a non-singular vector in $\langle \mathbf{x}, \mathbf{y} \rangle'$ and if $h = (\rho f \rho)^{-1} f (\rho f \rho) = \rho f^{-1} \rho f \rho f \rho$ then $h(\mathbf{x} + \mathbf{y}) = (\alpha - 1)\mathbf{x} - \alpha\mathbf{y}$ and $h(\mathbf{x} - \mathbf{y}) =$

$-(\alpha+1)\,\mathbf{x}-\alpha\mathbf{y}$; thus if $\alpha=1$ we may take $\mathbf{v}=\mathbf{x}-\mathbf{y}$ and if $\alpha=-1$ we may take $\mathbf{v}=\mathbf{x}+\mathbf{y}$, and $\mathbf{v}$ and h have the required properties.

Now consider the map $\hat{h}$ on $\langle\mathbf{v}\rangle'$ obtained by letting $\hat{h}(\mathbf{w})$ ($\mathbf{w}\in\langle\mathbf{v}\rangle'$) be the $\langle\mathbf{v}\rangle'$ component of $h(\mathbf{w})$, and let $\hat{G}_0$, $\hat{G}$ and $\hat{H}$ be respectively the special orthogonal, special general orthogonal and special linear groups on $\langle\mathbf{v}\rangle'$ (with respect to the restriction of Q where appropriate); then $\hat{h}\in\hat{H}$ and $\hat{G}_0$ has the same action as $SO(\langle\mathbf{v}\rangle')$.

If $\hat{h}\in\hat{G}_0$, then we can multiply h by an element of $SO(\langle\mathbf{v}\rangle')$ to obtain a non-trivial transvection h_1 centred on $\mathbf{v}$; we may now find a non-singular vector $\mathbf{u}\in\langle\mathbf{v}\rangle'$ that lies in the axis of h_1, so that if $U=\langle\mathbf{u}\rangle'$, then U is an $(n-1)$-dimensional non-isotropic subspace fixed by $h_1\in F\backslash(F\cap G)$ with each vector in U' fixed by h_1; as indicated above, this leads to the conclusion that $F=SL_n(K)$.

If $\hat{h}\in\hat{G}\backslash\hat{G}_0$, then as $\langle\mathbf{v}\rangle'$ is spanned by non-zero singular vectors and as $\hat{G}_0$ acts transitively on those non-zero singular vectors, there is such a vector $\mathbf{w}$ with $h(\mathbf{w})\notin\langle\mathbf{v}\rangle'$ and there is an element $\hat{g}\in\hat{G}_0$ (corresponding to some $g\in SO(\langle\mathbf{v}\rangle')$) such that $\hat{g}\hat{h}(\mathbf{w})=\mathbf{w}$, i.e., $gh(\mathbf{w})=\mathbf{w}+\lambda\mathbf{v}$ with $\lambda=\pm 1$. But now $(gh)^2$ fixes $\mathbf{v}$ without fixing $\langle\mathbf{v}\rangle'$ (so $(gh)^2\notin G$), and the corresponding element of $\hat{H}$ is $(\hat{g}\hat{h})^2$ which lies in $\hat{G}_0$. Thus we can now apply the argument from the previous case, with $(gh)^2$ in place of h.

Finally, if $\hat{h}\notin\hat{G}$, then by induction $\langle\hat{h},\hat{G}_0\rangle=\hat{H}$. Thus given an orthogonal base $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_{n-1}\}$ for $\langle\mathbf{v}\rangle'$ with $Q(\mathbf{v}_1)=-Q(\mathbf{v}_2)=Q(\mathbf{v})$, there exists $\hat{h}_1\in\langle\hat{h},\hat{G}_0\rangle$, corresponding to some $h_1\in\langle h, SO(\langle\mathbf{v}\rangle')\rangle$, such that $\hat{h}_1(\mathbf{v}_1)=\mathbf{v}_2$, $\hat{h}_1(\mathbf{v}_2)=-\mathbf{v}_1$ and $\hat{h}_1(\mathbf{v}_i)=\mathbf{v}_i$ for $i\geqslant 3$. It follows that $h_1(\mathbf{v})=\mathbf{v}$, $h_1(\mathbf{v}_1)=\mathbf{v}_2+\lambda_1\mathbf{v}$, $h_1(\mathbf{v}_2)=-\mathbf{v}_1+\lambda_2\mathbf{v}$ and $h_1(\mathbf{v}_i)=\mathbf{v}_i+\lambda_i\mathbf{v}$ $(i\geqslant 3)$ for some $\lambda_1,\lambda_2,\ldots,\lambda_{n-1}\in K$. Therefore $h^3(\mathbf{v})=\mathbf{v}$, $h^3(\mathbf{v}_1)=-\mathbf{v}_2+\lambda_2\mathbf{v}$, $h^3(\mathbf{v}_2)=\mathbf{v}_1-\lambda_1\mathbf{v}$, $h^3(\mathbf{v}_i)=\mathbf{v}_i$ $(i\geqslant 3)$ and $\mathbf{v}+\mathbf{v}_2$ is singular but $h^3(\mathbf{v}+\mathbf{v}_2)=(1-\lambda_1)\,\mathbf{v}+\mathbf{v}_1$ is non-singular, so $h^3\notin G$. We now take $U=\langle\mathbf{v}_{n-1}\rangle'$. The subspace U is non-isotropic of dimension $n-1$, and h^3 fixes U and every vector in $U'=\langle\mathbf{v}_{n-1}\rangle$, so as indicated at the beginning of the proof, an induction argument leads to the conclusion that $F=SL_n(K)$.

We have proved that if $F\leqslant SL_n(K)$ and $G_0\leqslant F$ but $F\nleqslant G$, then $F=SL_n(K)$. In other words, any proper subgroup of $SL_n(K)$ containing $SO_n(K)$ lies in $SGO_n(K)$. Thus we have proved Theorem I. ∎

Let M_1 be the subgroup of $M(Q)$ consisting of the multiplicators of elements of $SGO_n(K)$.

COROLLARY TO THEOREM I. *If $n\geqslant 3$, then $SGO_n(K)$ is a maximal subgroup of $SL_n(K)$, and the proper subgroups of $SL_n(K)$ containing $SO_n(K)$ are in one-to-one correspondence with the subgroups of M_1.*

Proof. The maximality of $SGO_n(K)$ in $SL_n(K)$ is immediate from Theorem I.

Let $\theta: G \to M_1$ be the map taking g to its multiplicator; then θ is an epimorphism with kernel G_0, so that the subgroups of G containing G_0 are in one-to-one correspondence with the subgroups of M_1. By Theorem I, the proper subgroups of $SL_n(K)$ containing G_0 lie in G, and the result follows. ∎

3. The Case $n=2$

We now assume that $n=2$; then $G<F$. Let $\mathbf{x}$ and $\mathbf{y}$ be singular vectors such that $B(\mathbf{x},\mathbf{y})=1$; we shall write the elements of $SL_2(K)$ as 2×2 matrices with respect to the base $\{\mathbf{x},\mathbf{y}\}$ of V. Let

$$h_\lambda=\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix},\qquad g_\lambda=\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix},\qquad g=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

where $\lambda\in K\backslash\{\mathbf{0}\}$; then any element of G may be written as either g_λ or gg_λ for some $\lambda\in K\backslash\{\mathbf{0}\}$ and h_λ normalises G. Alternatively G may be characterised as the subgroup of $SL_2(K)$ consisting of the matrices with two zero entries.

Proposition 5. *F contains a transvection centred on* $\mathbf{x}$, *except when* $|K|\leqslant 11$.

Proof. Let $f\in F\backslash G$. If f has a zero entry, then by multiplying f by suitable elements of G we may readily construct a transvection centred on $\mathbf{x}$, so we may suppose that all the entries of f are non-zero. Writing $f=(f_{ij})$, if $f_{11}=\beta$, then we may replace f by $g_\beta^{-1}f$ and thus assume that $f_{11}=1$. Let $\gamma=f_{12}$; then $h_\gamma^{-1}Fh_\gamma$ contains a transvection centred on $\mathbf{x}$ if and only if F does, so we could replace F by $h_\gamma^{-1}Fh_\gamma$ and f by $h_\gamma^{-1}fh_\gamma$. Thus we may assume that $f_{11}=f_{12}=1$, so

$$f=\begin{pmatrix}1 & 1\\ \alpha & \alpha+1\end{pmatrix}$$

for some $\alpha\in K\backslash\{0,-1\}$. Let K_0 be the prime subfield of K and let $K_0(\alpha)$ be the minimal subfield of K containing α; then $SGO_2(K_0(\alpha))$ and $SL_2(K_0(\alpha))$ may be embedded in G and $SL_2(K)$, respectively, as groups of matrices with respect to the base $\{\mathbf{x},\mathbf{y}\}$. Hence we need only construct a transvection centred on $\mathbf{x}$ in $\langle f, SGO_2(K_0(\alpha))\rangle$. Suppose that $K_0\neq GF(3)$ and that $\alpha\neq -4,-2,1,3$ and let

$$f^*=f^{-1}g_{(2/\alpha)}f^{-1}g_{(\alpha+4/\alpha+2)}fg_{3/4}f^{-1}g_{(\alpha-1/\alpha-3)}fgg_{(\alpha+1)/2}f.$$

Then writing $f^* = (f^*_{ij})$ we claim that $f^*_{22} = 0$ and that $f^*_{11} = 0$ if and only if $p(\alpha) = 0$ for some non-zero polynomial $p(t) \in K_0[t]$. First note that in proving the claim we may replace elements $g_{\gamma/\delta}$ by

$$\hat{g}_{\gamma/\delta} = \begin{pmatrix} \gamma^2 & 0 \\ 0 & \delta^2 \end{pmatrix}.$$

Next let $f_1 = f^{-1}\hat{g}_{(2/\alpha)}f^{-1}\hat{g}_{(\alpha+4/\alpha+2)}$ and let $f_2 = f\hat{g}_{3/4}f^{-1}\hat{g}_{(\alpha-1/\alpha-3)}$ $fg\hat{g}_{(\alpha+1)/2}f$; then

$$f_1 = \begin{pmatrix} (\alpha+4)^2(\alpha^3+4\alpha^2+8\alpha+4) & -(\alpha+2)^4 \\ -\alpha(\alpha+4)^2(\alpha+2)^2 & \alpha(\alpha+4)(\alpha+2)^2 \end{pmatrix}$$

and

$$f_2 = \begin{pmatrix} -(\alpha-1)^4(9-7\alpha)+7(\alpha-3)^2(4\alpha^2-(\alpha+1)^3) & 12(\alpha+1)(\alpha-1)^2(\alpha-3) \\ 7\alpha(\alpha+1)(\alpha-1)^4+(7\alpha+16)(\alpha-3)^2(4\alpha^2-(\alpha+1)^3) & 12(\alpha+1)(\alpha-1)^2(\alpha-3)(\alpha+4) \end{pmatrix}.$$

As $f_1 f_2$ is a scalar multiple of f^* we see that indeed $f^*_{22} = 0$ and that $f^*_{11} = 0$ if and only if $p(\alpha) = 0$ where $p(t) \in K_0[t]$ and $p(0) = -2^8 \cdot 3^2$. In fact

$$p(t) = 32[3t^6 + 9t^5 - 4t^4 - 23t^3 - 241t^2 - 228t - 72],$$

but we don't need this.

Hence if $K_0 \neq GF(3)$, then f^* has exactly one zero entry and so F contains a transvection centred on **x**, except when $\alpha = -4, -2, 1, 3$ or a root of $p(t)$.

Suppose that $K_0 = \mathbb{Q}$, let $\lambda \in K\backslash\{0, 1, -1\}$ such that $\lambda(\alpha+1) - \lambda^{-1}\alpha$, $\lambda^{-1}(\alpha+1) - \lambda\alpha \neq 0$ and let $\mu = (\lambda(\alpha+1) - \lambda^{-1}\alpha)$; then

$$f^{-1}g_\lambda f = \begin{pmatrix} \lambda(\alpha+1) - \lambda^{-1}\alpha & (\lambda-\lambda^{-1})(\alpha+1) \\ -(\lambda-\lambda^{-1})\alpha & \lambda^{-1}(\alpha+1) - \lambda\alpha \end{pmatrix}$$

so $g_\mu^{-1}f^{-1}g_\lambda f \in F\backslash G$. As at the beginning of this proof we may consider $h_\eta g_\mu^{-1}f^{-1}g_\lambda f h_\eta^{-1}$ in place of $g_\mu^{-1}f^{-1}g_\lambda f$ and $h_\eta F h_\eta^{-1}$ in place of F, where $\eta = \mu[(\lambda-\lambda^{-1})(\alpha+1)]^{-1}$. Now

$$h_\eta g_\mu^{-1}f^{-1}g_\lambda f h_\eta^{-1} = \begin{pmatrix} 1 & 1 \\ -(\lambda-\lambda^{-1})^2\alpha(\alpha+1) & 1-(\lambda-\lambda^{-1})^2\alpha(\alpha+1) \end{pmatrix}$$

so we may construct a transvection centred on **x** in $h_\eta F h_\eta^{-1}$ (and thus also in F) unless $-(\lambda-\lambda^{-1})^2\alpha(\alpha+1)$ is one of $-4, -2, 1, 3$ or is a root of $p(t)$.

But there are an infinite number of possible values of $(\lambda-\lambda^{-1})^2$. Hence F contains a transvection centred on $\mathbf{x}$.

If K_0 is finite of characteristic >11 and if $\alpha=-4, -2, 1, 3$ or a root of $p(t)$, then $K_0(\alpha)$ is finite. By Result 1, $SGO_2(K_0(\alpha))$ is therefore maximal in $SL_2(K_0(\alpha))$, so $\langle f, SGO_2(K_0(\alpha))\rangle$ contains a transvection centred on $\mathbf{x}$. Similarly if $K_0=GF(5)$, $GF(7)$ or $GF(11)$ and either K is finite, or α is a root of $p(t)$ but $\alpha\notin K_0$, then we can apply Result 1. Suppose that $5\leqslant |K_0|\leqslant 11$, that $\alpha\in K_0$ and that K is infinite; then there exists $\lambda\in K$ such that $[K_0(\lambda): K_0]>4$. Noting that $\lambda(\alpha+1)-\lambda^{-1}\alpha$, $\lambda^{-1}(\alpha+1)-\lambda\alpha\neq 0$, we may construct $h_\eta g_\mu^{-1} f^{-1} g_\lambda f h_\eta^{-1}$ as in the case $K_0=\mathbb{Q}$. Then since $-(\lambda-\lambda^{-1})^2\alpha(\alpha+1)\notin K_0$, $h_\eta F h_\eta^{-1}$ (and thus also F) contains a transvection centred on $\mathbf{x}$.

Finally, suppose that $K_0=GF(3)$. If K is finite, then F contains a transvection centred on $\mathbf{x}$, by Result 1. If K is infinite, then as above, for some $\lambda\in K$, $[K_0(\lambda): K_0]>4$ and so if $\alpha\in K_0$, then $-(\lambda-\lambda^{-1})^2\alpha(\alpha+1)\notin K_0$. Thus we may assume that $\alpha\notin K_0$. Let

$$f^{**}=f^{-1}g_{(\alpha-1/\alpha+1)}fg_\alpha f.$$

Then $f_{11}^{**}=0$; and $f_{22}^{**}=0$ if and only if

$$\alpha^4-\alpha^3+\alpha^2+1=0.$$

Hence f^{**} has exactly one zero entry and so F contains a transvection centred on $\mathbf{x}$, except when α is a root of the polynomial

$$p_1(t)=t^4-t^3+t^2+1.$$

Now $p_1(t)$ is irreducible over K_0, so if α is a root of $p_1(t)$, then $K_0(\alpha)$ is a finite field of order 81; hence by Result 1, $\langle f, SGO_2(K_0(\alpha))\rangle$ contains a transvection centred on $\mathbf{x}$. ∎

Remark. In the proof of Proposition 5, there appears a product of twelve matrices; this approach could be simplified or avoided in many cases, but would still appear necessary when K is a field of characteristic 0 in which -1 is a non-square.

Proof of Theorem II. By Proposition 5, there is a transvection $\tau\in F$ centred on $\mathbf{x}$. An argument used in the proof of Proposition 3 may now be applied: as F contains $g_\xi\tau g_\xi^{-1}$ for each $\xi\in K\backslash\{0\}$ and as every element of K may be expressed as the difference of two squares, F contains every transvection centred on $\mathbf{x}$. Therefore $\mathrm{Stab}_F\,\mathbf{x}$ acts transitively on the 1-dimensional subspaces of V other than $\langle\mathbf{x}\rangle$, so F acts transitively on the 1-dimensional subspaces of V, and hence F contains every transvection in $SL_2(K)$. As $SL_2(K)$ is generated by its transvections, it follows that $F=SL_2(K)$, so G is maximal in $SL_2(K)$. ∎

4. Related Results

In this section we consider some groups associated with $SGO_n(K)$, namely, $GO_n(K)$, $PGO_n(K)$, $PSGO_n(K)$, $SO_n(K)$ and $PSO_n(K)$, and consider when they may be maximal in $GL_n(K)$, $PGL_n(K)$, $PSL_n(K)$, $SL_n(K)$ and $PSL_n(K)$, respectively. We state conditions for maximality and interpret these conditions for algebraically closed fields and for $\mathbb{R}$ and $GF(q)$. We denote the centre of $GL_n(K)$ by Z and recall that it lies inside $GO_n(K)$.

Let J be the subgroup of K^* consisting of the determinants of elements of $GO_n(K)$. If $J < K^*$, then $GO_n(K)$ cannot be maximal in $GL_n(K)$, being contained in the subgroup $\{g \in GL_n(K): \det g \in J\}$ of $GL_n(K)$. If $J = K^*$ and if $GO_n(K) < E \leqslant GL_n(K)$, then $SGO_n(K) < E \cap SL_n(K)$, so that $GO_n(K)$ is maximal in $GL_n(K)$ if $SGO_n(K)$ is maximal in $SL_n(K)$; in the five exceptional cases of Theorem II, close inspection of Dickson's list of subgroups of $PSL_2(q)$ (cf. [2]) or of Wagner's clearer description of these subgroups (cf. [12]) yields the maximality of $PGO_2(q)$ in $PGL_2(q)$ when $q = 7, 9$ or 11, but shows that $PGO_2(3) < D_4 < PGL_2(3)$ and that $PGO_2(5) < S_4 < PGL_2(5)$, where D_4 is the dihedral group of order 8. We now need to determine when $J = K^*$. If n is odd, then $GO_n(K) = Z \cdot SO_n(K)$, so J consists of the nth powers of elements of K^*. If n is even, say, $n = 2m$, then J consists of the elements $\pm\lambda^m$ with $\lambda \in M(Q)$ (cf. [4]). Thus from Theorems I and II, we have

Theorem III. *If n is odd, then $GO_n(K)$ is maximal in $GL_n(K)$ if and only if*

$$\{\lambda^n : \lambda \in K^*\} = K^*.$$

If n is even, with $n = 2m$, and if $K \neq GF(3), GF(5)$ when $n = 2$, then $GO_n(K)$ is maximal in $GL_n(K)$ if and only if

$$\{\lambda^m, -\lambda^m : \lambda \in M(Q)\} = K^*.$$

As $Z \leqslant GO_n(K)$, the following result is immediate.

Theorem IV. *$PGO_n(K)$ is maximal in $PGL_n(K)$ if and only if $GO_n(K)$ is maximal in $GL_n(K)$; $PSGO_n(K)$ is maximal in $PSL_n(K)$ if and only if $SGO_n(K)$ is maximal in $SL_n(K)$.*

Now consider $SO_n(K)$. Clearly $SO_n(K)$ is maximal in $SL_n(K)$ if and only if $SGO_n(K)$ is maximal in $SL_n(K)$ and every element of $SGO_n(K)$ has multiplicator 1. Noting the information given about J above and noting that if $n = 2$ and $K = GF(q)$, then $-1 \in M(Q)$, we have the following:

THEOREM V. *If n is odd, then $SO_n(K)$ is maximal in $SL_n(K)$ if and only if* 1 *has no non-trivial nth roots in K.*

If n is even, with $n = 2m$, then $SO_n(K)$ is maximal in $SL_n(K)$ if and only if -1 *has no mth roots in $M(Q)$ and* 1 *has no non-trivial mth roots in $M(Q)$.*

An equivalent formulation of the second part of Theorem V would be that if n is even, then $SO_n(K)$ is maximal in $SL_n(K)$ if and only if 1 has no non-trivial nth roots in $M(Q)$.

A more interesting question is that of the maximality of $PSO_n(K)$ in $PSL_n(K)$. Denoting $Z \cap SL_n(K)$ by Z_1, we consider the equivalent question of the maximality of $SO_n(K) \cdot Z_1$ in $SL_n(K)$, which then becomes a question of whether or not $SGO_n(K) = SO_n(K) \cdot Z_1$. If n is odd, then the equality is immediate. Suppose that n is even, say, $n = 2m$, and let M_1 and M_2 be the subgroups of $M(Q)$ consisting of the multiplicators of elements of $SGO_n(K)$ and $SO_n(K) \cdot Z_1$, respectively; then $M_2 \leqslant (K^*)^2$. Given the structure of J, we may characterize M_1 as the subgroup of $M(Q)$ the mth powers of whose elements are ± 1, and we may characterize M_2 as the subgroup of $(K^*)^2$ the mth powers of whose elements are 1. If $g \in SGO_n(K)$ with multiplicator λ, then $g \in SO_n(K) \cdot Z_1$ if and only if $\lambda \in M_2$. Thus if $n = 2m$, then $SGO_n(K) = SO_n(K) \cdot Z_1$ if and only if -1 has no mth roots in $M(Q)$ and every mth root of 1 in $M(Q)$ is a square in K^*. As a direct consequence we have the following result, noting that $-1 \in M(Q)$ when n is even and $K = GF(q)$.

THEOREM VI. *If n is odd, then $PSO_n(K)$ is maximal in $PSL_n(K)$.*

If n is even, with $n = 2m$, then $PSO_n(K)$ is maximal in $PSL_n(K)$ if and only if -1 *has no mth roots in $M(Q)$ and every mth root of* 1 *in $M(Q)$ is a square in K^*.*

We noted in Section 1 that if $K = \mathbb{C}$, then $M(Q) = K^*$; more generally, the same is true of any algebraically closed field. From Theorems III, V and VI we obtain

THEOREM VII. *Let K be algebraically closed. Then*

(i) *$GO_n(K)$ is maximal in $GL_n(K)$;*

(ii) *$SO_n(K)$ is maximal in $SL_n(K)$ if and only if n is a power of an odd prime p and K has characteristic p;*

(iii) *$PSO_n(K)$ is maximal in $PSL_n(K)$ if and only if n is odd.*

If $K = \mathbb{R}$, then $M(Q) = K^*$ if $\nu = n/2$ and $(K^*)^2$ otherwise. We deduce

THEOREM VIII. *Let $K = \mathbb{R}$. Then*

(i) *$GO_n(K)$ is maximal in $GL_n(K)$;*

(ii) *$SO_n(K)$ is maximal in $SL_n(K)$ if and only if $v < n/2$;*

(iii) *$PSO_n(K)$ is maximal in $PSL_n(K)$ if and only if $v < n/2$.*

Now suppose that $K = GF(q)$; then K^* is cyclic of order $q-1$. Let α be a generator of K^*. For any positive integer k, there are no non-trivial kth roots of 1 in K^* if and only if $(k, q-1) = 1$; equivalently $\{\lambda^k : \lambda \in K^*\} = K^*$ if and only if $(k, q-1) = 1$. Both these statements may be deduced from consideration of when the map $K^* \to K^*$, $\lambda \mapsto \lambda^k$ is a bijection. Suppose that n is even, say, $n = 2m$; then $M(Q) = K^*$. Let $d = (m, q-1)$, $d_2 = (m, (q-1)/2)$. In determining when $\{\lambda^m, -\lambda^m : \lambda \in K^*\} = K^*$, there are several possibilities to consider; note that $|\{\lambda^m : \lambda \in K^*\}| = (q-1)/d$. If $d > 2$, then $d_2 > 1$ and $\{\lambda^m, -\lambda^m : \lambda \in K^*\} \neq K^*$ because $2(q-1)/d < q-1$. If $d = 2 = d_2$, then $m/2$ is odd, so $(\alpha^{(q-1)/4})^m = (-1)^{m/2} = -1$, whence $\lambda^m = -\mu^m$ for some $\lambda, \mu \in K^*$ and so $\{\lambda^m : \lambda \in K^*\} \cap \{-\lambda^m : \lambda \in K^*\} \neq \varnothing$; thus $\{\lambda^m, -\lambda^m : \lambda \in K^*\} \neq K^*$. If $d = 2$ and $d_2 = 1$, then $(q-1)/2$ is odd but m is even, so -1 has no mth roots in K^*, whence $\{\lambda^m : \lambda \in K^*\} \cap \{-\lambda^m : \lambda \in K^*\} = \varnothing$, i.e., $\{\lambda^m, -\lambda^m : \lambda \in K^*\} = K^*$. If $d = 1$, then $\{\lambda^m : \lambda \in K^*\} = K^*$, so certainly $\{\lambda^m, -\lambda^m : \lambda \in K^*\} = K^*$. Thus $\{\lambda^m, -\lambda^m : \lambda \in K^*\} = K^*$ if and only if $d_2 = 1$. Next we note that, as $(2m, q-1) \geq 2$, there will always be either an mth root of -1 in $M(Q)$ or a non-trivial mth root of 1 in $M(Q)$. It remains to determine when, if ever, -1 has no mth roots in $M(Q)$ and every mth root of 1 in $M(Q)$ is a square in K^*. If m/d_2 is odd, then for $r = (q-1)/2d_2$, $(\alpha^r)^m = (-1)^{m/d_2} = -1$, so -1 has an mth root in $M(Q)$. If m/d_2 is even, then $(q-1)/2d_2$ is odd and with $r = (q-1)/2d_2$, α^r is non-square, but $(\alpha^r)^m = (\alpha^{q-1})^{m/2d_2} = 1$, so 1 has an mth root that is a non-square. Hence it is never the case that -1 has no mth roots in $M(Q)$ and that every mth root of 1 in $M(Q)$ is a square in K^*. From Theorems III, V and VI, we deduce

THEOREM IX. *Let $K = GF(q)$ and let $n = 2m$ when n is even. Then*

(i) *If n is odd, then $GO_n(K)$ is maximal in $GL_n(K)$ if and only if $(n, q-1) = 1$.*
If n is even, then $GO_n(K)$ is maximal in $GL_n(K)$ if and only if $(m, (q-1)/2) = 1$, and $q \neq 3$ or 5 when $n = 2$.

(ii) *$SO_n(K)$ is maximal in $SL_n(K)$ if and only if n is odd and $(n, q-1) = 1$.*

(iii) *$PSO_n(K)$ is maximal in $PSL_n(K)$ if and only if n is odd.*

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