Irreducible characters of groups associated with finite nilpotent algebras with involution

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A B S T R A C T

An algebra group is a group of the form $P = 1 + J$ where $J$ is a finite-dimensional nilpotent associative algebra. A theorem of Z. Halasi asserts that, in the case where $J$ is defined over a finite field $F$, every irreducible character of $P$ is induced from a linear character of an algebra subgroup of $P$. If $(J, \sigma)$ is a nilpotent algebra with involution, then $\sigma$ naturally defines a group automorphism of $P = 1 + J$, and we may consider the fixed point subgroup $C_P(\sigma)$. Assuming that $F$ has odd characteristic $p$, we show that every irreducible character of $C_P(\sigma)$ is induced from a linear character of a subgroup of the form $C_Q(\sigma)$ where $Q$ is a $\sigma$-invariant algebra subgroup of $P$. As a particular case, the result holds for the Sylow $p$-subgroups of the finite classical groups of Lie type.

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1. Introduction, main results and consequences

Let $p$ be an odd prime, let $F$ be a finite field of characteristic $p$, and let $A$ be a finite-dimensional associative $F$-algebra (with identity). We recall that an involution on $A$ is a map $\sigma : A \to A$ satisfying the following conditions:

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(i) $\sigma(a + b) = \sigma(a) + \sigma(b)$ for all $a, b \in A$;
(ii) $\sigma(ab) = \sigma(b)\sigma(a)$ for all $a, b \in A$;
(iii) $\sigma^2(a) = a$ for all $a \in A$.

We note that an involution $\sigma$ is not required to be $F$-linear. However, the field $F = F \cdot 1$ is preserved by $\sigma$, and therefore $\sigma$ defines a field automorphism of $F$ which is either the identity or of order 2. We say that $\sigma$ is of the first kind if $\sigma$ fixes $F$, and of the second kind if $\sigma_F$ has order 2. In any case, we let $F^\sigma = \{ \alpha \in F \mid \sigma(\alpha) = \alpha \}$ denote the $\sigma$-fixed subfield of $F$, and consider $A$ as a finite-dimensional associative $F^\sigma$-algebra. We observe that $\sigma$ is of the second kind if and only if the field extension $F^{\sigma} \subseteq F$ has degree 2, and $\sigma : F \to F$ is the Frobenius map defined by the mapping $\alpha \mapsto \alpha^q$ where $q = |F^{\sigma}|$; hence, $F^\sigma = F^{q}$ and $F = F^{q^2}$. For simplicity, we will the bar notation $\bar{\alpha} = \alpha^q$ for $\alpha \in F$.

An important example occurs in the case where $A = M_n(F)$ is the $F$-algebra consisting of all $n \times n$ matrices with entries in $F$, endowed with the canonical transpose involution given by the mapping $a \mapsto a^t$ where $a^t$ denotes the transpose of $a \in M_n(F)$. More generally, let $q = |F^\sigma|$, let $F_{q} : M_n(F) \to M_n(F)$ be the Frobenius morphism defined by $F_{q}(aij) = (aij)^q$ for all $(aij) \in M_n(F)$, and define $a^* = F_{q}(a)^t$ for all $a \in M_n(F)$. Then, the mapping $a \mapsto a^*$ defines an involution on $M_n(F)$; we note that, if $F^\sigma = F$, then $a^* = a^t$ for all $a \in M_n(F)$. As usual, we will denote by $GL_n(F)$ the general linear group consisting of all invertible matrices in $M_n(F)$. If $\sigma : M_n(F) \to M_n(F)$ is any involution of the first kind, then there exists $u \in GL_n(F)$ with $u^t = \pm u$ and such that $\sigma(a) = u^{-1}a^t u$ for all $a \in M_n(F)$; moreover, the matrix $u$ is uniquely determined up to a factor in $F^\times$. On the other hand, if $\sigma : M_n(F) \to M_n(F)$ is any involution of the second kind, then there exists $u \in GL_n(F)$ with $u^* = u$ and such that $\sigma(a) = u^{-1}a^* u$ for all $a \in M_n(F)$; moreover, the matrix $u$ is uniquely determined up to a factor in $(F^\sigma)^\times$.

The proofs can be found in the book [16] by M.-A. Knus et al. (see, in particular, Propositions 2.19 and 2.20) where the complete classification of involutions is also given for arbitrary central $F$-algebras (see Propositions 2.7 and 2.20).] For simplicity of writing, we will the bar notation $\bar{\alpha} = \alpha^q$ for $\alpha \in F$.

In the general situation, let $A^\times$ denote the unit group of the $F$-algebra $A$. Then, for any involution $\sigma : A \to A$, the cyclic group $\langle \sigma \rangle$ acts on $A^\times$ as a group of automorphisms by means of $x^\sigma = \sigma(x^{-1})$ for all $x \in A^\times$. For any $\sigma$-invariant subgroup $H \subseteq A^\times$, we denote by $C_H(\sigma)$ the subgroup of $H$ consisting of all $\sigma$-fixed elements; that is,

$$C_H(\sigma) = \{ x \in H \mid x^\sigma = x \} = \{ x \in H \mid \sigma(x^{-1}) = x \}.$$  

In the case where $A = M_n(F)$, an arbitrary involution $\sigma : M_n(F) \to M_n(F)$ defines a group $C_{GL_n(F)}(\sigma)$ which is isomorphic to one of the well-known finite classical groups of Lie type (defined over $F$): the symplectic group $\text{Sp}_{2m}(q)$ if $\sigma$ is symplectic (and $F = F_q$), the orthogonal groups $O_{2m}^+(q)$, $O_{2m+1}(q)$, or $O_{2m+2}^-(q)$ if $\sigma$ is orthogonal (and $F = F_{q'}$), and the unitary group $U_n(q^2)$ if $\sigma$ is unitary (and $F = F_{q^2}$). [For the details on the definition of the classical groups, we refer to Chapter I the book [5] by R. Carter.] In fact, up to isomorphism, these groups may be defined by the involution $\sigma = \sigma_u$ where $u \in GL_n(F)$ is the matrix defined as follows; here, $J_m$ denotes the $m \times m$ matrix with 1’s along the anti-diagonal and 0’s elsewhere.

(i) For $\text{Sp}_{2m}(q)$, we choose $F = F_q$ and $u = \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix}$.
(ii) For $O_{2m}^+(q)$ or $O_{2m+1}(q)$, we choose $F = F_{q'}$ and $u = J_n$ where either $n = 2m$ or $n = 2m + 1$.
(iii) For $O_{2m+2}^-(q)$, we choose $F = F_q$ and $u = \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix}$ where $c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for $\varepsilon \in \mathbb{F}^\times - (\mathbb{F}^\times)^2$.
(iv) For $U_n(q^2)$, we choose $F = F_{q^2}$ and $u = J_n$. (In this case, we have $F^\sigma = F_q$.)

In this paper, we study the (complex) irreducible characters of the group $C_p(\sigma)$ in the case where $P$ is a $\sigma$-invariant algebra subgroup of $A^\times$. Following the terminology of [12], given any nilpotent
subalgebra $J$ of $A$, the algebra group associated to $J$ is the multiplicative group $1 + J$ of $A^\times$; notice that a subalgebra of $A$ is not required to contain the identity (it is simply a multiplicatively closed vector subspace of $A$). We note that $B = F \cdot 1 + J$ is a (local) subalgebra of $A$, and that $P = 1 + J$ is a (normal) Sylow $p$-subgroup of the unit group $B^\times$; in fact, $B^\times$ is isomorphic to the direct product $F^\times \times P$.

As a standard example, let $A = M_n(F)$, let $u \in GL_n(F)$ be any of the matrices listed above, and let $\sigma : M_n(F) \to M_n(F)$ be the involution given by $\sigma(a) = u^{-1}a^*u$ for all $a \in M_n(F)$. Let $P = U_n(F)$ be the (upper) unitriangular subgroup of $GL_n(F)$ consisting of all upper-triangular matrices with $1$'s on the main diagonal, and note that $P = 1 + J$ is the algebra subgroup of $GL_n(K)$ associated with the nilpotent subalgebra $J = u_n(F) \subseteq M_n(F)$ which consists of all upper-triangular matrices with $0$'s on the main diagonal; we note that $u_n(F)$ is the Jacobson radical of the Borel subalgebra $B = B_n(F)$ of $M_n(F)$ consisting of all upper-triangular matrices (hence, $B^\times$ is the standard Borel subgroup $B_n(F)$ of $GL_n(F)$). The nilpotent subalgebra $u_n(F)$ is clearly $\sigma$-invariant, and thus the unitriangular subgroup $P = U_n(F)$ is also $\sigma$-invariant. Therefore, we may consider the $\sigma$-fixed subgroup $C_P(\sigma)$; we note that, $P$ is a Sylow $p$-subgroup of $GL_n(F)$, whereas $C_P(\sigma)$ is a Sylow $p$-subgroup of the corresponding finite classical group.

The main result of this paper is as follows. (We note that our proof is valid only under the assumption of $p$ being odd; in fact, in the paper [17], G. Lusztig showed that a Sylow $p$-subgroup of $S_P(2^r)$ for $r > 1$ always has irreducible characters of degree $2^{r-1}$, and hence Theorem 1.4 below is false in this case.) Following [12], a subgroup $Q$ of an arbitrarily given algebra group $P = 1 + J$ is said to be algebra subgroup of $P$ if there exists a subalgebra $U$ of $J$ such that $Q = 1 + U$.

**Theorem 1.1.** Let $F$ be a finite field of odd characteristic, and let $(A, \sigma)$ be a finite-dimensional $F$-algebra with involution. Let $J$ be a $\sigma$-invariant nilpotent subalgebra of $A$, and let $\xi$ be an arbitrary irreducible character of $C_P(\sigma)$ where $P = 1 + J$. Then, there exists a $\sigma$-invariant algebra subgroup $Q \subseteq P$ and a linear character $\eta$ of $Q(\sigma)$ such that $\eta = \eta^C_P(\sigma)$.

As a consequence, we deduce that the degree of every irreducible character of $C_P(\sigma)$ is a power of $|F^\sigma|$ (see Theorem 1.3 below); to see this, it is enough to show that $|C_P(\sigma)|$ is a power of $|F^\sigma|$. A crucial tool is the well-known Cayley transform $\phi : J \to P$ defined on an arbitrary nilpotent subalgebra of $A$ by the rule $\phi(a) = (1 - a)(1 + a)^{-1}$ for all $a \in J$. Since $p$ is odd, the map $\phi$ is bijective. On the other hand, since $((1 + a)^{-1})^{-1} = 1 - a + a^2 - a^3 + \cdots$, it is clear that $\phi(a) = 1 - 2a + 2a^2 - 2a^3 + \cdots$ for all $a \in J$. Therefore, we easily deduce that $\phi(\sigma(a)) = \sigma(\phi(a))$ for all $a \in J$. In particular, we obtain the following elementary result.

**Lemma 1.2.** Let $J$ be a $\sigma$-invariant nilpotent subalgebra of $A$, and let

$$C_J(\sigma) = \{ a \in J \mid \sigma(a) = -a \}.$$

Then, the Cayley correspondence defines a bijection $\phi : C_J(\sigma) \to C_P(\sigma)$ where $P = 1 + J$. In particular, $|C_P(\sigma)|$ is a power of $|F^\sigma|$.

**Proof.** On the one hand, since $\phi(-a) = \phi(a)^{-1}$, we deduce that $\sigma(\phi(a)) = \phi(\sigma(a)) = \phi(-a) = \phi(a)^{-1}$ for all $a \in C_J(\sigma)$, and thus $\phi(C_J(\sigma)) \subseteq C_P(\sigma)$. On the other hand, let $x \in C_P(\sigma)$ be arbitrary, and let $a \in J$ be such that $\phi(a) = x$. Then, $\phi(\sigma(a)) = \sigma(x) = x^{-1} = \phi(a)^{-1} = \phi(-a)$, and so $\sigma(a) = -a$. The proof is complete; for the last assertion it is enough to observe that $C_J(\sigma)$ is a vector space over $F^\sigma$. □

A similar argument shows that $|C_P(\sigma) \cap Q|$ is a power of $|F^\sigma|$ for any algebra subgroup $Q \subseteq P$; in the terminology of [12], this means that $C_P(\sigma)$ is a strong subgroup of $P$ (considered as an algebra group over $F^\sigma$). More generally, if $P$ is any algebra group over (any field) $F$ and $H \subseteq P$ is a subgroup, we say that $H$ is a strong subgroup of $P$ if $|H \cap Q|$ is a power of $|F|$ for all algebra subgroups $Q \subseteq P$. In our situation, we observe that $P$ can be considered, in the obvious way, as an algebra group over $F^\sigma$. 

In fact, we may use results of [12] and [20] to deduce the following result; notice that the assertion on character degrees is also an obvious consequence of Theorem 1.1 (together with Lemma 1.2).

**Theorem 1.3.** Let \((A, \sigma)\) be a finite-dimensional \(F\)-algebra with involution, let \(J\) be a \(\sigma\)-invariant nilpotent subalgebra of \(A\), and let \(P = 1 + J\). Then, \(C_P(\sigma)\) is a strong subgroup of \(P\) (considered as an algebra group over \(F^\sigma\)). In particular, all irreducible characters of \(C_P(\sigma)\) have \(|F^\sigma|\)-power degree, and all conjugacy classes of \(C_P(\sigma)\) have \(|F^\sigma|\)-power cardinality.

**Proof.** We prove that \(|C_P(\sigma) \cap Q|\) is a power of \(|F^\sigma|\) for all \(F^\sigma\)-algebra subgroups \(Q \leq P\). The result then follows by [12, Theorem D] (for the irreducible characters) and by [20, Lemma 5] (for the conjugacy classes). Let \(U \subseteq J\) be an arbitrary \(F^\sigma\)-subalgebra. By the previous lemma, the Cayley transform clearly defines a bijective map \(\phi : C_J(\sigma) \cap U \rightarrow C_P(\sigma) \cap (1 + U)\), and the result follows because \(C_J(\sigma) \cap U\) is an \(F^\sigma\)-vector subspace of \(J\).

As a consequence, we deduce the following result; we refer that the same result was also obtained by M. Boyarchenko (see the preprint [3]; see also the PhD thesis [4]) for a large class of finite groups of Lie type using a rather different method. (Given any finite group \(G\) and any prime divisor \(p\) of \(|G|\), we denote by \(\text{Syl}_p(G)\) the set consisting of all Sylow \(p\)-subgroups of \(G\).

**Theorem 1.4.** Let \(F\) be a finite field of odd characteristic \(p\), let \(G\) be a finite classical group of Lie type defined over \(F\), and let \(P \in \text{Syl}_p(G)\). Then, all irreducible characters of \(P\) have \(q\)-power degree where \(q = |F|\).

Before we proceed with the proof of Theorem 1.1, we should mention that the corresponding result for algebra groups was proved by Z. Halasi in the paper [9] (although the theorem was first stated by E.A. Gutkin in [8]). We refer to this theorem as the Gutkin–Halasi’s Theorem; it asserts that every irreducible character of a finite algebra group is induced by a linear character of some algebra group over \(F\). Since every algebra group is the Sylow \(p\)-subgroup of a finite algebra group considered as being precisely the algebra groups over \(F\).

**Theorem 1.5.** Let \(A\) be a finite-dimensional algebra over a finite field \(F\) of characteristic \(p\), and let \(P \in \text{Syl}_p(A^\times)\). Then, \(P\) is an algebra subgroup of \(A^\times\); in other words, there exists a nilpotent subalgebra \(J \leq A\) such that \(P = 1 + J\). In particular, \(|P|\) is a power of \(|F|\).

**Proof.** Let \(B = \langle x | x \in P \rangle\) be the vector space (over \(F\)) spanned by \(P \subseteq A\). Then, \(B\) is a subalgebra of \(A\), and \(P\) is a Sylow \(p\)-subgroup of \(B^\times\). We claim that \(P = 1 + J(B)\) where \(J(B)\) denotes the Jacobson radical of \(B\). On the one hand, since \(1 + J(B)\) is a \(p\)-subgroup of \(B^\times\), there exists \(z \in B^\times\) such that \((1 + J(B))^z \leq P\), and thus \(1 + J(B) \leq P\) (because \(J(B)\) is an ideal of \(B\), hence \(J(B)^2 = J(B)\)). On the other hand, let \(F[P]\) be the group algebra of \(P\) over \(F\), and let \(\psi : F[P] \rightarrow B\) be the natural extension of the inclusion \(i : P \rightarrow B\) to a homomorphism of \(F\)-algebras. Since \(P\) is a finite \(p\)-group, the Jacobson radical of \(F[P]\) coincides with the augmentation ideal \(I(P)\) of \(F[P]\) (see [14, Proposition 52.4]); we recall that \(I(P)\) is the vector subspace of \(F[P]\) spanned by all the elements \(x - 1\) for \(x \in P\). It follows that \(\psi(I(P)) \subseteq J(B)\), and thus \(x - 1 \in J(B)\) for all \(x \in P\). The claim follows, and the proof is complete.

Since every algebra group is the Sylow \(p\)-subgroup of a well-determined local algebra (see the remarks above), we immediately deduce the following consequence.
Corollary 1.6. A finite group $P$ is an algebra group over $F$ if and only if $P \in \text{Syl}_p(A^\times)$ for some finite-dimensional $F$-algebra $A$.

By virtue of Theorem 1.5, the Gutkin–Halasi’s Theorem can be restated as follows; here, by a unitary subalgebra of $A$ we mean any subalgebra of $A$ containing the identity.

**Theorem** (Gutkin–Halasi). Let $A$ be a finite-dimensional algebra over a finite field $F$ of characteristic $p$, let $P \in \text{Syl}_p(A^\times)$, and let $\xi$ be an irreducible character of $P$. Then, there exists a unitary subalgebra $B \leq A$ and a Sylow $p$-subgroup $Q \in \text{Syl}_p(B^\times)$ with $Q \leq P$ such that $\xi = \eta^p$ for some linear character $\eta$ of $Q$. In particular, the degree $\xi(1)$ is a power of $|F|$.

In the case where $(A, \sigma)$ is a finite-dimensional algebra with involution (over a field of odd characteristic), we obtain a similar result as a consequence of Theorem 1.1; here, we naturally extend the notation $C_H(\sigma)$ to any subgroup $H \leq A^\times$ by setting $C_H(\sigma) = H \cap C_{A^\times}(\sigma)$.

**Theorem 1.7.** Let $F$ be a finite field of odd characteristic $p$, and let $(A, \sigma)$ be any finite-dimensional $F$-algebra with involution. Let $P \in \text{Syl}_p(A^\times)$, and let $\xi$ be an irreducible character of $C_P(\sigma)$. Then, there exists a $\sigma$-invariant unitary subalgebra $B \leq A$ and a Sylow $p$-subgroup $Q \in \text{Syl}_p(B^\times)$ with $Q \leq P$ such that $\xi = \eta^p$ for some linear character $\eta$ of $Q$. In particular, the degree $\xi(1)$ is a power of $|F|$.

**Proof.** Since $P$ is an algebra subgroup of $A^\times$ (by Theorem 1.5), it is enough to apply Theorem 1.1 to the $\sigma$-invariant algebra subgroup $P \cap P^\sigma$. The last assertion is an immediate consequence of Theorem 1.3. □

In particular, Theorem 1.1 (and the previous theorem) applies to any of the finite classical groups. In fact, as we remarked above, the Sylow $(p)$-subgroups of any finite classical group can be realised as the subgroup $C_P(\sigma)$ where $P = U_n(F)$ is the (upper) unitriangular subgroup of $GL_n(F)$ and $\sigma = \sigma_u$ is an involution of $A = M_n(F)$ defined by the appropriate matrix $u \in GL_n(F)$. Thus, in this particular situation, Theorem 1.1 (or Theorem 1.7) can be restated as follows.

**Theorem 1.8.** Let $F$ be a finite field of odd characteristic $p$, let $G$ be a classical algebraic group defined over the algebraic closure of $F$, and let $G = \bar{G}(F)$ be the corresponding finite classical group of Lie type defined over $F$. Let $P \in \text{Syl}_p(G)$, and let $\xi \in \text{Irr}(P)$. Then, there exists a closed connected subgroup $H$ of $G$ such that $\xi = \eta^p$ is induced from some linear character $\eta \in \text{Irr}(Q)$ of the subgroup $Q = P \cap H$ of $P$.

## 2. Proof of Theorem 1.1

In this section, we proceed with the proof of Theorem 1.1. In fact, we shall prove a slightly more general result. Firstly, we recall the Glauberman correspondence between $\sigma$-invariant irreducible characters of $P$ and irreducible characters of its subgroup $C_P(\sigma)$; our main reference is [11, Chapter 13]. As usual, we denote by $\text{Irr}(P)$ the set consisting of all irreducible characters of $P$ (and extend this notation to any finite group). For any character $\xi$ of $P$, we denote by $\xi^\sigma$ the character of $P$ given by $\xi^\sigma(x) = \xi(x^\sigma)$ for all $x \in P$, and set

$$\text{Irr}_\sigma(P) = \{\xi \in \text{Irr}(P) \mid \xi^\sigma = \xi\}.$$ 

Then, since $p$ is odd, the Glauberman correspondence asserts that there exists a uniquely defined bijective map

$$\pi_P : \text{Irr}_\sigma(P) \to \text{Irr}(C_P(\sigma))$$

such that, for any $\xi \in \text{Irr}_\sigma(P)$, the image $\eta = \pi_P(\xi)$ is the unique irreducible constituent of $\xi C_P(\sigma)$ with odd multiplicity (see [11, Theorem 13.1]). Given any $\sigma$-invariant subgroup $Q \leq P$, we shall write $\pi_Q$
to denote the Glauberman map \( \pi_Q : \text{Irr}_\sigma(Q) \to \text{Irr}(C_Q(\sigma)) \). We shall prove the following extension of Theorem 1.1.

**Theorem 2.1.** Let \( F \) be a finite field of odd characteristic, and let \((A, \sigma)\) be a finite-dimensional \( F \)-algebra with involution. Let \( J \) be a \( \sigma \)-invariant nilpotent subalgebra of \( A \), and let \( P = 1 + J \). Let \( \xi \) be an irreducible character of \( C_P(\sigma) \), and let \( \hat{\xi} \in \text{Irr}_\sigma(P) \) be such that \( \pi_P(\hat{\xi}) = \xi \). Then, there exists a \( \sigma \)-invariant algebra subgroup \( Q \leq P \) and a \( \sigma \)-invariant linear character \( \eta \) of \( Q \) such that \( \hat{\xi} = \eta^P \) and \( \xi = \eta^{C_P(\sigma)} \) where \( \eta \) is the linear character \( \eta = \pi_Q(\hat{\xi}) \) of \( C_Q(\sigma) \).

For the proof of this theorem, we will argue by induction on the dimension of \( J \). Given any \( \sigma \)-invariant nilpotent subalgebra \( J \) of \( A \), we consider the algebra subgroup \( N = 1 + J^2 \) of \( P = 1 + J \); in the terminology of [12], \( N \) is an ideal subgroup of \( P \) and, in particular, it is a normal subgroup of \( P \). Following [2], we say that an irreducible character \( \xi \in \text{Irr}(P) \) is strongly Heisenberg if \( \xi_N = e \vartheta \) for some (positive) integer \( e \) and some \( P \)-invariant character \( \vartheta \in \text{Irr}(N) \); hence, by [9, Theorem 1.3], \( \vartheta \) is linear, and so \( e = \xi(1) \). We note that, if \( \xi \) is \( \sigma \)-invariant, then \( \vartheta \) is also \( \sigma \)-invariant (by [11, Theorem 13.27]); it is clear that the subgroup \( N \) is \( \sigma \)-invariant. More generally, given any finite group \( G \), we say that \( \xi \in \text{Irr}(G) \) is a Heisenberg character if there exists a normal subgroup \( N \) such that \( G/N \) is abelian and \( \xi_N = e \vartheta \) for some (positive) integer \( e \) and some \( G \)-invariant linear character \( \vartheta \in \text{Irr}(N) \). The following conditions are equivalent for any \( \xi \in \text{Irr}(G) \):

(i) \( \xi \) is a Heisenberg character;

(ii) \( Z(\xi) = \{ x \in G \mid \xi(x) = \xi(1) \} \) is the centre of \( \xi \), then \( G/Z(\xi) \) is an abelian group.

It was proved in [2, Theorem 3.1] that every irreducible character \( \xi \in \text{Irr}(P) \) is induced from a strongly Heisenberg irreducible character of some algebra subgroup of \( P \). We next show that the same is true for \( \sigma \)-invariant irreducible characters; that is, every \( \xi \in \text{Irr}_\sigma(P) \) is induced from a strongly Heisenberg \( \eta \in \text{Irr}_\sigma(Q) \) of some \( \sigma \)-invariant algebra subgroup \( Q \leq P \). For the proof we need the following auxiliary result (see [10, Lemma 3.2]); henceforth, we fix the finite-dimensional \( F \)-algebra \( A \) and the involution \( \sigma : A \to A \).

**Lemma 2.2.** Let \( J \) be a \( \sigma \)-invariant nilpotent subalgebra of \( A \), and let \( P = 1 + J \). Let \( N \leq Q \) be \( \sigma \)-invariant normal subgroups of \( P \), let \( \xi \in \text{Irr}_\sigma(Q) \), and let \( \eta \in \text{Irr}_\sigma(N) \) be a constituent of \( \xi_N \). Then, there exists \( \zeta \in \text{Irr}_\sigma(Q) \) such that \( \langle \xi, \xi_Q \rangle \neq 0 \) and \( \langle \xi_N, \eta \rangle \neq 0 \).

The following result will also be very useful.

**Lemma 2.3.** Let \( J \) be a \( \sigma \)-invariant nilpotent subalgebra of \( A \), and let \( P = 1 + J \). Let \( U \subseteq J \) be any \( \sigma \)-invariant vector subspace such that \( J^2 \subseteq U \), and consider the \( \sigma \)-invariant ideal subgroup \( Q = 1 + U \) of \( P \). Let \( \eta \in \text{Irr}_\sigma(Q) \) be arbitrary. Then, the inertia group \( I_P(\eta) = \{ x \in P \mid \eta^x = \eta \} \) is a \( \sigma \)-invariant algebra subgroup of \( P \).

**Proof.** By [2, Lemma 3.2], we know that \( I_P(\eta) \) is an algebra subgroup of \( P \). The result follows because \( I_P(\eta) \) is clearly \( \sigma \)-invariant. \( \square \)

We are now able to prove the following reduction theorem (see [2, Theorem 3.1]).

**Proposition 2.4.** Let \( J \) be a \( \sigma \)-invariant nilpotent subalgebra of \( A \), let \( P = 1 + J \), and let \( \xi \in \text{Irr}_\sigma(P) \). Then, there exists a \( \sigma \)-invariant algebra subgroup \( Q \leq P \) and a strongly Heisenberg irreducible character \( \eta \in \text{Irr}_\sigma(Q) \) such that \( \xi = \eta^P \).

**Proof.** We proceed by induction on \( \dim J \). Suppose that \( \xi \) is not strongly Heisenberg, and let \( N = 1 + J^2 \). By [11, Theorem 13.27], there exists \( \vartheta \in \text{Irr}_\sigma(N) \) such that \( \langle \xi_N, \vartheta \rangle \neq 0 \). By [9, Theorem 1.3], \( \vartheta \) is not \( P \)-invariant, and thus \( I_P(\vartheta) \) is a proper \( \sigma \)-invariant algebra subgroup of \( P \) (by the
previous lemma). By Lemma 2.2, there exists \( \eta \in \text{Irr}_\sigma(I_P(\vartheta)) \) with \( \langle \eta, \xi_N \rangle \neq 0 \) and \( \langle \eta_N, \vartheta \rangle \neq 0 \). By Clifford’s correspondence (see [11, Theorem 6.11]), we must have \( \xi = \eta^P \), and the result follows by induction. \( \square \)

As a consequence, we may use Glauberman’s correspondence to prove the following result.

**Proposition 2.5.** Let \( J \) be a \( \sigma \)-invariant nilpotent subalgebra of \( A \), and let \( P = 1 + J \). Let \( \xi \in \text{Irr}(C_P(\sigma)) \) be arbitrary, and let \( \hat{\xi} \in \text{Irr}(P) \) be such that \( \pi_P(\hat{\xi}) = \xi \). Then, there exists a \( \sigma \)-invariant algebra subgroup \( Q \subseteq P \) and a strongly Heisenberg irreducible character \( \hat{\eta} \in \text{Irr}_\sigma(Q) \) such that \( \hat{\xi} = \hat{\eta}^P \) and \( \xi = \eta^{C_P(\sigma)} \) where \( \eta = \pi_Q(\hat{\eta}) \in \text{Irr}(C_Q(\sigma)) \). Moreover, \( \eta \) is a Heisenberg character of \( C_Q(\sigma) \).

For the proof, we will need the following consequence of [11, Theorems 13.27 and 13.28].

**Lemma 2.6.** Let \( J \) be a \( \sigma \)-invariant nilpotent subalgebra of \( A \), let \( P = 1 + J \), and let \( Q \) be a \( \sigma \)-invariant algebra subgroup of \( P \). Then:

(i) For any \( \xi \in \text{Irr}_\sigma(P) \), the restriction \( \xi_Q \) has a \( \sigma \)-invariant irreducible constituent.

(ii) For any \( \eta \in \text{Irr}_\sigma(Q) \), the induced character \( \eta^P \) has a \( \sigma \)-invariant irreducible constituent.

**Proof.** We proceed by induction on \( \text{dim} \, J \). Let \( N = 1 + J^2 \), and consider the subgroup \( QN \leq P \). It is clear that \( QN \) is a \( \sigma \)-invariant algebra subgroup of \( P \); moreover, since \( N \subseteq QN \), it is obvious that \( QN \) is a normal subgroup of \( P \) (in fact, it is an ideal subgroup). If \( QN = P \), then \( Q = N \) (by [12, Lemma 3.1]) and there is nothing to prove. Thus, we assume that \( QN \neq P \). By [11, Theorem 13.27], there exists \( \eta \in \text{Irr}_\sigma(QN) \) such that \( \langle \eta, \xi_Q \rangle \neq 0 \). By induction, we conclude that \( \eta_Q \) has a \( \sigma \)-invariant irreducible constituent, and thus (i) is proved. On the other hand, let \( \eta \in \text{Irr}_\sigma(Q) \) be arbitrary. Then, by induction, there exists \( \zeta \in \text{Irr}_\sigma(QN) \) such that \( \langle \zeta, \eta_Q \rangle \neq 0 \). Since \( QN \subseteq P \), [11, Theorem 13.28] asserts that \( \zeta^P \) has a \( \sigma \)-invariant irreducible constituent, and thus \( \zeta^P = (\eta^P)^{QN} \) also has a \( \sigma \)-invariant irreducible constituent. The lemma follows. \( \square \)

The following observation will also be very useful.

**Lemma 2.7.** Let \( J \) be a nilpotent subalgebra of \( A \), and let \( Q \) be a strong subgroup of \( P = 1 + J \) which contains \( N = 1 + J^2 \). Then, \( Q \) is an algebra subgroup of \( P \). In particular, if \( J \) is \( \sigma \)-invariant, then \( C_P(\sigma)N \) is an \( F^\sigma \)-algebra subgroup of \( P \).

**Proof.** Let \( u \in J \) be arbitrary, and let \( U = J^2 + Fu \). Then, \( 1 + U \) is an algebra subgroup of \( P \), and thus \( |Q \cap (1 + U)| \) is a power of \( |F| \). Since \( N \leq Q \cap (1 + U) \) and \( |1 + U : N| \leq |F| \), either \( 1 + U \leq Q \), or \( Q \cap (1 + U) = N \). The result follows; for the last assertion, one can apply the first part to the \( \sigma \)-algebra group \( P/N \cong 1 + J/J^2 \) and to its strong subgroup \( C_{P/N}(\sigma) \) (see Lemma 2.2), and use the fact that \( C_{P/N}(\sigma) = C_P(\sigma)N/N \) (see [7, Theorem 5.3.15]). \( \square \)

The proof of Proposition 2.5 will be complete once we prove the following result.

**Proposition 2.8.** Let \( J \) be a \( \sigma \)-invariant nilpotent subalgebra of \( A \), and let \( P = 1 + J \). Let \( \xi \in \text{Irr}(C_P(\sigma)) \) be arbitrary, and let \( \hat{\xi} \in \text{Irr}_\sigma(P) \) be such that \( \pi_P(\hat{\xi}) = \xi \). Moreover, let \( Q \subseteq P \) be a \( \sigma \)-invariant algebra subgroup such that \( \hat{\xi} = \hat{\eta}^P \) for some \( \hat{\eta} \in \text{Irr}_\sigma(Q) \). Then, \( \xi = \eta^{C_P(\sigma)} \) where \( \eta = \pi_Q(\hat{\eta}) \in \text{Irr}(C_Q(\sigma)) \).

**Proof.** We proceed by induction on \( \text{dim} \, J \). Let \( N = 1 + J^2 \), and consider the subgroup \( QN \subseteq P \) and the induced character \( \hat{\xi} = \hat{\eta}^{QN} \in \text{Irr}(QN) \). By Lemma 2.6, the irreducible character \( \hat{\xi} \) must be \( \sigma \)-invariant; moreover, we clearly have \( \hat{\xi} = \hat{\eta}^P \). If \( QN = P \), then \( Q = P \) (by [12, Lemma 3.1]), and there is nothing to prove. Thus, suppose that \( QN \) is a proper subgroup of \( P \), and let \( \xi = \pi_{QN}(\zeta) \in \text{Irr}(C_{QN}(\sigma)) \). By induction, we have \( \zeta = \eta^{C_Q(\sigma)} \), and thus it is enough to prove that \( \xi = \zeta^{C_P(\sigma)} \). To see this, we consider
the subgroup \( L = C_P(\sigma)QN \); by the previous lemma, \( C_P(\sigma)N \) is an \( F^\sigma \)-algebra subgroup of \( P \), and so \( L \) is also an \( F^\sigma \)-algebra subgroup of \( P \). Since \( \xi^L \) is irreducible, the inertia group \( I_1(\xi) \) equals \( QN \) (by Mackey's criterion; see [11, Exercise 6.1]). On the other hand, by Lemma 2.6 (and by the previous lemma), we have \( \hat{\xi}L \in \text{Irr}_\sigma(L) \), and \( \pi_L(\hat{\xi}^L) = (\pi_QN(\xi))^{C_L(\sigma)} = \xi^{C_L(\sigma)} \) (see [11, Exercise 13.14]).

Since \( C_P(\sigma) \leq L \), we have \( C_L(\sigma) = C_P(\sigma) \), and so \( \pi_L(\hat{\xi}^L) = \xi^{C_P(\sigma)} \). Finally, using [7, Theorem 5.3.15], we deduce that \( C_P/L(\sigma) = C_P(\sigma) \leq L = 1 \), and thus [11, Exercises 13.4 and 13.5] imply that \( \hat{\xi} = \pi_P(\xi) = \pi_L(\hat{\xi}^L) = \eta^{C_P(\sigma)} \), as required. \( \Box \)

We now finish the proof of Proposition 2.5.

**Proof of Proposition 2.5.** The first assertion follows by Proposition 2.4 and the previous proposition. For the last assertion, let \( U \leq J \) be the vector subspace such that \( Q = 1 + U \), and let \( N = 1 + U^2 \). Let \( \hat{\vartheta} \in \text{Irr}_\sigma(N) \) be such that \( \hat{\vartheta}N = e\hat{\vartheta} \) for \( e = \hat{\xi}(1) \), and let \( \hat{\vartheta} = \pi_N(\hat{\vartheta}) \in \text{Irr}(C_N(\sigma)) \). Since \( \hat{\vartheta} \) is linear, we must have \( \hat{\vartheta}C_N(\sigma) = \hat{\vartheta} \) (hence, \( \hat{\vartheta} \) is linear). Moreover, since \( \hat{\vartheta} \) is \( Q \)-invariant, we easily deduce that \( I_{C_Q(\sigma)}(\hat{\vartheta}) = \hat{\vartheta}C_Q(\sigma) \). The result follows because \( C_Q(\sigma)/C_N(\sigma) \cong C_Q(\sigma)N/N \) is abelian. \( \Box \)

Next, we consider strongly Heisenberg \( \sigma \)-invariant irreducible characters of the algebra group \( P = 1 + J \) where \( J \) is a \( \sigma \)-invariant nilpotent subalgebra of \( A \). We recall some of the techniques used in the paper [2]. Given any \( P \)-invariant (linear) character \( \vartheta \in \text{Irr}_\sigma(N) \) of \( N = 1 + J^2 \), we define the commutator pairing \( c_\vartheta : P/N \times P/N \to \mathbb{C}^\times \) by

\[
c_\vartheta(xN, yN) = \vartheta([x, y])
\]

for all \( x, y \in P \); as usual, we set \([x, y] = x^{-1}y^{-1}xy \) for \( x, y \in P \). It is well known that \( c_\vartheta \) defines an alternating bilinear form on \( P/N \). Moreover, there exists an algebra subgroup \( L \leq P \) containing \( N \) and such that \( L/N \) is a maximal isotropic subgroup of \( P/N \) with respect to \( c_\vartheta \) (see [2, Proposition 1.3]). In our situation, we have the following result.

**Lemma 2.9.** Let \( J \) be a \( \sigma \)-invariant nilpotent subalgebra of \( A \), let \( P = 1 + J \), and let \( N = 1 + J^2 \). Let \( \vartheta \in \text{Irr}_\sigma(N) \) be a \( P \)-invariant (linear) character, and let \( c_\vartheta : P/N \times P/N \to \mathbb{C}^\times \) be the commutator pairing associated with \( \vartheta \). Then, there exists a \( \sigma \)-invariant algebra subgroup \( L \leq P \) containing \( N \) such that \( L/N \) is a maximal isotropic subgroup of \( P/N \) with respect to \( c_\vartheta \).

**Proof.** Let \( S = \langle \sigma \rangle \), let \( F^\sigma[S] \) denote the group algebra of \( S \) over \( F^\sigma \), and consider the left \( F^\sigma[S] \)-module \( U = J/J^2 \); we recall that \( \sigma : J \to J \) is an \( F^\sigma \)-linear isomorphism. Since \( p \) is odd, Maschke’s Theorem guarantees that \( J/J^2 = U_1 \oplus \cdots \oplus U_n \) for some irreducible \( F^\sigma[S] \)-submodules \( U_1, \ldots, U_n \leq U \). On the other hand, since \( F \) contains a primitive square root of unity (because \( p \) is odd), every irreducible \( F^\sigma[S] \)-module is one-dimensional (see [7, Theorem 3.24]), and thus there exist vectors \( u_1, \ldots, u_n \in J/J^2 \) such that \( U_i = F^\sigma u_i \) for all \( 1 \leq i \leq m \); in fact, since \( \sigma \) has order 2, we have \( \sigma(u_i) = \pm u_i \) for all \( 1 \leq i \leq n \). Then, \( \{u_1, \ldots, u_n\} \) is a spanning set for \( J/J^2 \) considered as a vector space over \( F \), and so we can choose an \( F \)-basis \( \{v_1, \ldots, v_m\} \) for \( J/J^2 \) with \( v_1, \ldots, v_m \in \{u_1, \ldots, u_n\} \). For each \( 1 \leq i \leq m \), let \( V_i \leq J \) be the vector subspace such that \( V_i/J^2 = Fv_1 \oplus \cdots \oplus Fv_i \), and let \( N_i = 1 + V_i \leq P \). Then, we obtain a chain \( J/J^2 = V_0 \subset V_1 \subset \cdots \subset V_m = J \) of \( \sigma \)-invariant vector subspaces satisfying \( \dim V_i/V_{i-1} = 1 \) for all \( 1 \leq i \leq m \), and hence also a chain \( N = N_0 \supset N_1 \supset \cdots \supset N_m = P \) of \( \sigma \)-invariant ideal subgroups of \( P \) satisfying \( |N_i : N_{i-1}| = |F| \) for all \( 1 \leq i \leq m \). Now, for each \( 1 \leq i \leq m \), let \( L_i = \{x \in N_i \mid [x, N_i] \leq \ker(\vartheta) \} \) where \( \ker(\vartheta) = \{z \in N \mid \vartheta(z) = \vartheta(1) \} \) is the kernel of \( \vartheta \). By [2, Lemma 1.4], for each \( 1 \leq i \leq m \), \( L_i \) is an ideal subgroup of \( P \), and so the product \( L = L_1 \cdots L_m \) is also an ideal subgroup of \( P \). Moreover, since \( \vartheta \) is \( \sigma \)-invariant, each \( L_i \) is clearly \( \sigma \)-invariant, hence \( L \) is also \( \sigma \)-invariant. Finally, it is easy to show that \( L/N \) is a maximal isotropic subgroup of \( P/N \) with respect to \( c_\vartheta \) (for a proof see [6, Lemma 1.12.3]). \( \Box \)

We are now able to prove the following result.
Proposition 2.10. Let $J$ be a $\sigma$-invariant nilpotent subalgebra of $A$, let $P = 1 + J$, and let $N = 1 + J^2$. Let $\xi \in \text{Irr}_\sigma(P)$ be strongly Heisenberg, and let $\vartheta \in \text{Irr}_\sigma(N)$ be the linear character such that $\xi_N = e^\vartheta$ for $e = \xi(1)$. Then, there exists a $\sigma$-invariant algebra subgroup $L \leq P$ with $N \leq L$ and a linear character $\eta \in \text{Irr}_\sigma(L)$ such that $\xi = \eta^P$.

Proof. Let $L \leq P$ be a $\sigma$-invariant algebra subgroup containing $N$ and such that $L/N$ is a maximal isotropic subgroup of $P/N$ with respect to the commutator pairing $c_\vartheta$. It is well known that $\xi = \xi^P$ for some linear character $\xi \in \text{Irr}(L)$; moreover, we must have $\xi_N = \vartheta$. On the other hand, by Lemma 2.2, there exists $\eta \in \text{Irr}_\sigma(L)$ such that $(\eta, \xi_L) \neq 0$ and $(\eta_N, \vartheta) \neq 0$. By Gallagher’s Theorem (see [19, Corollary 6.14]), we must have $\eta = \omega \xi$ for some $\omega \in \text{Irr}(L)$ with $N \leq \text{ker}(\omega)$. Since $L/N$ is abelian, $\omega$ is linear, and thus $\eta$ is also linear. By degree considerations, we conclude that $\xi = \eta^P$, as required. □

The Glauberman correspondence can now be used to establish the following result. (The proof is an obvious application of the previous theorem and Proposition 2.8.)

Proposition 2.11. Let $J$ be a $\sigma$-invariant nilpotent subalgebra of $A$, let $P = 1 + J$, and let $N = 1 + J^2$. Let $\xi \in \text{Irr}_\sigma(C_P(\sigma))$, and let $\hat{\xi} \in \text{Irr}_\sigma(P)$ be such that $\xi = \pi_P(\hat{\xi})$. Suppose that $\hat{\xi}$ is strongly Heisenberg, and let $\vartheta \in \text{Irr}_\sigma(N)$ be the linear character such that $\hat{\xi}_N = e^\vartheta$ for $e = \hat{\xi}(1)$. Then, there exists a $\sigma$-invariant algebra subgroup $L \leq P$ with $N \leq L$ and a linear character $\hat{\eta} \in \text{Irr}_\sigma(L)$ such that $\hat{\xi} = \hat{\eta}^P$ and $\xi = \hat{\eta}^{\pi(\sigma)}$.

Remark. In the notation of the theorem, let $c_\sigma : P/N \times P/N \to \mathbb{C}^\times$ be the commutator pairing associated with $\hat{\sigma}$, so that $L/N$ is a maximal isotropic subgroup of $P/N$ with respect to $c_\sigma$. Then, the quotient group $C_{\sigma}(\sigma)/C_N(\sigma)$ is a maximal isotropic subgroup of $C_P(\sigma)/C_N(\sigma)$ with respect to commutator pairing $c_\sigma$ associated with the linear character $\vartheta = \pi_N(\hat{\sigma}) \in \text{Irr}(C_P(\sigma))$. In fact, it is clear that $C_{\sigma}(\sigma)/C_N(\sigma)$ is an isotropic subgroup of $C_P(\sigma)/C_N(\sigma)$ with respect to $c_\sigma$. On the other hand, let $\hat{\eta} \in \text{Irr}_\sigma(L)$ be a $\sigma$-invariant extension of $\hat{\sigma}$. Then, $\eta = \pi_L(\hat{\eta}) \in \text{Irr}(C_L(\sigma))$ is an extension of $\vartheta$. Moreover, since $I_P(\hat{\eta}) = L$ (because $L/N$ is maximal isotropic with respect to $c_\sigma$), we also have $I_{C_P(\sigma)}(\vartheta) = C_L(\sigma)$; we observe that, for any $x \in I_{C_P(\sigma)}(\vartheta)$, we have $\pi_N(x) = \vartheta^x \in \vartheta$, and thus $\vartheta^x = \vartheta$ (because the Glauberman map is bijective). It follows that $\eta^{C_P(\sigma)}$ is irreducible, hence $C_L(\sigma)/C_N(\sigma)$ is a maximal isotropic with respect to $c_\sigma$.

Using Proposition 2.10 together with Proposition 2.4, we clearly deduce the following result.

Proposition 2.12. Let $J$ be a $\sigma$-invariant nilpotent subalgebra of $A$, let $P = 1 + J$, and let $\xi \in \text{Irr}_\sigma(P)$ be arbitrary. Then, there exists a $\sigma$-invariant algebra subgroup $Q \leq P$ and a $\sigma$-invariant linear character $\hat{\eta} \in \text{Irr}_\sigma(Q)$ such that $\xi = \hat{\eta}^Q$.

This also completes the proof of our main Theorem 2.1; the desired conclusion follows immediately by Proposition 2.8. We observe that the proof of Theorem 1.1 is also complete.

3. Irreducible characters for large primes

As before, let $(A, \sigma)$ be a finite-dimensional (associative) $F$-algebra, and let $J$ be a $\sigma$-invariant nilpotent subalgebra of $A$. In the case where $J$ satisfies $J^P = 0$, the irreducible characters of the algebra group $P = 1 + J$ may be described by Kirillov’s method of coadjoint orbits (see [15, Proposition 2]; see also [19, Theorem 7.7]). In fact, in this situation, we may define the usual exponential map $\exp : J \to P$ by $\exp a = 1 + a + a^2/2 + \cdots + a^{p-1}/(p-1)!$ for all $a \in J$. As is well known, this map is bijective, and its inverse is the logarithm map $\ln : P \to J$ defined by $\ln(1 + a) = \ln((1 + a)^p)$. Then, the logome
\[ a - a^2/2 + \cdots + (-1)^p a^{p-1}/(p-1) \] for all \( a \in J \); moreover, the condition \( J^p = 0 \) assures that the Campbell–Hausdorff formula holds (see [13, p. 175]). We consider the adjoint action of \( P \) on \( J \) given by \( a^\gamma = x^{-1}ax \) for all \( a \in J \) and all \( x \in P \), and observe that the exponential map defines a permutation isomorphism between this action and the action of \( P \) on itself by conjugation: we clearly have \( \exp(a^\gamma) = \exp(a)^x \) for all \( a \in J \) and all \( x \in P \). On the other hand, let \( J^\circ \) be the set consisting of all linear characters of the additive group \( J^+ \) of \( J \); hence, \( J^\circ = \text{Irr}(J^+) \). By duality, the group \( P \) acts on \( J^\circ \) via the coadjoint action: if \( \lambda \in J^\circ \) and \( x \in P \), then \( \lambda^x = \lambda(a^{-1}) \) for all \( a \in J \). We refer to the orbits of this action as the coadjoint \( P \)-orbits on \( J^\circ \). The orbit of \( \lambda \in J^\circ \) will be denoted by \( O_{\lambda} \), and its stabiliser by \( C_\lambda \). For each \( P \)-orbit \( O \subseteq J^\circ \), we define the map \( \hat{\xi}_O : P \to \mathbb{C} \) by

\[
\hat{\xi}_O(\exp a) = |O|^{-1/2} \sum_{\hat{\lambda} \in O} \hat{\lambda}(a)
\]

for all \( a \in J \); in the case where \( O = O_{\lambda} \) is the \( P \)-orbit of \( \lambda \in J^\circ \), we simplify the notation and write \( \hat{\xi}_\lambda \) instead of \( \hat{\xi}_{O_{\lambda}} \). It is easy to see that these functions form an orthonormal basis in the unitary space \( \mathfrak{f}(P) \) consisting of all class functions of \( P \). In fact, Kazhdan's Theorem asserts that each function \( \hat{\xi}_O \) is an irreducible character of \( P \), and that every irreducible character is of the form \( \hat{\xi}_O \) for some \( P \)-orbit \( O \subseteq J^\circ \) (see [15, Proposition 2]).

Now, let \( U \) be a subalgebra of \( J \), and let \( Q = 1 + U \) be the corresponding algebra subgroup of \( P \); we note that \( Q = \exp(U) \). For a given linear character \( \lambda \in J^\circ \), let \( \mu = \lambda_U \) be the restriction of \( \lambda \) to \( U \). Then, we can consider the irreducible character \( \hat{\xi}_{\mu} \in \text{Irr}(Q) \) associated with the coadjoint \( Q \)-orbit \( O_{\mu} \subseteq U^\circ \), and the induced character \( \hat{\xi}_{\mu}^P \). By [18, Theorem 1], we have

\[
(\hat{\xi}_{\mu})^P = \hat{\xi}_{\lambda} \quad \iff \quad |P : Q| = \frac{|C_Q(\mu)|}{|C_P(\lambda)|};
\]

in other words, \( (\hat{\xi}_{\mu})^P = \hat{\xi}_{\lambda} \) if and only if \( (\hat{\xi}_{\mu})^P \) and \( \hat{\xi}_{\lambda} \) have the same value at the identity.

A particular situation occurs when the subalgebra \( U \subseteq J \) is a \( \lambda \)-polarisation, that is, a subalgebra of \( J \) which, as a vector subspace, is maximal with respect to the condition \( \lambda([U, U]) = 1 \); a vector subspace \( V \) of \( J \) satisfying \( \lambda([V, V]) = 1 \) is said to be \( \lambda \)-isotropic. By Witt's Theorem (see [1, Theorems 3.10 and 3.11]), every maximal \( \lambda \)-isotropic subspace of \( J \) has dimension equal to \( (1/2)(\dim J + \dim C_J(\lambda)) \) where \( C_J(\lambda) = \{ a \in J \mid [a, J] \subseteq \ker(\lambda) \} \). Since \( 1 + C_J(\lambda) = C_P(\lambda) \) is the centraliser of \( \lambda \), we conclude that \( |O_P(\lambda)| = |P : Q|^2 \) where \( Q = 1 + U \) is the algebra subgroup of \( P \) associated with any \( \lambda \)-polarisation \( U \subseteq J \); in particular, \( |O_P(\lambda)| \) is a square power of \( |F| \). On the other hand, if \( U \subseteq J \) is an arbitrary \( \lambda \)-polarisation and \( \mu = \lambda_U \) is the restriction of \( \lambda \) to \( U \), it is easy to see that \( O_Q(\mu) = (\mu) \) where \( Q = 1 + U \). Therefore, the irreducible character \( \hat{\xi}_{\mu} \in \text{Irr}(Q) \) is given by \( \hat{\xi}_{\mu}(\exp a) = \mu(a) = \lambda(a) \) for all \( a \in U \). As remarked above, [18, Theorem 1] completes the proof of the following result. (We observe that \( \lambda \)-polarisations exist for all \( \lambda \in J^\circ \); a construction can be found in [6, Section 1.12] (see also the proof of Lemma 3.2 below).)

**Proposition 3.1.** Let \( J \) be a nilpotent subalgebra of \( A \) satisfying \( J^p = 0 \), and let \( P = 1 + J \). Let \( \lambda \in J^\circ \), let \( U \subseteq J \) be a \( \lambda \)-polarisation, and let \( Q = 1 + U \). Then, \( \hat{\xi}_{\lambda} = (\hat{\xi}_{\mu})^P \) where \( \hat{\xi}_{\lambda} : Q \to \mathbb{C} \) is the linear character defined by \( \hat{\xi}_{\lambda}(\exp a) = \lambda(a) \) for all \( a \in U \).

Next, we consider the \( \sigma \)-invariant irreducible characters of \( P \), and use Glauberman's correspondence to identify the irreducible characters of the group \( C_P(\sigma) \). We note that Kirillov's method also describes the irreducible characters of \( C_P(\sigma) \) in terms of the coadjoint \( C_P(\sigma) \)-orbits on \( C_J(\sigma)^O \); the construction follows exactly the same steps as above, and depends on the Lie algebra structure of \( C_J(\sigma) \) considered as a vector space over the \( \sigma \)-fixed point field \( F^\sigma \). The purpose of this section is to illustrate how the Glauberman correspondence \( \pi_P : \text{Irr}(P) \to \text{Irr}(C_P(\sigma)) \) can be used to describe the irreducible characters of \( C_P(\sigma) \) in terms of the \( \sigma \)-invariant coadjoint \( P \)-orbits on \( J^\circ \). We start by
observing that every linear character \( \lambda \in C_J(\sigma)^0 \) is extendible to \( J \); in fact, every element \( a \in J \) decomposes uniquely as the sum \( a = \frac{1}{2}(a + \sigma(a)) + \frac{1}{2}(a - \sigma(a)) \), and hence \( \lambda \) is the restriction to \( C_J(\sigma) \) of the linear character \( \hat{\lambda} \in J^\circ \) defined by \( \hat{\lambda}(a) = \lambda(\frac{1}{2}(a - \sigma(a))) \) for all \( a \in J \). We have the following result.

**Lemma 3.2.** Let \( J \) be a \( \sigma \)-invariant nilpotent subalgebra of \( A \), let \( \lambda \in C_J(\sigma)^0 \), and let \( \hat{\lambda} \in J^\circ \) be defined by \( \hat{\lambda}(a) = \lambda(\frac{1}{2}(a - \sigma(a))) \) for all \( a \in J \). Then, there exists a \( \sigma \)-invariant \( \hat{\lambda} \)-polarisation \( U \leq J \). Moreover, for any \( \sigma \)-invariant \( \hat{\lambda} \)-polarisation \( U \leq J \), the vector subspace \((\text{over } F^\sigma)C_U(\sigma) \leq C_J(\sigma) \) is maximal \( \hat{\lambda} \)-isotropic.

**Proof.** For the first assertion, it is enough to choose a chain \( 0 = J_0 \subset J_1 \subset \cdots \subset J_m = J \) of \( \sigma \)-invariant ideals of \( J \) satisfying \( \dim J_k/J_{k-1} = 1 \) for all \( 1 \leq k < m \) (for the existence of this chain we refer to the proof of Lemma 2.9). Then, for each \( 1 \leq k \leq m \), \( C_{J_k}(\hat{\lambda}) = \{a \in J_k \mid [a, J_k] \leq \ker(\hat{\lambda})\} \) is a \( \sigma \)-invariant vector subspace of \( J \); in fact, it is straightforward to check that \( \hat{\lambda}((\sigma(a), b)) = \hat{\lambda}((a, \sigma(b))) \) for all \( a, b \in J \). By [6, Lemma 11.23], \( U = C_{J_1}(\hat{\lambda}) + \cdots + C_{J_m}(\hat{\lambda}) \) is a \( \lambda \)-polarisation of \( J \) which is clearly \( \sigma \)-invariant.

For the last assertion, let \( U \leq J \) be an arbitrary \( \sigma \)-invariant \( \hat{\lambda} \)-polarisation. Then, \( C_U(\sigma) \) is an \( F^\sigma \)-vector subspace satisfying \( \lambda((a, b)) = 1 \) for all \( a, b \in C_U(\sigma) \); we recall that \( C_U(\sigma) \) is a Lie subalgebra of \( U \). On the other hand, let \( a \in C_J(\sigma) \) be such that \( \lambda((a, c)) = 1 \) for all \( a \in C_U(\sigma) \), and let \( u \in U \) be arbitrary. Then, \( \hat{\lambda}((u, c)) = \hat{\lambda}(\frac{1}{2}((u, c) - \sigma((u, c)))) = \hat{\lambda}(\frac{1}{2}(u - \sigma(u)), c)) = 1 \), and so the vector subspace \( U + Fc \leq J \) is \( \hat{\lambda} \)-isotropic. By the maximality of \( U \), we conclude that \( c \in U \), and this completes the proof. \( \square \)

We are now able to identify the \( \sigma \)-invariant irreducible characters of \( P \), and hence the irreducible characters of \( C_P(\sigma) \).

**Proposition 3.3.** Let \( J \) be a \( \sigma \)-invariant nilpotent subalgebra of \( A \) satisfying \( J^0 = 0 \), and let \( P = J + 1 \). For each \( \lambda \in C_J(\sigma)^0 \), let \( \hat{\lambda} \in J^\circ \) be defined by \( \hat{\lambda}(a) = \lambda(\frac{1}{2}(a - \sigma(a))) \) for all \( a \in J \). Then,

\[
\begin{align*}
\text{(i)} & \quad \text{Irr}_\sigma(P) = \{\xi_\lambda \mid \lambda \in C_J(\sigma)^0\}; \\
\text{(ii)} & \quad \text{Irr}(C_P(\sigma)) = \{\xi_\lambda \mid \lambda \in C_J(\sigma)^0\} \text{ where } \xi_\lambda = \pi_P(\xi_\hat{\lambda}) \text{ for } \lambda \in C_J(\sigma)^0.
\end{align*}
\]

**Proof.** It is enough to prove (i) because \( \text{Irr}(C_P(\sigma)) = \pi_P(\text{Irr}_\sigma(P)) \) (by Glauberman’s Theorem). Let \( \lambda \in C_J(\sigma)^0 \) be arbitrary, let \( U \leq J \) be a \( \sigma \)-invariant \( \hat{\lambda} \)-polarisation, and consider the \( \sigma \)-invariant algebra group \( Q = 1 + U \) of \( P \). Then, the linear character \( \hat{\varphi}_\lambda \) of \( Q \) is \( \sigma \)-invariant; in fact, since \( (\exp a)^{-1} = \exp(-a) \), we have \( \exp(a)^\sigma = \exp(-\sigma(a)) \) for all \( a \in J \), and thus

\[
\hat{\varphi}_\lambda((\exp a)^\sigma) = \hat{\lambda}(\sigma(a)) = \lambda(\frac{1}{2}(\sigma(a) + a)) = \hat{\lambda}(a) = \hat{\varphi}_\lambda(\exp a)
\]

for all \( a \in J \). It follows that the induced character \( \xi_\lambda = (\hat{\varphi}_\lambda)^P \) is also \( \sigma \)-invariant. Conversely, let \( \hat{\mu} \in J^\circ \) be such that \( \xi_{\hat{\mu}} \in \text{Irr}_\sigma(P) \), and let \( O_P(\hat{\mu}) \) be the coadjoint \( P \)-orbit which contains \( \hat{\mu} \). Let \( \hat{\mu}^\sigma \in J^\circ \) be defined by \( \hat{\mu}^\sigma(a) = \hat{\mu}(-\sigma(a)) \) for all \( a \in J \). It is straightforward to show that \( (\mu^\sigma)^P = P \) is a \( P \)-orbit on \( J^\circ \), and thus \( \xi_{\hat{\mu}}((\exp(\sigma(a))) = \hat{\varphi}_{\mu^\sigma}(\exp a) \) for all \( a \in J \). It follows that \( \hat{\mu}^\sigma \in O_P(\hat{\mu}) \), and thus the mapping \( \hat{\varphi} \mapsto \hat{\varphi}^\sigma \) defines an action of the cyclic group \( \langle \sigma \rangle \) on the \( P \)-orbit \( O_P(\hat{\mu}) \). By Glauberman’s Lemma (see [11, Lemma 13.8]), we conclude that there exists \( \hat{\lambda} \in O_P(\hat{\mu}) \) such that \( \hat{\lambda}^\sigma = \hat{\lambda} \); that is, such that \( \hat{\lambda}(\sigma(a)) = \hat{\lambda}(a) \) for all \( a \in J \). Therefore, we have \( \hat{\lambda}(\frac{1}{2}(a + \sigma(a))) = 1 \), and so \( \hat{\lambda}(a) = \hat{\lambda}(\frac{1}{2}(a - \sigma(a))) \) for all \( a \in J \). It is thus enough to define \( \lambda \in C_J(\sigma)^0 \) by \( \lambda = \hat{\lambda}_{C_J(\sigma)} \). \( \square \)

As a consequence of Proposition 2.8, we easily deduce the following result.
Let \( U \) be a \( \sigma \)-invariant nilpotent subalgebra of \( A \) satisfying \( J^0 = 0 \), and let \( P = 1 + J \). Let \( \lambda, \mu \in C_j(\sigma)^\circ \), and let \( \xi_\lambda, \xi_\mu \in \text{Irr}(\text{C}_P(\sigma)) \) be as above. Then, \( \xi_\lambda = \xi_\mu \) if and only if \( \lambda \) and \( \mu \) lie in the same \( \text{C}_P(\sigma) \)-orbit on \( C_j(\sigma)^\circ \).

Proof. Let \( \lambda, \mu \in C_j(\sigma)^\circ \) be arbitrary, and let \( \hat{\lambda}, \hat{\mu} \in J^0 \) be defined as before; hence, by the definition, we have \( \hat{\xi}_\lambda = \pi_P(\hat{\xi}_\lambda) = \pi_P(\hat{\xi}_\mu) \). Since the Glauberman correspondence is bijective, we conclude that \( \hat{\xi}_\lambda = \hat{\xi}_\mu \) if and only if \( \xi_\lambda = \xi_\mu \). As above, we consider the action of \( \sigma \) on \( J^0 \) given by \( \hat{\nu}^\sigma(a) \) for all \( \hat{\nu} \in J^0 \) and all \( a \in J \). Since \( O_P(\hat{\lambda}) \) is clearly \( \sigma \)-invariant (because \( \hat{\lambda}^\sigma = \hat{\lambda} \)), \( \text{C}_Q(\sigma) \) is maximal \( \lambda \)-isotropic (by \text{Lemma 3.2.2}), we have \( O_{\text{C}_Q(\sigma)}(\lambda) = \{ \lambda \} \), and also \( |O|^\frac{1}{2} = |C_P(\sigma) : \text{C}_Q(\sigma)| \) (by Witt's Theorem). Let \( \hat{\sigma}_\lambda \) be the linear character of \( \text{C}_Q(\sigma) \) defined by \( \hat{\sigma}_\lambda(\exp a) = \hat{\lambda}(a) \) for all \( a \in C_U(\sigma) \). Then, by \text{[18, Theorem 1]}, we conclude that
\[
(\hat{\sigma}_\lambda)_{\text{C}_P(\sigma)}(\exp a) = \frac{1}{|O|^\frac{1}{2}} \sum_{\mu \in 0} \mu(a)
\]
for all \( a \in C_j(\sigma) \), and the result follows because \( \xi_\lambda = (\hat{\sigma}_\lambda)_{\text{C}_P(\sigma)} \) (by \text{Proposition 3.4}).
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