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Boundedness and Convergence to Zero of Solutions of a Forced Second-Order Nonlinear Differential Equation

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Sufficient conditions for continuability, boundedness, and convergence to zero of solutions of $(a(t)x')' + h(t, x, x') + q(t)f(x)g(x') = e(t, x, x')$ are given.

1. INTRODUCTION

In this paper we discuss the boundedness and covergence to zero of solutions of the forced second-order nonlinear differential equation

$$(a(t)\, x')' + h(t, x, x') + q(t) f(x)\, g(x') = e(t, x, x'). \tag{*}$$

Problems of this type for less general equations have been studied by many authors particularly when $e(t, x, x') = 0$. Recent contributions to this area include [1–29]. In addition to relaxing the conditions that most other authors require on the functions in (*), none of the results in this paper explicitly require that the forcing term $e(t, x, x')$ be "small."

In Section 2 we present some new continuability and boundedness results for Eq. (*). In addition to obtaining some further boundedness results in Section 3, we obtain sufficient conditions for solutions of (*) to converge to zero. We will relate the results here to the recent work of Grimmer [10], Hammett [11], and Londen [20]. We conclude the paper with some extensions of results of Wong [27, 28] and the present authors [7].

2. CONTINUABILITY AND BOUNDEDNESS

Consider the equation

$$(a(t)\, x')' + h(t, x, x') + q(t) f(x)\, g(x') = e(t, x, x'), \tag{1}$$

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where $a, q\colon [t_0, \infty) \to R$, $f, g\colon R \to R$, and $h, e\colon [t_0, \infty) \times R^2 \to R$ are continuous, $a(t) > 0$, $q(t) > 0$, and $g(x') > 0$. It will be convenient to write Eq. (1) as the system

$$x' = y,$$
$$y' = (-a'(t)\, y - h(t, x, y) - q(t) f(x) g(y) + e(t, x, y))/a(t). \tag{2}$$

Let $q'(t)_+ = \max\{q'(t), 0\}$ and $q'(t)_- = \max\{-q'(t), 0\}$ so that $q'(t) = q'(t)_+ - q'(t)_-$. Define $F(x) = \int_0^x f(s)\, ds$, $G(y) = \int_0^y [s/g(s)]\, ds$ and assume that there is a continuous function $r\colon [t_0, \infty) \to R$ such that

$$|\, e(t, x, y)| \leqslant r(t), \tag{3}$$

$$h(t, x, y)\, y \geqslant 0, \tag{4}$$

and there are nonnegative constants m and n such that

$$|\, y\, |/g(y) \leqslant m + nG(y). \tag{5}$$

THEOREM 1. *If conditions* (3)–(5) *hold,* $a'(t) \geqslant 0$, $F(x)$ *is bounded from below, and* $G(y) \to \infty$ *as* $|\, y\, | \to \infty$, *then all solutions of* (2) *can be defined for all* $t \geqslant t_0$.

Proof. Suppose that $(x(t), y(t))$ is a solution of (2) with finite escape time, i.e., there exists $T > t_0$ such that $\lim_{t \to T^-}[|\, x(t)| + |\, y(t)|] = +\infty$. Since $F(x)$ is bounded from below, $F(x) \geqslant -K$ for some $K > 0$. Define $V(x, y, t) = G(y)/q(t) + (F(x) + K)/a(t)$; then $V' = -G(y)\, q'(t)/q^2(t) - a'(t)\, y^2/g(y)\, q(t)\, a(t) - h(t, x, y)\, y/g(y)\, q(t)\, a(t) + e(t, x, y)\, y/g(y)\, q(t)\, a(t) - (F(x) + K)\, a'(t)/a^2(t) \leqslant q'(t)_-\, G(y)/q^2(t) + r(t)\, |\, y\, |/g(y)\, q(t)\, a(t) \leqslant q'(t)_-\, G(y)/q^2(t) + mr(t)/q(t)\, a(t) + nr(t)\, G(y)/q(t)\, a(t)$. Integrating and noting that $r(t)/q(t)\, a(t)$ is bounded on $[t_0, T]$ we have $G(y(t))/q(t) \leqslant V(t) \leqslant K_1 + \int_{t_0}^t \{[q'(s)_-/q(s) + nr(s)/a(s)] \times G(y(s))/q(s)\}\, ds$ for some $K_1 > 0$. From Gronwall's inequality we have $G(y(t))/q(t) \leqslant K_1 \exp \int_{t_0}^t [q'(s)_-/q(s) + nr(s)/a(s)]\, ds \leqslant K_1 \exp \int_{t_0}^T [q'(s)_-/q(s) + nr(s)/a(s)]\, ds \leqslant K_2 < \infty$. Thus $G(y(t))$ is bounded on $[t_0, T)$ so $y(t) = x'(t)$ is bounded on $[t_0, T)$. An integration shows that $x(t)$ is also bounded on $[t_0, T)$ and so we have a contradiction to the assumption that $(x(t), y(t))$ is a solution of (2) with finite escape time.

Remark. If $e(t, x, y) \equiv 0$ in Theorem 1, then condition (5) can be dropped.

Remark. We can drop the condition on $a'(t)$ by requiring a stronger condition on $g(y)$, namely, that there are positive constants M and k such that

$$y^2/g(y) \leqslant MG(y) \qquad \text{for } |\, y\, | \geqslant k. \tag{6}$$

The proof of this result involves more details than the proof of Theorem 1, so we omit it noting only that (6) implies (5).

The above continuability theorem improves other known results of this type for Eq. (1) in that our conditions on f and g are less restrictive than those usually required. We have not asked that $xf(x) > 0$ if $x \neq 0$ or $F(x) \geqslant 0$ as most authors (see for example Baker [1] or Burton and Grimmer [3]), but only require that $F(x)$ be bounded from below. Also, conditions (5) and (6) are less restrictive than bounding g from above and below or asking that $y^2/g(y) \leqslant MG(y)$ for all y (see [3]). These comments apply to the boundedness results in this paper as well.

Next, we given some sufficient conditions for all solutions of (1) to be bounded. The first two of these, as well as some of the theorems in the next section, serve to further illustrate the interplay between the roles of $a(t)$ and $g(x')$ in Eq. (1).

THEOREM 2. *Suppose* (4) *and* (5) *hold with* $n > 0$,

$$a'(t) \geqslant 0 \qquad \text{and} \qquad a(t) \leqslant a_2 \tag{7}$$

and $|e(t, x, y)| \leqslant a(t)\, q'(t)/nq(t)$. *If*

$$F(x) \to \infty \qquad \text{as} \qquad |x| \to \infty, \tag{8}$$

then all solutions of (1) *are bounded.*

Proof. Since $F(x) \to \infty$ as $|x| \to \infty$, $F(x)$ is bounded from below, say $F(x) \geqslant -K$ for some $K > 0$. Letting $V(x, y, t) = q(t)\,(F(x) + K)/a(t) + G(y)$ we have $V' \leqslant (F(x) + K)\, q'(t)/a(t) + e(t, x, y)\, y/g(y)\, a(t) \leqslant q'(t)\, [q(t)\,(F(x) + K)/a(t) + |y|/ng(y)]/q(t)$. Integrating V' and using (5) we have $|y(t)|/g(y(t)) + nq(t)\,(F(x(t)) + K)/a(t) \leqslant m + nV(t) \leqslant m + nV(t_0) + \int_{t_0}^t \{q'(s)\,[nq(s)\,(F(x(s)) + K)/a(s) + |y(s)|/g(y(s))]/q(s)\}\, ds$. Applying Gronwall's inequality we have $|y(t)|/g(y(t)) + nq(t)\,(F(x(t)) + K)/a(t) \leqslant K_1 \exp \int_{t_0}^t [q'(s)/q(s)]\, ds = K_1 q(t)/q(t_0)$. Hence $nq(t)\, F(x(t))/a(t) \leqslant K_1 q(t)/q(t_0)$ so $F(x(t))$ is bounded for $t \geqslant t_0$. The conclusion of the theorem follows from (8).

THEOREM 3. *Suppose* (4), (6), *and* (8) *hold,*

$$\int_{t_0}^{\infty} [a'(s)_-/a(s)]\, ds < \infty \qquad \text{and} \qquad a(t) \leqslant a_2 \tag{9}$$

and $|e(t, x, y)| \leqslant a(t)\, q'(t)/Mq(t)$. *Then all solutions of* (1) *are bounded.*

Proof. Condition (6) implies that there exists $A > 0$ such that $y^2/g(y) \leqslant A + MG(y)$ for all y. Notice also that if $|y| \leqslant 1$, then $|y|/g(y) \leqslant B$ for some $B > 0$, and if $|y| \geqslant 1$, then $|y|/g(y) \leqslant y^2/g(y)$ so $|y|/g(y) \leqslant B + y^2/g(y)$ for all y. By (8), $F(x) \geqslant -K$ for some $K > 0$. If $M \geqslant 1$, define $V(x, y, t) = q(t)\,(F(x) + K)/a(t) + G(y) + A + B$. Then $V' \leqslant q'(t)\,[q(t)\,(F(x) + K)/a(t) + |y|/Mg(y)]/q(t) + a'(t)_-\, [q(t)\,(F(x) + K)/a(t) + y^2/g(y)]/a(t) \leqslant q'(t) \times$

$[q(t)(F(x) + K)/a(t) + (A + B)/M + G(y)]/q(t) + a'(t)_- [q(t)(F(x) + K)/a(t) + A + MG(y)]/a(t) \leqslant (q'(t)/q(t) + Ma'(t)_-/a(t)) [q(t)(F(x) + K)/a(t) + A + B + G(y)]$ since $M \geqslant 1$. Integrating, we have $V(t) \leqslant V(t_0) + \int_{t_0}^t [q'(s)/q(s) + Ma'(s)_-/a(s)] V(s)\, ds$ so $V(t) \leqslant V(t_0) \exp \int_{t_0}^t [q'(s)/q(s) + Ma'(s)_-/a(s)]\, ds \leqslant K_1 \exp \int_{t_0}^t [q'(s)/q(s)]\, ds = K_1 q(t)/q(t_0)$. The boundedness of $x(t)$ follows as in the proof of the previous theorem.

If $M < 1$, define $V(x, y, t) = q(t)(F(x) + K)/a(t) + G(y) + (A + B)/M$. Then $V' \leqslant q'(t) [q(t)(F(x) + K)/a(t) + (A + B)/M + G(y)] q(t) + a'(t)_- \times [q(t)(F(x) + K)/a(t) + A + MG(y)]/a(t) \leqslant (q'(t)/q(t) + a'(t)_-/a(t)) V$ since $M < 1$. The remainder of the proof follows as before.

The following corollary is a rather immediate consequence of the previous two theorems.

COROLLARY 4. *If, in addition to the hypotheses of either Theorem* 2 *or Theorem* 3, *we have* $q(t) \leqslant q_2$ *and* $G(y) \to \infty$ *as* $|y| \to \infty$, *then all solutions of system* (2) *are bounded.*

Proof. From the proof of Theorem 2 we have $V' \leqslant q'(t) [V + m/n]/q(t)$ so $V(t) \leqslant V(t_0) + \int_{t_0}^t [q'(s) V(s)/q(s)]\, ds + m \ln[q(t)/q(t_0)]/n$. Hence $V(t) \leqslant K_2 \exp \int_{t_0}^t [q'(s)/q(s)]\, ds \leqslant K_2 q_2/q(t_0) < \infty$. The boundedness of $y(t)$ then follows from the boundedness of $G(y(t))$. A similar proof holds for Theorem 3.

Notice that Theorems 2 and 3 do not explicitly require that $e(t, x, x')$ be small; most authors ask that $e(t, x, x') \equiv e(t)$ and $\int_{t_0}^\infty |e(s)|\, ds < \infty$. However, in order to obtain that solutions of system (2) are bounded, it was necessary to bound $q(t)$ from above. This immediately implies that $\int_{t_0}^\infty |e(s, x(s), x'(s))|\, ds < \infty$. It would be interesting to see if, under condition (7) or (9), solutions of (2) can be bounded without requiring $q(t)$ to be bounded from above. The authors are not aware of such a result for even the equation $(a(t) x')' + q(t) f(x) = 0$.

The previous two theorems offer alternative generalizations of a result obtained by the authors in [8]. The following theorem is patterned somewhat after a theorem in [9]; however, it is not a direct generalization of that result.

THEOREM 5. *Suppose conditions* (4), (6), (8), *and* (9) *hold and there is a continuous function* $r: [t_0, \infty) \to R$ *and a constant* $w > 0$ *such that* $|e(t, x, y) y| \leqslant q(t) g(y)/r^w(t)$, $\int_{t_0}^\infty [r'(s)_-/r(s)]\, ds < \infty$, $\int_{t_0}^\infty [1/r^w(s)]\, ds < \infty$, $H(t) = r(t)/q(t)$ *is bounded, and* $\int_{t_0}^\infty [H'(s)_-/H(s)]\, ds < \infty$. *Then all solutions of* (1) *are bounded.*

Proof. As before, $F(x) \geqslant -K$ for some $K > 0$ so let $V(x, y, t) = G(y)/r(t) + (F(x) + K)/a(t) H(t)$. Then $V' \leqslant r'(t)_- G(y)/r^2(t) + a'(t)_- y^2/g(y) r(t) a(t) + e(t, x, y) y/g(y) r(t) a(t) + H'(t)_- (F(x) + K)/a(t) H^2(t) + a'(t)_- (F(x) + K)/a^2(t) H(t) \leqslant (r'(t)_-/r(t) + H'(t)_-/H(t) + a'(t)_-/a(t)) V + a'(t)_- y^2/g(y) r(t) a(t) + q(t)/r^{1+w}(t) a(t)$. Now the condition $\int_{t_0}^\infty [r'(s)_-/r(s)]\, ds < \infty$ implies $r(t) \geqslant r_1 > 0$, and similarly, $H(t) \geqslant H_1 > 0$ and $a(t) \geqslant a_1 > 0$. Thus choosing A as in the proof of Theorem 3, we have $V' \leqslant [r'(t)_-/r(t) + H'(t)_-/H(t) + (M + 1) \times$

$a'(t)_{-}/a(t)]\, V + Aa'(t)_{-}/a(t)\, r_1 + 1/r^w(t)\, H_1 a_1$. Integrating and applying Gronwall's inequality we again obtain that $V(t)$ is bounded. The conclusion of the theorem follows as before.

Remark. If in Theorem 5 we replace condition (9) by condition (7), then (6) can be dropped.

3. Boundedness and Convergence to Zero

In this section we obtain some further boundedness results as well as sufficient conditions for solutions of (1) to converge to zero. The quotient $H(t) = r(t)/q(t)$ plays a significant role in some of these results. Other authors, for example, Chang [4], Jones [13, 14], Lalli [19], Wong [26], and Zarghamee and Mehri [29], utilized the quotient $a(t)/q(t)$. The present authors [7–9] obtained some results of this type for less general equations, and in [8, 9] $H(t)$ was required to be monotonic. Wong [27, 28] gave sufficient conditions for all oscillatory solutions of a less general unforced version of (1) to converge to zero. In Theorems 12 and 13 we extend these results to Eq. (1).

The theorems in this section only pertain to the continuable solutions of (1). Since the previous section contained some sufficient conditions for solutions to be continuable, we could combine those results with the ones in this section and thus eliminate this provision. We will use the same classification of solutions that was used in [7–9]. That is, a solution $x(t)$ of (1) will be called nonoscillatory if there exists $t_1 \geqslant t_0$ such that $x(t) \neq 0$ for $t \geqslant t_1$; the solution will be called oscillatory if for any given $t_1 \geqslant t_0$ there exist t_2 and t_3 satisfying $t_1 < t_2 < t_3$, $x(t_2) > 0$, and $x(t_3) < 0$; and it will be called a Z-type solution if it has arbitrarily large zeros but is ultimately nonnegative or nonpositive.

The following lemma will be used in proving the convergence to zero of the nonoscillatory solutions of (1). We make the following additional assumptions on Eq. (1). Assume that:

(i) $xf(x) > 0$ if $x \neq 0$ and $f(x)$ is bounded away from zero if x is bounded away from zero;

(ii) condition (3) holds and $r(t)/q(t) \to 0$ as $t \to \infty$;

(iii) if x is bounded, then there exists a continuous function k and $t_1 \geqslant t_0$ such that $|h(t, x, y)| \leqslant k(t)\, g(y)$ for (t, x, y) in $[t_1, \infty) \times R^2$ and $k(t)/q(t) \to 0$ as $t \to \infty$;

(iv) $g(y) \geqslant c > 0$, $\int_{t_0}^{\infty} q(s)\, ds = \infty$, and $\int_{t_0}^{\infty} [1/a(s)]\, ds = \infty$.

Lemma 6. *If* (i)–(iv) *hold and* $x(t)$ *is a bounded nonoscillatory solution of* (1), *then* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (1), say $0 < x(t) < B$ for $t \geqslant T \geqslant t_0$, and let $k(t)$ and $t_1 \geqslant T$ be determined by (iii). Suppose that $\liminf_{t\to\infty} x(t) \neq 0$. Then there exists $t_2 \geqslant t_1$ such that $x(t)$ is bounded away from zero for $t \geqslant t_2$. Hence by (i), $f(x(t)) \geqslant A > 0$ for $t \geqslant t_2$. Choose $t_3 \geqslant t_2$ so that $r(t)/cq(t) < A/4$ and $k(t)/q(t) < A/4$ for $t \geqslant t_3$. From (1) we have $(a(t)\,x')'/g(x') = e(t, x, x')/g(x') - h(t, x, x')/g(x') - q(t)\,f(x) \leqslant r(t)/c + k(t) - Aq(t) \leqslant q(t)\,[r(t)/cq(t) + k(t)/q(t) - A] \leqslant -Aq(t)/2$ for $t \geqslant t_3$. Thus $(a(t)\,x')' \leqslant -Acq(t)/2 < 0$ for $t \geqslant t_3$. Integrating, we have $a(t)\,x'(t) \leqslant a(t_3)\,x'(t_3) - \int_{t_3}^{t} [Acq(s)/2]\,ds \to -\infty$ as $t \to \infty$, so there exists $t_4 \geqslant t_3$ such that $x'(t) < 0$ for $t \geqslant t_4$. Since $(a(t)\,x')' < 0$, $x'(t) < a(t_4)\,x'(t_4)/a(t)$ for $t \geqslant t_4$. Hence $x(t) < x(t_4) + a(t_4)\,x'(t_4) \int_{t_4}^{t} [1/a(s)]\,ds \to -\infty$ as $t \to \infty$, contradicting the fact that $x(t) > 0$ for $t \geqslant T$. A similar argument holds if $x(t) < 0$ for $t \geqslant T$.

The following two examples illustrate that condition (iii) above is essential.

EXAMPLE 1. Consider the equation

$$x'' + tx' + x/t = 1/t^2 + 2/t^3, \qquad t > 0.$$

Here $g(x') \equiv 1$ and $h(t, x, x') = tx'$ and we see that all the hypotheses of Lemma 6 are satisfied except (iii) since we do not have $|\,h(t, x, x')| \leqslant k(t)\,g(x')$. This equation has the bounded nonoscillatory solution $x(t) = (t + 1)/t$ which does not have $\liminf_{t\to\infty} x(t) = 0$.

EXAMPLE 2. The equation

$$x'' + tx' + x[1 + (x')^2]/t = (t^4 + 2t^3 + t + 1)/t^6, \qquad t > 0,$$

satisfies all the conditions of Lemma 6 except (iii). Here $|\,h(t, x, x')| = t\,|\,x'\,| \leqslant t[1 + (x')^2] = tg(x')$, but $k(t)/q(t) = t^2 \not\to 0$ as $t \to \infty$. Again, $x(t) = (t + 1)/t$ is a bounded nonoscillatory solution of this equation.

Recently Hammett [11] obtained sufficient conditions for the nonoscillatory solutions of (1) to converge to zero in case

$$h(t, x, x') \equiv 0, \qquad g(x') \equiv 1, \qquad e(t, x, x') \equiv e(t),$$

and

$$\int_{t_0}^{\infty} |\,e(s)|\,ds < \infty.$$

In addition to making other improvements, Grimmer [10] was able to relax Hammett's condition on the size of $e(t)$ by only requiring that $E(t) = \int_{t_0}^{t} e(s)\,ds$ be bounded. Although Londen [20] weakened some of Hammett's other hypotheses, he still required $\int_{t_0}^{t} |\,e(s)|\,ds < \infty$. Our results on the convergence to zero of the nonoscillatory solutions of (1), namely Theorems 7–9 below, will allow for large forcing terms; we may even have $e(t) \to \infty$ as $t \to \infty$. The theorems obtained here are not direct generalizations of those in [10, 11] or [20].

In fact, as we will show by some examples, our results are in some sense independent of those in [10, 11, and 20].

In what follows it will be convenient to have the following notation at our disposal.

Condition W. If $x(t)$ is a nonoscillatory or Z-type solution of (1), then $\lim_{t\to\infty} x(t) = 0$.

Also, we define $p(t) = \exp(-\int_{t_0}^{t} [q'(s)_-/q(s)]\, ds)$ and $b(t) = \exp(-\int_{t_0}^{t} [a'(s)_-/a(s)]\, ds)$ and notice that $p(t) \leqslant 1$ and $b(t) \leqslant 1$.

THEOREM 7. *Suppose conditions* (3), (4), (8), *and* (9) *hold*,

$$\int_{t_0}^{\infty} [q'(s)_-/q(s)]\, ds < \infty, \tag{10}$$

$$\int_{t_0}^{\infty} [r(s)/q(s)]\, ds < \infty, \tag{11}$$

and there is a positive constant N such that

$$y^2/g(y) \leqslant N. \tag{12}$$

Then all solutions of (1) *are bounded. If, in addition* (i)–(iv) *hold, then Condition* W *holds.*

Proof. From (8), $F(x) \geqslant -K$ for some $K > 0$. Let $V(x, y, t) = b(t)\, p(t)\, [(F(x) + K)/a(t) + G(y)/q(t)]$; then $V' = b(t)\, p(t)\, \{-(F(x) + K) \times a'(t)/a^2(t) - G(y)\, q'(t)/q^2(t) - a'(t)\, y^2/g(y)\, q(t)\, a(t) - h(t, x, y)\, y/g(y)\, q(t)\, a(t) + e(t, x, y)\, y/g(y)\, q(t)\, a(t) - [(F(x) + K)/a(t) + G(y)/q(t)]\, (a'(t)_-/a(t) + q'(t)_-/q(t))\} \leqslant b(t)\, p(t)\, \{-(F(x) + K)\, a'(t)_+/a^2(t) - G(y)\, q'(t)_+/q^2(t) - (F(x) + K) \times q'(t)_-/q(t)\, a(t) - G(y)\, a'(t)_-/q(t)\, a(t) + a'(t)_- y^2/g(y)\, q(t)\, a(t) + e(t, x, y)\, y/g(y) \times q(t)\, a(t)\} \leqslant b(t)\, p(t)\, \{a'(t)_-\, y^2/g(y)\, q(t)\, a(t) + r(t)\, |y|/g(y)\, q(t)\, a(t)\}$. Now (9) and (10) imply that $q(t) \geqslant q_1 > 0$, $p(t) \geqslant p_1 > 0$, $a(t) \geqslant a_1 > 0$ and $b(t) \geqslant b_1 > 0$. Also, $|y|/g(y)$ is bounded for $|y| \leqslant 1$ and $|y|/g(y) \leqslant y^2/g(y)$ if $|y| \geqslant 1$ so $|y|/g(y) \leqslant N_1$ for all y.

Integrating V' we have $V(t) \leqslant V(t_0) + (N/q_1) \int_{t_0}^{t} [a'(s)_-/a(s)]\, ds + (N_1/a_1) \int_{t_0}^{t} [r(s)/q(s)]\, ds \leqslant K_1 < \infty$ for all $t \geqslant t_0$. Hence $F(x(t)) \leqslant K_1 a(t)/b(t)\, p(t) \leqslant K_1 a_2/b_1 p_1$ for $t \geqslant t_0$ and so by (8), $x(t)$ is bounded.

Next let $x(t)$ be a nonoscillatory or Z-type solution of (1). Note that by (i) we can choose $K = 0$. Since $\liminf_{t\to\infty} |x(t)| = 0$ by Lemma 6, if $x(t)$ is ultimately monotonic, we are done. If $x(t)$ is not ultimately monotonic let $\epsilon > 0$ be given and choose $t_1 \geqslant t_0$ so that (iii) is satisfied for $t \geqslant t_1$, $y(t_1) = 0$, $F(x(t_1)) < a_1 b_1 p_1 \epsilon/3a_2$, $\int_{t_1}^{\infty} [a'(s)_-/a(s)]\, ds \leqslant b_1 p_1 q_1 \epsilon/3a_2 N$ and $\int_{t_1}^{\infty} [r(s)/q(s)]\, ds < a_1 b_1 p_1 \epsilon/3a_2 N_1$. Then integrating V' for $t \geqslant t_1$, we have $F(x(t)) \leqslant a(t)\, V(t)/b(t) p(t) \leqslant a_2 V(t)/b_1 p_1 \leqslant a_2 V(t_1)/b_1 p_1 + (a_2 N/q_1 b_1 p_1) \times \int_{t_1}^{t} [a'(s)_-/a(s)]\, ds + (a_2 N_1/a_1 b_1 p_1) \int_{t_1}^{t} [r(s)/q(s)]\, ds < \epsilon$ for $t \geqslant t_1$. This implies that $\lim_{t\to\infty} x(t) = 0$ since by (i) $F(x(t)) \to 0$ if and only if $x(t) \to 0$.

THEOREM 8. *Suppose conditions* (3)–(4), (7)–(8), *and* (10)–(11) *hold, and there is a positive constant L such that*

$$|y|/g(y) \leqslant L. \tag{13}$$

Then all solutions of (1) *are bounded. Under the additional assumptions* (i)–(iv), *Condition* W *holds.*

Proof. We will use the same notation for constants introduced in the proof of Theorem 7. Let $V(x, y, t) = p(t)\ [(F(x) + K)/a(t) + G(y)/q(t)]$; then $V' \leqslant p(t)\, e(t, x, y)\, y/g(y)\, q(t)\, a(t)$. Now $a'(t) \geqslant 0$ implies $a(t) \geqslant a_1 > 0$, so integrating V' we obtain $V(t) \leqslant V(t_0) + (L/a_1) \int_{t_0}^{t} [r(s)/q(s)]\, ds < \infty$ so $x(t)$ is bounded.

Now let $x(t)$ be a nonoscillatory or Z-type solution of (1) and $\epsilon > 0$ be given. Following the argument used in the proof of the previous theorem, choose $t_1 \geqslant t_0$ so that (iii) is satisfied, $y(t_1) = 0$, $F(x(t_1)) < a_1 p_1 \epsilon/2a_2$, and $\int_{t_1}^{\infty} [r(\ )/q(s)]\, ds < a_1 p_1 \epsilon/2a_2 L$. Integrating, we have $F(x(t)) \leqslant a(t)\ V(t)/p(t) \leqslant a_2 F(x(t_1))/a_1 p_1 + (a_2 L/a_1 p_1) \int_{t_1}^{t} [r(s)/q(s)]\, ds < \epsilon$ for $t \geqslant t_1$. Hence $\lim_{t\to\infty} x(t) = 0$.

THEOREM 9. *If conditions* (3)–(4), (6), *and* (8)–(10) *hold*, $g(y) \geqslant c > 0$, *and*

$$\int_{t_0}^{\infty} [r(s)/(q(s))^{1/2}]\, ds < \infty, \tag{14}$$

then all solutions of (1) *are bounded. Moreover, if* (i)–(iv) *hold, then Condition* W *is satisfied.*

Proof. Defining V as in the proof of Theorem 7, differentiating, and integrating we have $b(t)\ p(t)\ G(y(t))/q(t) \leqslant V(t_0) + \int_{t_0}^{t} \{b(s)\ p(s)\ [a'(s)_-\ y^2(s) + r(s)\ |y(s)|]/g(y(s))\ q(s)\ a(s)\}\ ds$. If $|y|/(q(t))^{1/2} \geqslant 1$, then $|y|/(q(t))^{1/2} \leqslant y^2/q(t) < y^2/q(t) + 1$; if $|y|/(q(t))^{1/2} \leqslant 1$, then $|y|/(q(t))^{1/2} \leqslant 1 + y^2/q(t)$. By (6), $y^2/g(y) \leqslant A + MG(y)$ for all y. Hence $b(t)\ p(t)\ y^2(t)/g(y(t))\ q(t) \leqslant Ab(t) \times p(t)/q(t) + Mb(t)\, p(t)\ G(y(t))/q(t) \leqslant Ab(t)\, p(t)/q(t) + MV(t_0) + M \int_{t_0}^{t} \{b(s) \times p(s)\ [a'(s)_-/a(s) + r(s)/(q(s))^{1/2}\ a(s)]\ y^2(s)/g(y(s))\ q(s)\}\ ds + M \int_{t_0}^{t} [b(s)\ p(s) \times r(s)/g(y(s))\ (q(s))^{1/2}\ a(s)]\ ds$. Now $M \int_{t_0}^{t} [b(s)\ p(s)\ r(s)/g(y(s))\ (q(s))^{1/2}\ a(s)]\ ds \leqslant (M/ca_1) \int_{t_0}^{t} [r(s)/(q(s))^{1/2}]\ ds \leqslant K_1 < \infty$ so $b(t)\ p(t)\ y^2(t)/g(y(t))\ q(t) \leqslant K_2 \times \exp \int_{t_0}^{t} [a'(s)_-/a(s) + r(s)/(q(s))^{1/2}\ a_1]\ ds \leqslant K_3 < \infty$. Thus $V(t) \leqslant V(t_0) + K_3 \int_{t_0}^{t} [a'(s)_-/a(s) + r(s)/(q(s))^{1/2}\ a_1]\ ds + K_1 \leqslant K_4 < \infty$. Thus all solutions are bounded.

Let $x(t)$ be a nonoscillatory or Z-type solution of (1) and let $\epsilon > 0$ be given. Choose $t_1 \geqslant t_0$ so that (iii) is satisfied, $y(t_1) = 0$, $F(x(t_1)) < a_1 b_1 p_1 \epsilon/3a_2$, $\int_{t_1}^{\infty} [a'(s)_-/a(s)]\ ds < b_1 p_1 \epsilon/3a_2 K_3$, and $\int_{t_1}^{\infty} [r(s)/(q(s))^{1/2}]\ ds < a_1 b_1 p_1 c\epsilon/3a_2(cK_3 + 1)$. Then $F(x(t)) \leqslant a(t)\ V(t)/b(t)\, p(t) \leqslant a_2 V(t_1)/b_1 p_1 + (a_2 K_3/b_1 p_1) \int_{t_1}^{t} [a'(s)_-/a(s)]\ ds + (a_2 K_3/a_1 b_1 p_1) \int_{t_1}^{t} [r(s)/(q(s))^{1/2}]\ ds + (a_2/a_1 b_1 p_1 c) \int_{t_1}^{t} [r(s)/(q(s))^{1/2}]\ ds < \epsilon$ for $t \geqslant t_1$. Again we see that Condition W holds.

We will now consider some examples which will show the relationship between our results and those in [10, 11, 20].

EXAMPLE 3. The equation

$$x'' + t^3[(x')^2 + 1]\, x = t, \qquad t > 0,$$

satisfies Theorems 7 and 8 and the equation

$$x'' + t^5 x = t \sin t, \qquad t > 0,$$

satisfies Theorem 9, but none of the results in [10, 11] or [20] apply to either of these equations. On the other hand, the equation

$$(tx')' + tx = 1/t^2, \qquad t > 0,$$

satisfies Theorem 1 in [10], the Theorem in [11], and Theorem 2 in [20], but none of the results in this paper apply.

EXAMPLE 4. Consider the equation

$$x'' + x^3 = e(t), \qquad t > 1,$$

where $e(t) = (6t^2 + 1 + 3 \sin t + 3 \sin^2 t + \sin^3 t + 6t^2 \sin t - t^4 \sin t - 4t^3 \cos t)/t^6$. From the results in [10, 11, 20] we can conclude that all nonoscillatory solutions converge to zero. In addition to obtaining this same conclusion from Theorem 9 above, we also have that all Z-type solutions converge to zero, and here $x(t) = (1 + \sin t)/t^2$ is such a solution. Moreover, notice that Theorem 9 also applies to the damped equation

$$x'' + x'e^x/[(x')^2 + 1] \ln t + x^3 = e(t), \qquad t > 1,$$

where $e(t)$ is as above, whereas the results in [10, 11] or [20] do not.

Remark. Notice that by the results in [10, 11] or [20] all nonoscillatory solutions of

$$x'' + x = 2e^{-t}, \qquad t > 0,$$

converge to zero, but the equation

$$x'' + e^t x' + x = 2e^{-t}, \qquad t > 0,$$

has the nonoscillatory solution $x(t) = 1 + e^{-t}$ which does not converge to zero. This is somewhat surprising since one often expects that the addition of positive damping (in the sense that (4) holds) preserves such properties.

The remainder of the theorems in this paper give sufficient conditions for the oscillatory and Z-type solutions of (1) to converge to zero. The first two of these, like the previous three theorems, extend results contained in [7].

THEOREM 10. *Assume that conditions* (3)–(4) *and* (9)–(12) *hold,*

$$xf(x) > 0 \qquad \text{if} \qquad x \neq 0, \tag{15}$$

$$q(t) \to \infty \qquad \text{as} \qquad t \to \infty, \tag{16}$$

and,

$$\int_0^{\pm\infty} [s/g(s)]\, ds < \infty. \tag{17}$$

If $x(t)$ *is an oscillatory or Z-type solution of* (1), *then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be an oscillatory or Z-type solution of (1). From (17) we have $G(y) \leqslant K_1$ for some $K_1 > 0$ and (12) implies that $|y|/g(y) \leqslant N_1$ for all y. Now let $\epsilon > 0$ and choose $t_1 \geqslant t_0$ so that $a_2K_1/b_1p_1q(t) < \epsilon/3$ for $t \geqslant t_1$, $x(t_1) = 0$, $\int_{t_1}^{\infty} [a'(s)_-/a(s)]\, ds < b_1p_1q_1\epsilon/3a_2N$, and $\int_{t_1}^{\infty} [r(s)/q(s)]\, ds < a_1b_1p_1\epsilon/3a_2N_1$. Define V as in the proof of Theorem 7 with $K = 0$, differentiate, and then integrate for $t \geqslant t_1$ to obtain $F(x(t)) \leqslant a_2V(t)/b_1p_1 \leqslant (a_2/b_1p_1)\{G(y(t_1))/q(t_1) + \int_{t_1}^{\infty} [Na'(s)_-/a(s)\, q_1 + N_1r(s)/q(s)\, a_1]\, ds\} < \epsilon$ for $t \geqslant t_1$. Thus $\lim_{t\to\infty} x(t) = 0$.

THEOREM 11. *If conditions* (3)–(4), (7), (10)–(11), (13), *and* (15)–(17) *hold, and* $x(t)$ *is an oscillatory or Z-type solution of* (1), *then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be an oscillatory or Z-type solution of (1) and define V as in the proof of Theorem 8 with $K = 0$. Now $G(y) \leqslant K_1$ for some $K_1 > 0$ so for a given $\epsilon > 0$ choose $t_1 \geqslant t_0$ such that $G(y(t))/q(t) < p_1\epsilon/2a_2$ for $t \geqslant t_1$, $x(t_1) = 0$, and $\int_{t_1}^{\infty} [r(s)/q(s)]\, ds < a_1p_1\epsilon/2a_2L$. Then differentiating and integrating V we have $F(x(t)) \leqslant a_2V(t_1)/p_1 + (a_2L/a_1p_1) \int_{t_1}^{t} [r(s)/q(s)]\, ds < \epsilon$ for $t \geqslant t_1$ so $\lim_{t\to\infty} x(t) = 0$.

The next two theorems generalize some results obtained by Wong [27, 28]. We need to make the following additional assumptions on Eq. (1). Assume that

$$0 < c < g(y) < C \qquad \text{and} \qquad |a'(t)| \leqslant a_3, \tag{18}$$

$$H(t) = r(t)/q(t) \to 0 \qquad \text{as} \qquad t \to \infty, \tag{19}$$

$$xf(x) \geqslant dF(x) \tag{20}$$

for some positive constant d, and there is a continuous function k: $[t_0, \infty) \to R$ such that

$$|h(t, x, y)| \leqslant k(t) \qquad \text{and} \qquad k(t)/q(t) \to 0 \qquad \text{as} \qquad t \to \infty. \tag{21}$$

THEOREM 12. *Suppose conditions* (3)–(4), (8)–(10), (14)–(16), *and* (18)–(21) *hold. If*

$$\int_{t_0}^{t} |(q^{-1}(s))'''|\, ds = o(\ln q(t)), \qquad t \to \infty, \tag{22}$$

then every oscillatory or Z-type solution $x(t)$ *of* (1) *satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. For $t \geqslant z \geqslant t_0$ define $V_z(x, y, t) = F(x)/a(t) + G(y)/q(t) + \int_z^t [h(s, x(s), y(s))\, y(s)/g(y(s))\, q(s)\, a(s)]\, ds - \int_z^t [e(s, x(s), y(s))\, y(s)/g(y(s))\, q(s) \times a(s)]\, ds$. We will first show that $V_z(t)$ approaches a finite limit as $t \to \infty$. Now $V_z' = -a'(t) F(x)/a^2(t) - q'(t)\, G(y)/q^2(t) - a'(t)\, y^2/g(y)\, q(t)\, a(t) \leqslant a'(t)_- F(x)/a^2(t) + q'(t)_-\, G(y)/q^2(t) + a'(t)_-\, y^2/cq(t)\, a(t)$. Integrating, we have $F(x(t))/a(t) + G(y(t))/q(t) \leqslant V_z(z) + \int_z^t [a'(s)_- F(x(s))/a^2(s) + q'(s)_-\, G(y(s))/q^2(s) + a'(s)_- \times y^2(s)/cq(s)\, a(s)]\, ds + \int_z^t [r(s) \mid y(s) \mid /ca_1 q(s)]\, ds$. Condition (18) implies that $y^2 \leqslant 2CG(y)$, and since $\mid y \mid/(q(t))^{1/2} \leqslant (y^2/q(t) + 1)/2$, we have $\mid y \mid/(q(t))^{1/2} \leqslant CG(y)/q(t) + 1$. Thus $F(x(t))/a(t) + G(y(t))/q(t) \leqslant V_z(z) + \int_z^t [a'(s)_- F(x(s))/a^2(s) + q'(s)_-\, G(y(s))/q^2(s)]\, ds + \int_z^t [2CG(y(s))a'(s)_-/cq(s)a(s)]\, ds + \int_z^t [Cr(s)\, G(y(s))/ca_1(q(s))^{3/2}]\, ds + \int_z^t [r(s)/ca_1(q(s))^{1/2}]\, ds \leqslant K_1 + \int_z^t [(1 + 2C/c)\, a'(s)_-/a(s) + q'(s)_-/q(s) + Cr(s)/ca_1(q(s))^{1/2}]\, [F(x(s))/a(s) + G(y(s))/q(s)] \times ds$. By Gronwall's inequality, (9), (10), and (14) we have $F(x(t))/(a(t) + G(y(t))/q(t) \leqslant K_2 < \infty$. It follows that $\mid y(t) \mid/(q(t))^{1/2} \leqslant CG(y(t))/q(t) + 1 \leqslant CK_2 + 1 = K_3$. Integrating V_z' we have $V_z(t) \leqslant V_z(z) + K_2 \int_z^t [a'(s)_-/a(s) + q'(s)_-/q(s)] + K_3^2 \int_z^t [a'(s)_-/ca(s)]\, ds \leqslant K_4$. Thus $\int_z^t [h(s, x(s), y(s))\, y(s)/g(y(s)) \times q(s)\, a(s)]\, ds \leqslant V_z(t) + \int_z^t [e(s, x(s), y(s))\, y(s)/g(y(s))\, q(s)\, a(s)]\, ds \leqslant K_4 + K_3 \int_z^t [r(s)/ca_1(q(s))^{1/2}]\, ds \leqslant K_5$.

Rewriting V_z' in the form $V_z' = -F(x)\, [a'(t)_+ - a'(t)_-]/a^2(t) - G(y) \times [q'(t)_+ - q'(t)_-]/q^2(t) - y^2[a'(t)_+ - a'(t)_-]/g(y)\, q(t)\, a(t)$ and integrating we obtain $\int_z^t [a'(s)_+ F(x(s))/a^2(s)]\, ds + \int_z^t [q'(s)_+\, G(y(s))/q^2(s)]\, ds + \int_z^t [y^2(s) \times a'(s)_+/g(y(s))\, q(s)\, a(s)]\, ds \leqslant V_z(z) + \int_z^t [a'(s)_- F(x(s))/a^2(s)]\, ds + \int_z^t [q'(s)_- \times G(y(s))/q^2(s)]\, ds + \int_z^t [y^2(s)\, a'(s)_-/g(y(s))\, q(s)\, a(s)]\, ds + \int_z^t [e(s, x(s), y(s))\, y(s)/g(y(s))\, q(s)\, a(s)]\, ds$. From what we have already done we see that each of the integrals on the right-hand side of the above inequality converges, so each of the integrals on the left-hand side converges as well. Therefore, if we integrate this form of V_z', we will have that $V_z(t)$ is equal to a constant plus the sum of three convergent integrals, and it follows that $V_z(t)$ has a finite limit as $t \to \infty$.

Now let $x(t)$ be an oscillatory or Z-type solution of (1). Then by Theorem 9 $x(t)$ is bounded, say $\mid x(t) \mid \leqslant B$. Let $R(t) = 1/q(t)$, $N = 2C + dc$, and $P(t) = NV_z(t) + R''(t)\, x^2/2 - R'(t)\, xy$. Then $P'(t) = -Na'(t)\, F(x)/a^2(t) - Nq'(t) \times G(y)/q^2(t) - Na'(t)\, y^2/g(y)\, q(t)\, a(t) + R'''(t)\, x^2/2 - q'(t)\, a'(t)\, xy/q^2(t)\, a(t) - q'(t)\, h(t, x, y)\, x/q^2(t)\, a(t) - q'(t) f(x)\, g(y)\, x/q(t)\, a(t) + q'(t)\, e(t, x, y)\, x/q^2(t)\, a(t) - q'(t)\, y^2/q^2(t)$. First we see that $-Na'(t)\, F(x)/a^2(t) - Na'(t)\, y^2/g(y)\, q(t)\, a(t) \leqslant NK_2 a'(t)_-/a(t) + 2NCG(y)\, a'(t)_-/cq(t)\, a(t) \leqslant NK_2(1 + 2C/c)\, a'(t)_-/a(t) = k_1 a'(t)_-/a(t)$. Next, we have $-q'(t)\, a'(t)\, xy/a(t)\, q^2(t) \leqslant \mid q'(t) \mid a_3 B \mid y \mid/q^2(t)\, a_1 \leqslant a_3 BK_3 q'(t)_+/(q(t))^{3/2}\, a_1 + a_3 BK_3 q'(t)_-/(q(t))^{3/2}\, a_1 \leqslant a_3 BK_3 q'(t)/(q(t))^{3/2}\, a_1 + 2a_3 BK_3 q'(t)_-/(q(t))^{3/2}\, a_1 \leqslant a_3 BK_3 q'(t)/(q(t))^{3/2}\, a_1 + k_2 q'(t)_-/q(t)$ since (10) bounds $q(t)$, and hence $(q(t))^{1/2}$, from below. Since $x(t)$ is bounded, $\mid f(x(t)) \mid$ is bounded so $W(t) = -Nq'(t)\, G(y)/q^2(t) - q'(t) f(x)\, g(y)\, x/q(t)\, a(t) + q'(t)\, y^2/q^2(t) \leqslant -Nq'(t)\, G(y)/q^2(t) - dcq'(t)_+ F(x)/q(t)\, a(t) + f(x)\, g(y)\, xq'(t)_-/q(t)\, a(t) + 2CG(y)\, q'(t)_+/q^2(t) \leqslant -Nq'(t)\, G(y)/q^2(t) - dcq'(t)\, F(x)/q(t)\, a(t) + k_3 q'(t)_-/q(t)$

$+ 2Cq'(t)\, G(y)/q^2(t) + 2CK_2q'(t)_-/q(t) \leqslant -dcq'(t)\, G(y)/q^2(t) - dcq'(t)\, F(x)/q(t)\, a(t) + k_4q'(t)_-/q(t)$.

Let $\epsilon > 0$ be given. Since $x(t)$ is either oscillatory or Z-type, the integrals in the definition of $V_z(t)$ converge, and conditions (19) and (21) hold, there exists $T \geqslant z$ such that $y(T) = 0$, $\int_T^\infty [h(s, x(s), y(s))\, y(s)/g(y(s))\, q(s)\, a(s)]\, ds < \epsilon/8$, $\int_T^\infty | e(s, x(s), y(s))\, y(s)/g(y(s))\, q(s)\, a(s)|\, ds < \epsilon/8$, and $B[r(t) + k(t)]/q(t)\, a_1 < dc\epsilon/8$ for $t \geqslant T$. Then $U(t) = W(t) + q'(t)\, x\{[e(t, x, y) - h(t, x, y)]/q(t)\}/q(t)\, a(t) \leqslant -dcq'(t)\, V_T(t)/q(t) + [dcq'(t)/q(t)] \int_T^t \{[h(s, x(s), y(s)) - e(s, x(s), y(s))]\, y(s)/g(y(s))\, q(s)\, a(s)\}\, ds + k_4q'(t)_-/q(t) + dc\epsilon\, | q'(t)|/8q(t) \leqslant -dcq'(t)_+\, V_T(t)/q(t) + dcq'(t)_-\, V_T(t)/q(t) + dc\epsilon\, | q'(t)|/4q(t) + dc\epsilon q'(t)_+/8q(t) + k_5q'(t)_-/q(t)$. Since there exists A such that $\lim_{t\to\infty} V_T(t) = A$, we will suppose that $A \geqslant 3\epsilon/4$ and let $\{t_n\}$ be an increasing sequence of zeros of $y(t)$ such that $t_1 = T$ and $t_n \to \infty$ as $n \to \infty$. Then there exists $j \geqslant 1$ such that $V_T(t) > 5\epsilon/8$ for $t \geqslant t_j$. Now $V_T(t)$ is bounded so for $t \geqslant t_j$ we have $U(t) \leqslant -5dc\epsilon q'(t)_+/8q(t) + dc\epsilon q'(t)_+/4q(t) + dc\epsilon q'(t)_+/8q(t) + k_6q'(t)_-/q(t) = -dc\epsilon q'(t)_+/4q(t) + k_6q'(t)_-/q(t) \leqslant -dc\epsilon q'(t)/4q(t) + k_6q'(t)_-/q(t)$.

We then have $P'(t) \leqslant k_1a'(t)_-/a(t) + a_3BK_3q'(t)/(q(t))^{3/2}\, a_1 + | R'''(t)|\, B^2/2 - dc\epsilon q'(t)/4q(t) + k_7q'(t)_-/q(t)$, so $dc\epsilon q'(t)/4q(t) + P'(t) \leqslant k_1a'(t)_-/a(t) + k_8q'(t)/(q(t))^{3/2} + | R'''(t)|\, B^2/2 + k_7q'(t)_-/q(t)$. Integrating for $n > j$ we obtain $dc\epsilon \ln(q(t_n))/4 \leqslant dc\epsilon \ln(q(t_j))/4 + P(t_j) - P(t_n) + k_1 \int_{t_j}^{t_n} [a'(s)_-/a(s)]\, ds + 2k_8/(q(t_j))^{1/2} - 2k_8/(q(t_n))^{1/2} + (B^2/2) \int_{t_j}^{t_n} | R'''(s)|\, ds + k_7 \int_{t_j}^{t_n} [q'(s)_-/q(s)]\, ds \leqslant N\, | V_T(t_n)| + | R''(t_n)|\, B^2/2 + (B^2/2) \int_{t_j}^{t_n} | R'''(s)|\, ds + k_9$. Since $V_T(t_n)$ is bounded and $| R''(t_n)| \leqslant | R''(t_j)| + \int_{t_j}^{t_n} | R'''(s)|\, ds$, we have $dc\epsilon \ln(q(t_n))/4 \leqslant k_{10} + B^2 \int_{t_j}^{t_n} | R'''(s)|\, ds$ for each $n \geqslant j$. In view of (16) and (22) this is impossible so we must have $\lim_{t\to\infty} V_T(t) = A < 3\epsilon/4$. Thus there exists $T_1 \geqslant T$ such that $V_T(t) < 7\epsilon/8$ for $t \geqslant T_1$ and hence $F(x(t))/a(t) < 7\epsilon/8 + \int_T^t | e(s, x(s), y(s))\, y(s)/g(y(s))\, q(s)\, a(s)|\, ds < \epsilon$ for $t \geqslant T_1$. Since $a(t) \leqslant a_2$, we have $F(x(t)) \to 0$ as $t \to \infty$ which implies that $x(t) \to 0$ as $t \to \infty$, and the proof of the theorem is now complete.

Wong [28] proved the previous theorem and Theorem 13 below for the case $a(t) \equiv 1$, $h(t, x, x') \equiv 0 \equiv e(t, x, x')$, $g(x') \equiv 1$, and $f(x) = x^{2n-1}$ where n is a positive integer. In [27] he extended the results in [28] to included functions $f(x)$ which satisfy conditions (8), (15), and (20).

THEOREM 13. *Assume that conditions* (3)–(4), (7), (8), (14)–(16), *and* (18)–(21) *hold. If for every w with $\frac{1}{2} < w < 1$ we have*

$$\int_{t_0}^\infty [q'(s)_-/q^w(s)]\, ds < \infty \tag{23}$$

and

$$\int_{t_0}^t |(q^{-w}(s))'''\, |\, ds = o(q^{1-w}(t)), \qquad t \to \infty, \tag{24}$$

then every oscillatory or Z-type solution $x(t)$ of (1) *satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be an oscillatory or Z-type solution of (1). First note that conditions (16) and (23) imply that (10) holds so by Theorem 9 $x(t)$ is bounded, say $|x(t)| \leqslant B$. With no loss in generality we may assume that $C > dc$. Let $N = (4C - dc)/2dc$ and $w = Ndc/(2C + dc)$. Then $Ndc = 2C - dc/2 > 3C/2$ and $1/(2C + dc) > 1/3C$ so $w > \frac{1}{2}$. Since $Ndc < 2C$, we also have that $w < 1$. Let $\epsilon > 0$ be given and define $V_z(t)$ as in the proof of the previous theorem. Once again $V_z(t)$ converges so there exists $z \geqslant t_0$ such that $\int_z^\infty [h(s, x(s), y(s))\, y(s)/g(y(s))\, q(s)\, a(s)]\, ds < N\epsilon(1 - w)/15w$, $\int_z^\infty |\, e(s, x(s), y(s))\, y(s)/g(y(s))\, q(s)\, a(s)|\, ds < \min\{N\epsilon(1 - w)/15w, \epsilon/8\}$, and $B[r(t) + k(t)]/q(t)\, a_1 < Ndc\epsilon(1 - w)/15w$ for $t \geqslant z$.

The constants K_i used below are those developed in the proof of the convergence of $V_z(t)$ in the previous theorem.

Let $T(t) = 1/q^w(t)$ and $K(t) = NdcT(t)\, q(t)\, V_z(t) + T''(t)\, x^2/2 - T'(t)\, xy$. Then $K'(t) \leqslant Ndc(1 - w)\, q'(t)\, V_z(t)/q^w(t) - Ndcq'(t)\, G(y)/q^{1+w}(t) + T'''(t)\, x^2/2 - wa'(t)\, q'(t)\, xy/a(t)\, q^{1+w}(t) - wq'(t)\, h(t, x, y)\, x/a(t)\, q^{1+w}(t) - wq'(t)\, f(x) \times g(y)\, x/a(t)\, q^w(t) + wq'(t)\, e(t, x, y)\, x/a(t)\, q^{1+w}(t) + wq'(t)\, y^2/q^{1+w}(t)$. Now $b = w + \frac{1}{2} > 1$, so $-wa'(t)\, q'(t)\, xy/a(t)\, q^{1+w}(t) \leqslant wa_3B\, |\, q'(t)|\, |\, y\, |/a_1(q(t))^{1/2}\, q^b(t) \leqslant wa_3BK_3\, |\, q'(t)|/a_1q^b(t) \leqslant c_1q'(t)_+/q^b(t) + c_1q'(t)_-/q^b(t) \leqslant c_1q'(t)/q^b(t) + 2c_1q'(t)_-/(q(t))^{1/2}\, q^w(t) \leqslant c_1q'(t)/q^b(t) + c_2q'(t)_-/q^w(t)$ since $(q(t))^{1/2}$ is bounded from below. Since $x(t)$ is bounded, $f(x(t))$ is bounded, so $-wq'(t)\, f(x)\, g(y)\, x/a(t)\, q^w(t) + wq'(t)\, y^2/q^{1+w}(t) \leqslant -wdcq'(t)_+ F(x)/a(t)\, q^w(t) + c_3q'(t)_-/q^w(t) + 2Cwq'(t)_+\, G(y)/q^{1+w}(t) \leqslant -wdcq'(t)\, F(x)/a(t)\, q^w(t) + c_3q'(t)_-/q^w(t) + 2Cwq'(t)\, G(y)/q^{1+w}(t) + 2Cwq'(t)_-\, G(y)/q^{1+w}(t) \leqslant -wcdq'(t)F(x)/a(t)\, q^w(t) + c_3q'(t)_-/q^w(t) + 2Cwq'(t) \times G(y)/q^{1+w}(t) + 2CwK_2q'(t)_-/q^w(t)$. Hence $W(t) = Ndc(1 - w)\, q'(t)\, V_z(t)/q^w(t) - Ndcq'(t)\, G(y)/q^{1+w}(t) - wq'(t)\, f(x)\, g(y)\, x/a(t)\, q^w(t) + wq'(t)\, y^2/q^{1+w}(t) \leqslant Ndc(1 - w)\, q'(t)\, V_z(t)/q^w(t) + (2Cw - Ndc)\, q'(t)\, G(y)/q^{1+w}(t) - wdcq'(t)\, F(x)/a(t)\, q^w(t) + c_4q'(t)_-/q^w(t) \leqslant Ndc(1 - w)\, q'(t)\, V_z(t)/q^w(t) - wdcq'(t)\, G(y)/q^{1+w}(t) - wdcq'(t)\, F(x)/a(t)\, q^w(t) + c_4q'(t)_-/q^w(t) \leqslant Ndc[1 - w(1 + 1/N)]\, q'(t)\, V_z(t)/q^w(t) + [wdcq'(t)/q^w(t)] \int_z^t \{[h(s, x(s), y(s)) - e(s, x(s), y(s))]\, y(s)/g(y(s))\, q(s)\, a(s)\}\, ds + c_4q'(t)_-/q^w(t)$. Letting $D = Ndc[1 - w(1 + 1/N)]$ we have $U(t) = W(t) + wq'(t)\, x\{[e(t, x, y) - h(t, x, y)]/q(t)\}/a(t)\, q^w(t) \leqslant Dq'(t)_+\, V_z(t)/q^w(t) - Dq'(t)_- V_z(t)/q^w(t) + dc\, |\, q'(t)|\, [2N\epsilon(1 - w)/15]\, q^w(t) + c_4q'(t)_-/q^w(t) + Ndc\epsilon(1 - w)\, |\, q'(t)|/15q^w(t)$.

Since $V_z(t)$ converges, suppose that $\lim_{t\to\infty} V_z(t) = L \geqslant 3\epsilon/4$. Now $x(t)$ is an oscillatory or Z-type solution so we let $\{t_n\}$ be an increasing sequence of zeros of $y(t)$ such that $t_n \to \infty$ as $n \to \infty$ and $4L/5 \leqslant V_z(t) \leqslant 6L/5$ for $t \geqslant t_1$. Then $U(t) \leqslant 6DLq'(t)_+/5q^w(t) + c_5q'(t)_-/q^w(t) + 4NdcL(1 - w)\, |\, q'(t)|/15q^w(t) \leqslant 6DLq'(t)/5q^w(t) + c_6q'(t)_-/q^w(t) + 4NdcL(1 - w)\, q'(t)/15q^w(t)$. Hence $K'(t) \leqslant 6DLq'(t)/5q^w(t) + 4NdcL(1 - w)\, q'(t)/15q^w(t) + |\, T'''(t)|\, B^2/2 + c_1q'(t)/q^b(t) + c_7q'(t)_-/q^w(t)$. Integrating from t_1 to t_n we have $K(t_n) \leqslant 6DLq^{1-w}(t_n)/5(1 - w) + 4NdcLq^{1-w}(t_n)/15 + (B^2/2) \int_{t_1}^{t_n} |\, T'''(s)|\, ds + c_1q^{1-b}(t_n)/(1 - b) + c_7 \int_{t_1}^{t_n} [q'(s)_-/q^w(s)]\, ds + c_8$. Now $K(t_n) = NdcT(t_n)\, q(t_n)\, V_z(t_n) + T''(t_n)\, x^2(t_n)/2 \geqslant 4NdcLq^{1-w}(t_n)/5 + T''(t_n)\, x^2(t_n)/2$, and since $b > 1$ and $[1 - w(1 + 1/N)]/$

$(1-w)=\frac{1}{3}$, we have $2NdcLq^{1-w}(t_n)/15 \leqslant B^2 \int_{t_1}^{t_n} | T'''(s)| \, ds + c_9$, which is impossible in view of (16) and (24). Therefore $\lim_{t\to\infty} V_z(t) = L < 3\epsilon/4$. Hence there exists $T_1 \geqslant z$ such that $V_z(t) < 7\epsilon/8$ for $t \geqslant T_1$ so $F(x(t))/a(t) < 7\epsilon/8 + \int_{T_1}^{t} | e(s, x(s), y(s)) \, y(s)/g(y(s)) \, q(s) \, a(s)| \, ds < \epsilon$ for $t \geqslant T_1$. Since $a(t)$ is bounded from above, we have that $F(x(t)) \to 0$ as $t \to \infty$ which implies that $x(t) \to 0$ as $t \to \infty$ completing the proof of the theorem.

By combining various theorems in this section we could obtain results which would guarantee that all solutions of (1) tend to zero as $t \to \infty$. We leave the formulation of such results to the reader.

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