Equivalence of the Euler and Lagrangian Equations of Gas Dynamics for Weak Solutions

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This paper demonstrates the equivalence of the Euler and the Lagrangian equations of gas dynamics in one space dimension for weak solutions which are bounded and measurable in Eulerian coordinates. The precise hypotheses include all known global solutions on \( \mathbb{R} \times \mathbb{R}^+ \). In particular, solutions containing vacuum states (zero mass density) are included. Furthermore, there is a one-to-one correspondence between the convex extensions of the two systems, and the corresponding admissibility criteria are equivalent. In the presence of a vacuum, the definition of weak solution for the Lagrangian equations must be strengthened to admit test functions which are discontinuous at the vacuum. As an application, we translate a large-data existence result of DiPerna for the Euler equations for isentropic gas dynamics into a similar theorem for the Lagrangian equations. © 1987 Academic Press, Inc.

1. INTRODUCTION

There are two different systems of partial differential equations for one-dimensional flow of a compressible, inviscid, non-heat-conducting gas, each resulting from a particular choice of independent space coordinate. If we let \( x \) be a linear coordinate on physical space, and let \( t \) be time, we obtain the Euler equations [2]:

\[
\begin{align*}
(a) \quad & \rho_t + (\rho u)_x = 0, \\
(b) \quad & (\rho u)_t + (\rho u^2 + p(\rho, S))_x = 0, \\
(c) \quad & (\rho e(\rho, S) + \rho u^2/2)_t + ((\rho e(\rho, S) + \rho u^2/2 + p(\rho, S)) u)_x = 0, \\
(d) \quad & (\rho S)_t + (\rho u S)_x \geq 0,
\end{align*}
\]

(1.1)
describing the conservation of mass, momentum, and energy, and the increase of entropy across shock waves, respectively. Here \( \rho, u, \) and \( S \) are the mass density, velocity, and entropy, respectively, and \( e \) and \( p \) are inter-

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nal energy per unit mass and the pressure, expressed as functions of $\rho$ and $S$. If we choose, instead of $x$, a material coordinate:

$$ y = \int_{x(t)}^x \rho(s, t) \, ds, $$

where $x(t)$ is a well-defined particle path satisfying $x'(t) = u(x(t), t)$, then we obtain the Lagrangian equations [2],

(a) $\tau_t - u_x = 0$,

(b) $u_t + \tilde{p}(\tau, S)_x = 0$,

(c) $(\tilde{e}(\tau, S) + u^2/2)_t + (u\tilde{p}(\tau, S))_x = 0$,

(d) $S_t \geq 0$,

where $\tau = 1/\rho$ is the specific volume, $\tilde{p}(\tau, S) = p(1/\tau, S)$, and $\tilde{e}(\tau, S) = e(1/\tau, S)$.

A tedious calculation using the chain rule and product rule shows that (1.1) and (1.3) are equivalent for classical solutions [2]. However, solutions of these equations are known to develop discontinuities, which represent shock waves. Consequently one must consider weak solutions. Even though we may define the weak derivatives $D$, or $F$, for any of the conserved densities $D$ or any flux $F$ in (1.1) or (1.3), the product rule and chain rule do not hold in any sense that permits us to say that (1.1) and (1.3) are equivalent for weak solutions; but see [11]. One may check that the Rankine-Hugoniot conditions for shock wave solutions of (1.1) and (1.3) are equivalent [2], however, this is not sufficient to prove mathematically that the Cauchy problems are equivalent.

In this paper we give a simple and elegant proof that (1.1) and (1.3) are equivalent for weak bounded measurable solutions on $\mathbb{R} \times \mathbb{R}^+$. To be precise, we have

**Theorem 1.** The change of variables (1.2) induces a one-to-one correspondence between $L^\infty$ weak solutions of (1.1) satisfying $\|S\|_\infty < \infty$, $\|u\|_\infty < \infty$, $0 < \delta \leq \rho(x, t) \leq M < \infty$ a.e. for some $\delta$ and $M$, and $L^\infty$ weak solutions of (1.3) satisfying $0 < \epsilon \leq \tau(y, t) \leq N < \infty$ a.e. for some $\epsilon$ and $N$. In addition, if $p(0, S) = 0$ and $e(0, S)$ is finite for all finite $S$, then there is a one-to-one correspondence between equivalence classes of bounded measurable solutions of (1.1) for which

$$ \int_0^\infty \rho(x, t) \, dx = \int_{-\infty}^0 \rho(x, t) \, dx = \infty, $$

and equivalence classes of weak solutions of (1.3), in which $\tau$ is a Radon measure on $\mathbb{R} \times \mathbb{R}^+$ that dominates two-dimensional Lebesgue measure $m_2$ in
the sense that for some $K > 0$, $\tau(E) \geq Km_2(E)$ for any subset $E$ of $\mathbb{R} \times \mathbb{R}^+$, and in which $u$ and $S$ are bounded.

The equivalence classes mentioned above pertain to the following equivalence relations: In Eulerian coordinates, two solutions are equivalent if the mass densities are equal a.e. with respect to $m_2$, and $u$ and $S$ are equal a.e. with respect to $\rho$. In Lagrangian coordinates, two solutions are equivalent if the specific volumes $\tau$ are equal as measures and $u$ and $S$ are equal a.e. with respect to $m_2$.

The measure theoretic notation used throughout this paper is that of Federer [5], because we use several theorems from this book. In particular the word "measure" refers to an outer measure; in fact all of the measures used in this paper, except for one dimensional Hausdorff measure, are Radon measures. Measurability of a set refers to Carathéodory's definition. We recall the definition of a Radon measure from [5].

**Definition 1.** By a Radon measure we mean a measure $\phi$, over a locally compact Hausdorff space $X$, with the following three properties:

(i) If $K$ is a compact subset of $X$, then $\phi(K) < \infty$.

(ii) If $V$ is an open subset of $X$, then $V$ is $\phi$ measurable and $\phi(V) = \sup \{ \phi(K): K$ is a compact subset of $V \}$. 

(iii) If $A$ is any subset of $X$ then 

$$\phi(A) = \inf \{ \phi(V): V$ is open, $A$ is a subset of $V \}.$$ 

Thus a Radon measure is finite on compact sets and has nice regularity properties. Note that given any locally Lebesgue integrable function $f$, the (outer) measure $f m_n$ on $\mathbb{R}^n$ defined by:

$$(f m_n)(E) = \inf \left\{ \int_V f \, dx: V$ is an open set containing $E \right\}$$ 

is a Radon measure. In addition, the Riesz representation theorem gives a natural correspondence between signed Radon measures and distributions of order 0 [4]. As is common in distribution theory, we will make no distinctions between the function $f$, the measure $f m_n$, and the distribution $\phi \rightarrow \int f \phi \, dx$.

We will make extensive use of the following change of variable formulae, which we have specialized from [5].

**Formula 1.** [5, p. 54, 2.2.17, 2.4.18]. If $\phi$ is a measure on $X$, and $T: X \rightarrow Y$, then there is a measure $T_\ast \phi$ on $Y$, defined by $(T_\ast \phi)(E) = \phi(T^{-1}(E))$. If $T$ is a proper map between closed subsets $X$ and $Y$ of $\mathbb{R}^n$, and
\( \phi \) is a Radon measure, then \( T_\# \phi \) is a Radon measure, and for any function \( f: Y \to \mathbb{R} \), \( f \) is \( T_\# \phi \) measurable if and only if \( f \circ T \) is \( \phi \) measurable, and
\[
\int f \, d(T_\# \phi) = \int f \circ T \, d\phi.
\]

Formula 1 is similar to the familiar formula for the expectation of a function of a random variable. To prove the \( \phi \) measurability of \( f \circ T \), given the \( T_\# \phi \) measurability of \( f \), one must adapt [5, 2.4.18] to the case where \( \phi(X) \) may not be finite, but where \( \phi \) is a Radon measure, \( T \) is proper, and the spaces \( X \) and \( Y \) are countable unions of compact sets. However, this is an easy exercise.

**Formula 2.** [5, 2.10.9, 3.1.6, 3.2.3]. Let \( X \) and \( Y \) be subsets of \( \mathbb{R}^n \), and let \( T: X \to Y \) be Lipschitz. Then \( T \) is differentiable a.e., so that the Jacobian \( JT \) is defined a.e. Let \( N(T, y) \) be the number (possibly infinite) of points \( x \) in \( X \) such that \( T(x) = y \). Then:

1. If \( A \) is an \( m_n \) measurable set, then
   \[
   \int_A JT \, dx = \int N(T|_A, y) \, dy.
   \]
2. If \( u \in L^1(\mathbb{R}^n) \), then
   \[
   \int u(x) \, JT \, dx = \int \sum \{u(x): T(x) = y\} \, dy.
   \]

In Section 3 we will show that in the presence of vacuum states, the definition of weak solution in Lagrangian coordinates must be strengthened in order for equivalence to hold. One must eliminate certain nonphysical weak solutions by requiring the definition of weak solution to hold with test functions whose distributional gradient is a measure which is absolutely continuous with respect to the specific volume, \( \tau \). In other words, we must admit test functions which are discontinuous at the vacuum.

Theorem 1 is a special case of the following general theorem which applies to many other important systems of conservation laws, including the isentropic gas dynamics equations, and the shallow water equations.

**Theorem 2.** Let
\[
U_t + F(U)_x = 0,
\]
\[
U(x, t) = (u_1, \ldots, u_n) \quad (x, t) \in \mathbb{R}^n,
\]
\[
F(U) = (f_1, \ldots, f_n) \quad (U) \in \mathbb{R}^n
\]
be a system of conservation laws. For any bounded measurable solution of
(1.5), with \( u_1(x, t) \geq 0 \), let \( y(x, t) \) satisfy
\[
\frac{\partial y}{\partial x} = u_1(x, t), \quad \frac{\partial y}{\partial t} = -f_1(U(x, t)),
\]
in the sense of distributions. Then \( T: (x, t) \to (y(x, t), t) \) is a Lipschitz-continuous transformation, which induces a one-to-one correspondence between \( L^\infty \) weak solutions of (1.5) on \( \mathbb{R} \times \mathbb{R}^+ \) satisfying \( 0 < \varepsilon \leq u_1(x, t) \leq M < \infty \) for some \( \varepsilon \) and \( M \), and \( L^\infty \) weak solutions of
\[
(1/u_1)_t - (f_1(U)/u_1)_y = 0, \quad [(u_2, \ldots, u_n)/u_1]_t + [(f_2, \ldots, f_n)(U) - f_1(U)(u_2, \ldots, u_n)/u_1] = 0 \tag{1.7}
\]
on \( \mathbb{R} \times \mathbb{R}^+ \) satisfying \( \varepsilon \leq u_1(x, t) \leq M \). In addition, if \( F(U)/u_1 \) is bounded for \( u_1 > 0 \), then there is a one-to-one correspondence between equivalence classes of bounded measurable solutions of (1.5) for which \( u_j/u_1 \) is bounded for \( j = 2, \ldots, n \) and
\[
\int_0^\infty u_1(x, t) \, dx = \int_{-\infty}^0 u_1(x, t) \, dx = \infty, \tag{1.8}
\]
and equivalence classes of weak solutions of (1.7) for which \( v_1 = 1/u_1 \) is a Radon measure which dominates Lebesgue measure, and \( v_j = u_j/u_1 \) is bounded for \( j = 2, \ldots, n \). If \( \eta(U) \) is any convex extension of (1.5), i.e., there is a flux \( q(U) \) such that \( D\eta DF = Dq \), so that \( \eta + q_x = 0 \) for classical solutions, then any solution of (1.5) satisfying
\[
\eta(U)_t + q(U)_x \leq 0 \tag{1.9}
\]
corresponds to a solution of (1.7) satisfying
\[
\tilde{\eta}(V)_t + \tilde{q}(V)_x \leq 0, \tag{1.10}
\]
where \( V = (v_1, \ldots, v_n) \), \( \tilde{\eta}(V) = \eta(U)/u_1 \), and \( \tilde{q}(V) = q(U) - f_1(U) \tilde{\eta}(V) \). Furthermore \( \eta \) is convex if and only if \( \tilde{\eta} \) is convex as a function of \( V \). Thus Lax's generalized entropy condition [7] holds for a solution of (1.5) if and only if it holds for the corresponding solution of (1.7).

As in Theorem 1, in Theorem 2 the definition of weak solution for (1.7) must be strengthened, in the presence of a \( u_1 \)-vacuum, to admit discontinuous test functions. The equivalence relation referred to is similar to that used in Theorem 1.

As a consequence of these theorems, theorems on the existence, uniqueness, and behavior of solutions for one system may be carefully trans-
lated into theorems on solutions of the other. For example, in [3], DiPerna used compensated compactness methods to prove the existence of global solutions, via the limit of vanishing diffusion, to the Eulerian isentropic gas dynamics equations

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + \rho^{1+2/n})_x &= 0, \\
\rho(x, 0) &= \rho_0(x), \\
u(x, 0) &= u_0(x),
\end{align*}
\]

for any \( n \in \mathbb{N} \), with large initial data satisfying \( 0 < \varepsilon \leq \rho_0(x) \leq M < \infty, \|u_0\|_\infty < \infty \), and \( (\rho_0 - \bar{\rho}, u_0 - \bar{u}) \in H^2(\mathbb{R}) \cap C^2(\mathbb{R}) \), for some constants \( \bar{\rho} > 0 \) and \( \bar{u} \). As an application of our results, we may translate the existence part of this result as follows:

**Theorem 3.** Let \( \tau_0(y) \) and \( u_0(y) \) satisfy \( (\tau_0 - \bar{\tau}, u_0 - \bar{u}) \in H^2(\mathbb{R}) \cap C^2(\mathbb{R}) \), and \( 0 < \varepsilon \leq \tau_0(y) \leq M < \infty \) for all \( y \in \mathbb{R} \), and some constants \( \bar{\tau} > 0 \) and \( \bar{u} \). Then there is a global weak solution to the Cauchy problem

\[
\begin{align*}
\tau_t - u_y &= 0, \\
u_t + (\tau^{-1-2/n})_y &= 0, \\
\tau(y, 0) &= \tau_0(y), \\
u(y, 0) &= u_0(y),
\end{align*}
\]

which satisfies

\[
\eta(\tau, u)_t + q(\tau, u)_y \leq 0
\]

for all convex functions \( \eta \) which satisfy \( D\eta D(-u, \tau^{-1-2-n}) = Dq \).

With slight modification, our results may be applied to initial-boundary value problems, such as the “piston problem,” which have been studied in Lagrangian coordinates [8, 10] for ideal polytropic gases \( p(\tau, S) = a^2\tau^{-\gamma} \exp((\gamma - 1) S/R) \), in the quadrant \( t \geq 0, y \geq 0 \), or the vertical strip \( t \geq 0, 0 \leq y \leq 1 \). In the quadrant the boundary conditions are

\[
u(0, t) = u_i(t), \quad t \geq 0
\]

or

\[
p(\tau(0, t), S(0, t)) = p_i(t), \quad t \geq 0.
\]

In the strip \( 0 \leq y \leq 1 \) the boundary conditions are (1.13) together with similar boundary conditions at \( y = 1 \). The existence theorems in [8, 10] can
be translated into existence theorems for (1.1), or (1.11) with $1 + 2/n$ replaced by $\gamma > 1$, with boundary conditions of the form:

Given $u_i(t) \in BV$ there is a unique Lipschitz $x_i(t)$ such that $x_i'(t) = u_i(t)$. The boundary condition is $u(x_i(t), t) = u_i(t)$. (1.14a)

Given $p_i(t)$, $0 < p_0 \leq p_i(t) \leq M < \infty$, $p[\tau(x_i(t), t), S(x_i(t), t)] = p_i(t)$ for the curve $x_i(t)$ satisfying $x_i'(t) = u_i(t), t, x_i(0) = 0$. (1.14b)

We omit the details of the proof of this equivalence, and merely note that the solutions found in [8, 10] do not have vacuum states, and have bounded spatial variation. The trace $u(x_i(t), t)$ exists for a.e. $t$ because $u(\cdot, t) \in BV$ for a.e. $t$.

Remark. The central theme, or motto, of these results is that the choice of coordinate system is not important, provided that proper physical laws are observed.

A related paper, [11], has appeared, in which it is shown that the Principle of Virtual Work, i.e., the definition of weak solutions to systems of conservation laws in several space dimensions, is equivalent to the integral laws of motion, using some similar techniques.

2. THE TRANSFORMATION

The proofs of Theorems 1 and 2 are almost identical and we concentrate on Theorem 1.

The transformation from (1.1) to (1.3) is effected by a change of space coordinate, (1.2). Actually this is simply a classical formula for a solution to the gradient system

$$\frac{\partial y}{\partial x} = \rho(x, t), \quad \frac{\partial y}{\partial t} = -(\rho u)(x, t).$$

This system is consistent because $\rho = -(\rho u)_x$. Hence, it has a solution, $(y, t) = T(x, t)$, in the space of distributions. If $0 < \varepsilon \leq \rho(x, t) \leq M < \infty$ and $u \in L^\infty$ then $T$ is a bi-Lipschitz homeomorphism from $\mathbb{R} \times \mathbb{R}^+$ onto itself.

The transformation proceeds via the change of variables formula for integrals. The Jacobian of the transformation is $\rho$. Consider one of the conservation laws (1.1), written as $D_i + F_i = 0$. The weak formulation of this equation, with initial conditions $D(x, 0) = D_0(x)$, is

$$\int_{t > 0} \int \phi_i D + \phi_i^t F \, dx \, dt + \int_{t = 0} \phi D_0 \, dx = 0, \quad (2.1)$$
for all $C^1$ test functions $\phi$ with compact support. We observe that for such weak solutions, (2.1) also holds for Lipschitz test functions. Indeed, if $\phi$ is Lipschitz with compact support, convolution with a standard smoothing kernel produces a $C^\infty$ function with compact support, $\phi_\varepsilon$, such that $\nabla \phi_\varepsilon \to \nabla \phi$ in $L^1$ as $\varepsilon \to 0$, so that

$$
\int_{t > 0} \int (\phi_\varepsilon)_t D + (\phi_\varepsilon)_x F \, dx \, dt + \int_{t = 0} \phi_\varepsilon D_0 \, dx
$$

$$
\to \int_{t > 0} \phi_t D + \phi_x F \, dx \, dt + \int_{t = 0} \phi D_0 \, dx.
$$

Change of variables transforms (2.1) to

$$
0 = \int_{t > 0} \left[ (\phi_t \circ T^{-1})(D \circ T^{-1}) + (\phi_x \circ T^{-1})(F \circ T^{-1}) \right] \frac{1}{(\rho \circ T^{-1})} 
$$

$$
\quad + \int_{t = 0} (\phi \circ T^{-1})(D_0 \circ T^{-1}) \frac{1}{(\rho_0 \circ T^{-1})} 
$$

$$
\quad - \int_{t > 0} \left[ (\phi, - (\bar{\rho} \bar{u}) \phi_x, D + (\phi_x \bar{\rho}) \bar{F} \right] \frac{1}{\bar{\rho}} 
$$

$$
\quad \frac{1}{\bar{\rho} \rho_0} \frac{1}{\rho_0} dy \, dt + \int_{t = 0} \bar{\phi} \bar{D}_0 \frac{1}{\bar{\rho}_0} 
$$

where $\phi = \phi \circ T^{-1}$, $\bar{D} = D \circ T^{-1}$, etc. Since $T$ is a bi-Lipschitz homeomorphism of $\mathbb{R} \times \mathbb{R}^+$ onto itself, the induced map $\phi \to \phi \circ T^{-1} = \bar{\phi}$ is a bijection on the set of Lipschitz test functions on $\mathbb{R} \times \mathbb{R}^+$. Therefore we have that $\rho \bar{D}_t \rho \bar{D}_y = 0$ in the weak sense.

Thus (1.1) is transformed to:

$$
1, + 0, = 0 \quad \text{cons. of mass},
$$

$$
u, + \bar{\rho}(\tau, S), = 0 \quad \text{cons. of momentum},
$$

$$
(u^2/2 + \bar{e}(\tau, S)), + (\bar{\rho} u), = 0 \quad \text{cons. of energy}.
$$

We seem to be missing (1.3a). This equation follows from the conservation of volume, $D = 1$, $F = 0$,

$$
(1/\rho)_t - u_x = 0.
$$

One defines $\tau$ to be the specific volume $1/\rho$, and we have (1.2a).

The weak formulation of the increase of entropy, (1.1d), is

$$
\int_{t > 0} \phi, (\rho S + \phi_x \rho u) \, dx \, dy + \int_{t = 0} \phi \rho_0 S_0 \, dx \leq 0
$$

for all nonnegative $C^1$ test functions $\phi$. Clearly this inequality is preserved in the transformation, and we obtain (1.2d). In addition, if $(\eta, q)$ is any
convex extension of \((1.1)\), including \((-\rho S, -\rho u S)\), then any solution of
\((1.1)\) satisfies \(\eta_t + q_x \leq 0\) if and only if the corresponding solution of \((1.3)\)
satisfies \((\eta/\rho)_t + (q - u \eta)_x \leq 0\). Furthermore,
\[
\bar{\eta}(\tau, u, e + u^2/2) = \tau \eta(1/\tau, u/\tau, (e + u^2/2)/\tau)
\]
is convex, as follows. Since \(\eta\) is convex, we know that the convex set
\(E = \{(z, U): z \geq \eta(U)\}\) is an intersection of half spaces. Consequently \(\eta\) is a
supremum of affine functions,
\[
\eta(\rho, \rho u, \rho e + \rho u^2/2) = \sup_x (c_{0x} + c_{1x} \rho + c_{2x} \rho u + c_{3x} \rho (e + u^2/2)). \tag{2.2}
\]
Then
\[
\bar{\eta}(1/\rho, u, e + u^2/2) = \eta(\rho, \rho u, \rho e + \rho u^2/2)/\rho
\]
\[
= \sup_x (c_{0x}/\rho + c_{1x} + c_{2x} u + c_{3x} (e + u^2/2)), \tag{2.3}
\]
and thus is a convex function. Clearly this argument is reversible and generalizes for Theorem 2. Thus Lax's generalized entropy condition \([7]\) holds for the transformed solution of \((1.3)\) if and only if it holds for the
corresponding solution of \((1.1)\).

In \([6]\) it is shown that the convexity of \(-S\) as a function of \(\tau, u,\) and
\(E = u^2/2 + e\), is equivalent to the convexity of \(E\) as a function of \(\tau, u,\) and \(S\).
The above discussion is similar to a simple geometric explanation and
proof of this fact, due to Andrew Majda; namely that a function is convex
if and only if through each point on its graph there is a sub-tangent hyper-
plane such that the graph lies on one side of the hyperplane:
\[
E \quad E_0 \geq c_1 (\tau - \tau_0) + c_2 (u - u_0) + c_3 (S - S_0).
\]
In this case \(c_3 = (\partial E/\partial S)_{u,z} = T\) = temperature and is always positive. Thus
\[-(S - S_0) \geq (1/c_3)(c_1 (\tau - \tau_0) + c_2 (u - u_0) - (E - E_0)),\]
and we see that \(-S\) is convex as a function of \(\tau, u,\) and \(E\). Essentially we
are looking at the same graph, but from a different direction.

**Remark.** The level curves \(y = \text{constant}\) yield, rather easily, the "particle
paths" of the solution.

### 3. In Case of Vacuums...

Liu and Smoller \([9]\) have demonstrated solutions of the Eulerian isen-
tropic gas dynamics equations, which contain vacuum states. In this section
we show how our results extend to arbitrary bounded measurable solutions of (1.1), including those containing vacuum states, provided (1.4) holds, \( \rho(0, S) = 0 \), and \( e(0, S) \) is finite. This condition is satisfied by ideal polytropic gases.

In this case the transformation \( T \) is still Lipschitz and, for fixed \( t \), \( y \) is a monotone function of \( x \). However, \( T \) is no longer one-to-one, and may in fact map sets of positive measure in \((x, t)\) space, namely the vacuum regions, into sets of zero measure in \((y, t)\) space. Hence we may no longer regard \( \tau \) as a function; however, it has a natural expression as a measure, namely \( \tau = T_* m_2 \).

**Lemma 1.** If (1.4) holds at \( t = 0 \), then \( T \) is proper and onto. It then follows from Formula 1 that \( \tau \) is a Radon measure.

**Proof.** Suppose that initially there is no half line of \( \mathbb{R} \) containing finite mass, i.e., (1.4) holds for \( t = 0 \). For any interval \([a, b] \subset \mathbb{R} \), \( t_1 > 0 \), \( \varepsilon > 0 \), \( \delta > 0 \), choose smooth functions \( \phi(t) \) and \( \psi(x) \) with compact support, such that \( \psi(x) = 1 \) for \( x \in [a, b] \), \( \phi(t) = 1 \) for \( t \in [0, t_1] \), \( \psi \geq 0 \), \( \phi \geq 0 \), \( \int |\phi'| = \int |\psi'| = 2 \), \( \int \phi < t_1 + \delta \), and \( \int \psi < b - a + \varepsilon \). Then since \( \rho_1 + (\rho u)_x = 0 \) weakly,

\[
\int \int \phi'(t) \psi(x) \rho(x, t) \phi(t) \psi(x)(\rho u)(x, t) \, dx \, dt \leq \int \psi(x) \rho_0(x) \, dx = 0.
\]

Let \( \delta \to 0 \); then we have

\[
\left| \int \psi(x)(\rho(x, t_1) - \rho_0(x)) \, dx \right| \leq 2 \| \rho u \|_{\infty} t_1,
\]

for any Lebesgue point \( t_1 \) of all the locally integrable functions \( t \to \int \psi(x) \rho(x, t) \, dx \), where \( \psi \) ranges over a sequence \( \psi_n \) as described above, with \( \varepsilon_n \to 0 \). We then have

\[
\left| \int_a^b \rho(x, t_1) - \rho_0(x) \, dx \right| \leq 2 \| \rho u \|_{\infty} t_1,
\]

for almost all \( t_1 \). Hence for fixed \( a \),

\[
\sup_{b > a} \left| \int_a^b \rho(x, t) - \rho_0(x) \, dx \right| \leq 2 \| \rho u \|_{\infty} t_1.
\]

Since by hypothesis

\[
\int_0^\infty \rho_0(x) \, dx = \infty,
\]

505/68/1-9
we must also have

$$\int_{0}^{\gamma} \rho(x, t_1) \, dx = \infty$$

for almost all $t_1$. Thus for almost all $t_1$, $y(\cdot, t_1)$ maps $\mathbb{R}$ onto itself, and $y(\cdot, t_1)^{-1}[a, b]$ is a closed interval of finite volume, hence compact. Since $t(x, t) = t$ and $T$ is Lipschitz, we see that $T$ is a proper map of $\mathbb{R} \times \mathbb{R}^+$ onto itself, and hence $\tau$ is a Radon measure.

**Lemma 2.** $T^\#(\rho) = m_2$.

**Proof.** Since $\rho$ is a Radon measure, it follows from Lemma 1 and Formula 1 that $T^\#(\rho)$ is a Radon measure. For any test function $\phi$,

$$T^\#(\rho)(\phi) = \int (\phi \circ T) \rho \, dx \, dt = \int (\phi \circ T) \partial y/\partial x \, dx \, dt = \int \phi \, dy \, dt,$$

where we have used Formulae 1 and 2. Thus on Baire sets, here the same as Borel sets, $T^\#(\rho) = m_2$. It follows from the regularity properties of Radon measures (see Definition 1) that $T^\#(\rho)$ is Lebesgue outer measure.

We note that for a.e. $t$, $y(x_1, t) = y(x_2, t)$ if and only if $\rho(x, t) = 0$ a.e. in $x_1 < x < x_2$. Consequently we may unambiguously define $\tilde{\rho}(y, t)$, $\tau$ a.e., such that $\tilde{\rho}(T(x, t)) = \rho(x, t)$. It follows from Formula 1 that $\tilde{\rho}$ is $\tau$-measurable. In particular, we see, using Lemma 2 and Formula 1, that $m_2 = T^\# \rho = \tilde{\rho}^\tau$, and thus $m_2$ is absolutely continuous with respect to $\tau$.

We decompose $\tau$ into its singular and absolutely continuous (a.c.) parts, with respect to $m_2$: $\tau_\sigma$ and $\tau_{a.c.}$, and we decompose $\mathbb{R} \times \mathbb{R}^+$ into Borel sets $V$ and $V^\sigma$ such that $\tau_\sigma(V^\sigma) = m_2(V) = 0$. Denote the density of $\tau_{a.c.}$ by $\tau$. Since $\tilde{\rho}\tau = m_2$, we must have $\tilde{\rho}\tau = 1$, $m_2$ – a.e. Since $p$ and $\rho e$ vanish at the vacuum, we may safely evaluate these nonlinear functions on $\tau$, except on the vacuum set $V$, where we set them equal to zero.

We will see later that Lagrangian densities, other than $\tau$, may be changed on sets of $m_2$-measure zero, so that they need only be defined $m_2$-a.e.; whereas fluxes, other than that of volume, must be defined $\tau$-a.e. Note that in (1.3), $e$ appears only in a density, so that the value of $e$ in a vacuum is irrelevant. Although $S$ appears as an argument of $\rho$, we have assumed that $\rho(0, S) = 0$. Therefore the value of $S$ in a vacuum is also irrelevant. Similarly the value of $u$ in a vacuum is irrelevant. Thus we may assume that all of the densities $D$ and fluxes $F$ vanish when $\rho = 0$, and we may define $(\tilde{u}, \tilde{D}, \tilde{F})(y, t)$, $\tau$-a.e., such that $(\tilde{u}, \tilde{D}, \tilde{F})(T(x, t)) = (u, D, F)(x, t)$. Note that the functional relationships between $(\tilde{D}, \tilde{F})$ and $(\tau, \tilde{u}, \tilde{S})$ still hold in an acceptable way.
LEMMA 3. \( \tau_t - \tilde{u}_y = 0 \), weakly.

Proof. Let \( \phi \) be a \( C^1 \) test function. Let \( \tau_0 = (T|_{t=0}) \# m_1 \). Then, using Formula 1,

\[
\int_{t>0} \int \phi_t \, dt - \phi_y \, dy \, dt + \int_{t=0} \phi \, d\tau_0
\]

\[
= \int_{t>0} \int (\phi_t - \phi_y \tilde{u}) \, dt + \int_{t=0} \phi \, d\tau_0
\]

\[
= \int_{t>0} \int \phi \circ T - (\phi_y \circ T) \rho u \, dx \, dt + \int_{t=0} \phi \circ T \, dx
\]

\[
= \int_{t>0} (\phi \circ T) \, dx \, dt + \int_{t=0} \phi \circ T \, dx = 0,
\]

because, since \( T \) is Lipschitz and proper, \( \phi \circ T \) is a Lipschitz test function. Thus \( \tau_t - u_y - 0 \), weakly.

The following example shows that in the presence of a vacuum, we must strengthen the definition of weak solution for (1.3b), (1.3c), (1.3d). Let \( \tau_0 = 1 + \delta_0 \), where \( \delta_0 \) is the Dirac delta measure at \( y = 0 \), and let \( u = 0 \) and \( S = 1 \). Then \( p \) is a nonzero constant for \( y \neq 0 \), and is zero at \( y = 0 \). However since \( p \) is equal, a.e., to a constant, \( p_y = 0 \) in the sense of distributions. Hence we have described a steady solution to (1.3) or (1.12). This is clearly unphysical, since in Eulerian coordinates we have a vacuum of length one, and hence \( p_x = (\text{const.})(\delta_1 - \delta_0) \).

One may view this example as showing that Lagrangian coordinates are unphysical when a vacuum is present. However, we now show in what way the two coordinate systems are equivalent.

We motivate our new definition of weak solution by examining how Eulerian test functions pull back to Lagrangian coordinates. Given \( \tau \) and \( u \) satisfying \( \tau_t - u_y = 0 \) in the usual weak sense, with \( \tau \) a positive Radon measure, and \( u \in L^\infty \), define \( x(y, t) \) such that

\[
\frac{\partial x}{\partial y} = \tau, \quad \frac{\partial x}{\partial t} = u.
\]

Since \( \tau \) is a Radon measure, we have that \( x \in BV_{loc} \), and for \( t \) fixed, \( x \) is a monotone increasing function of \( y \). Thus \( Q \), defined by \( Q(y, t) = (x(y, t), t) \) has a unique monotone left inverse \( T \), i.e., \( T(Q(y, t)) = (y, t) \). If \( \tau \) dominates \( m_2 \), then \( T \) is Lipschitz continuous.

Let \( \phi \) be an Eulerian test function. Then \( \phi \circ Q \) is a Lagrangian test function which is discontinuous, but \( BV \). By [11] we have \( (\phi \circ Q)_x = (\phi_t \circ Q)^\wedge + (\phi_x \circ Q)^\wedge u \), and \( (\phi \circ Q)_\tau = (\phi_x \circ Q)^\wedge \tau \), where \( (\phi_t \circ Q)^\wedge \) and
(ϕ ∘ Q) are defined at the regular points of Q. The regular points of Q are those points (y₀, t₀) at which the half-space approximate limits
\[ l_{±} x(y₀, t₀) \] 
exist for some a ∈ ℝ², such that
\[ \lim_{r → 0} \frac{1}{πr²} \int_{Q} m_{2} \{ (y, t) : |(y - y₀, t - t₀)| < r, \ |x(y, t) - l_{±} x(y₀, t₀)| > r, \} = 0. \]
Since x is locally bounded and in BV, almost all (y, t), with respect to one-
dimensional Hausdorff measure, are regular points of x, and of Q [11]. At
such points, (∇ϕ ∘ Q) is defined by
\[ (∇ϕ ∘ Q)^{\chi}(y, t) = \int_{0}^{1} (∇ϕ(l_{±} Q(y, t)(1 - s) + l_{±} Q(y, t) s) ds. \]
In this case we see that (ϕ ∘ Q) is a measure which is absolutely continuous
with respect to τ, and (ϕ ∘ Q) is a function. This motivates the following
definition.

**Definition 2.** We say that (τ, u, S) is a weak solution of (1.3), if τ is a
Radon measure on ℝ × ℝ⁺, and u and S are bounded τ-measurable functions
such that (1.3a) holds in the sense of distributions, and the weak formulation
of (1.3b), (1.3c), (1.3d) holds with all test functions ϕ with compact support
such that ϕₙ = fₙ, and ϕₙ = g, with f, g ∈ L∞(τ).

**Remark.** Since, in the weak formulation of (1.3b), (1.3c), (1.3d), we
integrate Dϕₙ, where D is u, e + u²/2, or S, and since ϕₙ ∈ L∞, we see that D
may be changed on sets of m₂-measure zero. However, the corresponding
fluxes must be defined τ-a.e.

**Lemma 4.** If τ is a Radon measure on ℝ × ℝ⁺ such that τ(E) ≥ km₂(E),
for all E ⊆ ℝ × ℝ⁺, u ∈ L∞(τ), ϕ is a function with compact support, and
ϕₙ = fₙ, ϕₙ = g, where f, g ∈ L∞(τ), then there is at least one function ψ such
that ψ is Lipschitz with compact support, and ψ ∘ Q = ϕ.

**Proof.** We construct ψ as the limit, as ε → 0, of ψₙ = ϕₙ ∘ Tₑ, where jₑ is a
smoothing kernel, ϕₑ = ϕ ∗ jₑ, xₑ = x ∗ jₑ, and Qₑ(y, t) = (xₑ(y, t), t). Then
Qₑ is a diffeomorphism. Let Tₑ = Qₑ⁻¹. We have \( \partial xₚ / \partial y = τₑ \) ∗ jₑ, \( \partial xₚ / \partial t = uₑ = u ∗ jₑ \), \( \partial ϕₑ / \partial y = (fτ) ∗ jₑ = (fτ)ₑ \), and \( \partial ϕₑ / \partial t = gₑ \). Then
\[ \left\| \frac{∂ψₑ}{∂x} \right\|_∞ = \left\| \frac{fτ}{τₑ} \right\|_∞ \leq \left\| f \right\|_∞, \]
\[ \left\| \frac{∂ψₑ}{∂t} \right\|_∞ = \left\| gₑ + (fτ)ₑ uₑ / τₑ \right\|_∞ \leq \left\| gₑ \right\|_∞ + \left\| f \right\|_∞ \left\| u \right\|_∞. \]
Since $\psi_n$ is also uniformly bounded we have $\psi_n \to \psi$ uniformly on compact sets, for some subsequence $\psi_n$, and some Lipschitz function $\psi$, by the Arzela–Ascoli theorem. Since $\psi_n$ have uniformly bounded support, $\psi$ has compact support. We now show that $\psi \circ Q = \phi$, $m_2$ a.e. Since $\psi_n \circ Q_n = \phi_n$, which tends to $\phi$ in $L^1(m_2)$, and $Q_n \to Q$ in $L^1_{\text{loc}}$, we have, passing to another subsequence, $\phi = \lim \psi_n \circ Q_n = \psi \circ Q$ a.e., since $\psi_n \to \psi$ uniformly.

**Lemma 5.** Let $(D, F)$ be a density-flux pair of (1.1b), (1.1c) from a weak solution of (1.1) wherein $\rho, u, v$, and $S$ are bounded. Suppose $p(0, S) = 0$ and $e(0, S)$ is finite for finite $S$. Then $\tilde{u}, \tilde{D}, \tilde{F}$, as defined above, satisfy $(\tau \tilde{D})_t + (\tilde{F} - \tilde{u} \tilde{D})_x = 0$ in the sense of Definition 2.

**Proof.** We have, using Formula 1,

\[
\int_{t>0} (\psi \circ Q)_t \tilde{D} \, dt + (\psi \circ Q)_t (\tilde{F} - \tilde{u} \tilde{D}) \, dy \, dt + \int_{t=0} (\psi \circ Q) \, \tilde{D}_0 \, dt_0
\]

\[
= \int_{t>0} \left[ (\psi \circ Q)^\wedge + (\psi_x \circ Q)^\wedge \tilde{u} \right] \tilde{D} + (\psi_x \circ Q)^\wedge (\tilde{F} - \tilde{u} \tilde{D}) \, dt
\]

\[
+ \int_{t=0} (\psi \circ Q) \, \tilde{D}_0 \, dt_0
\]

\[
= \int_{t>0} ((\psi \circ Q)^\wedge \circ T) \, D + ((\psi_x \circ Q)^\wedge \circ T) \, F \, dx \, dt
\]

\[
+ \int_{t=0} (\psi \circ Q \circ T) \, D_0 \, dx.
\]

We now show that $D(\psi \circ Q)^\wedge \circ T = D\psi$, and $F(\psi_x \circ Q)^\wedge \circ T = F\psi_x$, $m_2$ a.e., and that $D_0(\psi \circ Q \circ T) = D_0\psi$, $m_1$ a.e. on the $x$-axis. Then the above equals

\[
\int_{t>0} \psi_x D + \psi_x F \, dx \, dt + \int_{t=0} \psi D_0 \, dx = 0,
\]

and we are done.

For $(x, t)$ such that $\rho(x, t) = 0$, we have that $\tilde{D} = 0 = 0$ and $\nabla \psi$ is bounded, so equality holds in this case. Since $Q$ is approximately continuous $m_2$ a.e., i.e., $l_x = l_{-x}$ at almost all regular points of $Q$, we have $(\nabla \psi \circ Q)^\wedge = \nabla \psi \circ Q$ a.e. But $m_2 = \hat{\rho} \tau$, so if

\[
E = \{ (y, t): (\nabla \psi \circ Q)^\wedge \neq \nabla \psi \circ Q \},
\]
we have

\[ 0 = m_2(E) = \int_E \rho \, dt = \int_{T^{-1}(E)} \rho \, dx \, dt. \]

Hence \((\nabla \psi \circ Q) \circ T = (\nabla \psi \circ Q) \circ T, \rho - \text{a.e.}\) We now show that 
\((\nabla \psi \circ Q) \circ T = \nabla \psi, \rho - \text{a.e.}\) At points of approximate continuity of \(Q\) we have \(Q_n \to Q\), where \(Q_n\) is the smoothing of \(Q\) used above. If \(T(x_0, t_0) = (y_0, t_0)\) is such a point,

\[
|Q_n(T_n(x_0, t_0)) - Q(T(x_0, t_0))| \\
\leq \iint \left| j_n \left[ T_n(x_n, t_0) - (y, t) \right] \left[ Q(y, t) - Q(y_0, t_0) \right] \right| \, dy \, dt.
\]

Since \(\{T_n\}\) is uniformly Lipschitz, we may pass to another subsequence such that \(T_n \to T\) uniformly on compact sets; one easily checks that the limit is the left inverse of \(Q\). Choose \(R\) small so that for all \(r < R\),

\[
m_2 \{ (y, t): |Q(y, t) - Q(y_0, t_0)| > \delta_1, \quad |(y - y_0, t - t_0)| < r \} < 2 \pi r^2
\]

and choose \(n\) large so that \(|T_n(x_0, t_0) - (y_0, t_0)| < R/2\), and

\[
\text{supp}(j_n) \subset \{ (y, t): |(y, t)| < R/2 \}.
\]

Then

\[
|Q_n(T_n(x_0, t_0)) - Q(T(x_0, t_0))| < \delta_2 \pi r^2 \|j_n\|_\infty \|Q - Q(y_0, t_0)\|_\infty, B(R) + \delta_1,
\]

where \(B(R)\) denotes the ball of radius \(R\) centered at \((y_0, t_0)\), and \(2r = \text{diam(supp}(j_n))\), so that \(r^2 \|j_n\|_\infty\) is bounded independent of \(n\). Thus,

requiring \(\delta_1\) and \(\delta_2\) to be arbitrarily small, we see that \(Q(T(x_0, t_0)) = (x_0, t_0)\). Thus \((\nabla \psi \circ Q) \circ T = (\nabla \psi \circ Q) \circ T = \nabla \psi, \rho - \text{a.e.}\) Since \(D = F = 0\) where \(\rho = 0\), we see that \((D, F) \cdot (\nabla \psi \circ Q) \circ T = (D, F) \cdot \nabla \psi, m_2 - \text{a.e.}\)

Using the fact that \(x(y, 0)\) is monotone, and using methods similar to the above argument, one may also show that \((Q \circ T)(x, 0) = x, \rho_0 - \text{a.e.}\) Since \(D_0 = 0\) whenever \(\rho_0 = 0\), we have that \(D_0(\psi \circ Q \circ T) = D_0 \psi, m_1 - \text{a.e.}\).

We now suppose that \((\tau, u, S)\) satisfy (1.3) according to Definition 2. We construct \(Q\) as in (3.1) and \(T\) such that \(T(Q(y, t)) = (y, t)\). Let \(\rho = Q_{\#} m_2\). Let \(\tilde{\rho}\) be the density of \(m_2\) with respect to \(\tau\), and let \(\bar{\rho}\) be the density of \(\tau_{\#} m_2\) with respect to \(m_2\), so that \(\tilde{\rho} = 1, m_2 - \text{a.e.}\)

**Lemma 6.** \(\rho = \partial y / \partial x = \tilde{\rho} \circ T\).
Proof. Since \( T \circ Q \) is the identity map, \( T_\# Q_\# m_2 = T_\# (\rho) = m_2 \). Furthermore on any test function \( \phi \), using Formulae 1 and 2,

\[
\int \phi \,dT_\# \left( \frac{\partial y}{\partial x} \right) = \int \phi \circ T \frac{\partial y}{\partial x} \, dx \, dt \\
= \int \phi(y, t) \, N(T, (y, t)) \, dy \, dt.
\]

Let

\[ A = \{(y, t); N(T, (y, t)) > 1\} \]

and let \( B = T^{-1}(A) \). Then if \( (x, t) \in B \), there is \( (x', t) \in B \) such that \( T(x, t) = T(x', t) \) and \( x' \neq x \). Since \( T \) is monotone, we must have \( \frac{\partial y}{\partial x} = 0 \) on the line segment from \( (x, t) \) to \( (x', t) \). Consequently for each \( t \),

\[ B \cap \left\{(x, t); \frac{\partial y}{\partial x} \neq 0 \text{ or does not exist}\right\} \]

is at most countable. Hence by Fubini's theorem \( \frac{\partial y}{\partial x} = 0 \) a.e. on \( B \). Consequently

\[
\int_A N(T, (y, t)) \, dy \, dt = \int_B \frac{\partial y}{\partial x} \, dx \, dt = 0
\]

and thus we see that \( T_\# \frac{\partial y}{\partial x} = m_2 \) on test functions, and that \( m_2(A) = 0 \). Thus we see, as in Lemma 2, that these two Radon measures are equal.

Thus \( T_\# \frac{\partial y}{\partial x} = T_\# (\rho) = \tilde{\rho} \tau \). This means that \( \frac{\partial y}{\partial x} = \rho = (\tilde{\rho} \circ T) \) as measures on sets of the form \( F = T^{-1}(E) \). For other sets \( F \), note that \( T^{-1}(T(F)) \setminus F \subset B \), and since \( \frac{\partial y}{\partial x}(B) = 0 \), we have that

\[
\frac{\partial y}{\partial x} (T^{-1}(T(F))) = \frac{\partial y}{\partial x} (B) = \frac{\partial y}{\partial x} (F) = \frac{\partial y}{\partial x} (T^{-1}(T(F))).
\]

Thus \( \frac{\partial y}{\partial x}(F) = \frac{\partial y}{\partial x}(T^{-1}(T(F))) \). Also note that \( \rho(F) = m_2(Q^{-1}(F)) \), and

\[
\rho(T^{-1}(T(F))) = m_2(Q^{-1}(T^{-1}(T(F)))) = m_2(T(F)).
\]

But \( T(F) \setminus A \subset Q^{-1}(F) \subset T(F) \), and \( m_2(A) = 0 \), so \( m_2(Q^{-1}(F)) = m_2(T(F)) \).

Thus \( \rho = \frac{\partial y}{\partial x} = \tilde{\rho} \circ T \) as Radon measures.

Lemma 7. \( T_\# m_2 = \tau \).

Proof. Similar to that of Lemma 2.
Let \((u, S)(x, t) = (u, S)(T(x, t))\). Then for any \(C^1\) testfunction \(\phi\), and any density-flux pair \((D, F)\) of (1.3),

\[
\iint_{t > 0} \phi \Delta d\rho + \phi_x u \, d\rho + \int_{t > 0} \phi D_0 \, d\rho_0 = \iint_{t > 0} (\phi \circ Q)(D \circ Q) + (\phi_x \circ Q)(u \circ Q) \, dy \, dt + \int_{t > 0} (\phi \circ Q)(D_0 \circ Q) \, dy.
\]

Note that \((D, u) \circ Q = (D, u) \circ T \circ Q = (D, u)\). Furthermore

\[
\iint_{t > 0} \phi_x F \, dx \, dt = \iint_{t > 0} F \, d(T_* (\phi_x m_2)).
\]

We evaluate \(T_* (\phi_x m_2)\) as follows. On any smooth function \(\sigma\) with compact support,

\[
T_* (\phi_x m_2)(\sigma) = (\phi_x m_2)(\sigma \circ T) = - (\phi m_2)(\sigma \circ T)_x = - (\phi m_2)(\sigma_y \circ T) \rho
\]

\[
= - \iint_{t > 0} \phi(\sigma_y \circ T) \, d\rho = - \iint_{t > 0} (\phi \circ Q)_y \sigma_y \, dy \, dt = (\phi \circ Q)_y(\sigma).
\]

Thus, as in Lemma 2, we see that \(T_* (\phi_x m_2) = (\phi \circ Q)_y\), as signed Radon measures, so that

\[
\iint_{t > 0} F\phi_x \, dx \, dt = \iint_{t > 0} F(\phi \circ Q)_y \, dy \, dt.
\]

We then have

\[
\iint_{t > 0} \phi \Delta d\rho + \phi_x(F \, dx \, dt + u \, d\rho) + \int_{t > 0} \phi D_0 \, d\rho_0 = \iint_{t > 0} (\phi \circ Q)_y D + (\phi \circ Q)_x F \, dy \, dt + \int_{t > 0} (\phi \circ Q)_y D_0 \, dy = 0,
\]

since \(\phi \circ Q\) is a discontinuous test function such that \((\phi \circ Q)_x = g\), and \((\phi \circ Q)_y = ft\), with \(f, g \in L^\infty(\tau)\). Thus \((\rho D)_t + (F + \rho u D)_x = 0\), weakly, and we see that \((\rho, u, S)\) is a weak solution of (1.1).

### 4. Other Coordinates

It should be clear that other transformations are possible. In particular, for (1.1), we could integrate the energy density \(\rho u^2/2 + \rho e\) instead of \(\rho\) to obtain \(y\). By Theorem 2, this yields an equivalent system of conservation.
laws, the nontrivial ones being the conservation of volume, mass, and momentum.

One may also consider a transformation of both space and time. A simple sufficient condition, for the transformed system to be an autonomous system of conservation laws, is that the differential of the transformation $T$ should be a function of the conserved unknown, $U$. If $T(x, t) = (x', t')$, and

$$DT = \left( \frac{\partial T}{\partial x}, \frac{\partial T}{\partial t} \right) = (G(U), H(U)), $$

and $JT = \det(DT)$, then the transformed system is

$$\frac{\partial}{\partial t'} \left( \left( U \frac{\partial t'}{\partial t} + F(U) \frac{\partial t'}{\partial x} \right) \frac{l}{JT} \right) + \frac{\partial}{\partial x'} \left( \left( U \frac{\partial x'}{\partial t} + F(U) \frac{\partial x'}{\partial x} \right) \frac{l}{JT} \right) = 0. $$

Given a matrix $(G(U), H(U))$, a necessary condition for the existence of a transformation $T$ such that $DT = (G(U), H(U))$ is that $G(U) = H(U)$, weakly. Thus $(G(U), H(U))$ must be of the form $A \cdot (U, -F(U)) + (b, c)$, where $A$ is a constant $2 \times n$ matrix and $b$ and $c$ are constant 2-vectors. Of course the transformation $T$ must be one-to-one, Cauchy data must be prescribed on a space-like curve, and the forward time direction must be properly chosen to respect the entropy condition. Presumably these are the only transformations which produce an autonomous system of conservation laws which is equivalent, for weak solutions, to the original system.

The existence of many equivalent coordinate system suggests a manifold structure. The fact that affine functions of the conserved densities are transformed into other affine functions, and that convex functions are transformed into other convex functions (2.2), (2.3) suggests that this structure is intrinsically affine. The value of this insight, if any, remains to be determined.

REFERENCES