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# Concentration on minimal submanifolds for a singularly perturbed Neumann problem

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## Abstract

We consider the equation  $-\varepsilon^2 \Delta u + u = u^p$  in  $\Omega \subseteq \mathbb{R}^N$ , where  $\Omega$  is open, smooth and bounded, and we prove concentration of solutions along  $k$ -dimensional minimal submanifolds of  $\partial\Omega$ , for  $N \geq 3$  and for  $k \in \{1, \dots, N-2\}$ . We impose Neumann boundary conditions, assuming  $1 < p < (N-k+2)/(N-k-2)$  and  $\varepsilon \rightarrow 0^+$ . This result settles in full generality a phenomenon previously considered only in the particular case  $N=3$  and  $k=1$ .

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## 1. Introduction

In this paper we study concentration phenomena for the problem

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \quad (P_\varepsilon)$$

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where  $\Omega$  is a smooth bounded domain of  $\mathbb{R}^N$ ,  $p > 1$ , and where  $\nu$  denotes the unit normal to  $\partial\Omega$ . Given a smooth embedded non-degenerate minimal submanifold  $K$  of  $\partial\Omega$ , of dimension  $k \in \{1, \dots, N - 2\}$ , we prove existence of solutions of  $(P_\varepsilon)$  concentrating along  $K$ . Since the solutions we find have a specific asymptotic profile, which is described below, a natural restriction on  $p$  is imposed, depending on the dimension  $N$  and  $k$ , namely  $p < (N - k + 2)/(N - k - 2)$ .

Problem  $(P_\varepsilon)$  or some of its variants (including the presence of non-homogeneous terms, different boundary conditions, etc.) arise in several contexts, as the nonlinear Schrödinger equation or from modeling reaction–diffusion systems, see for example [1,17,41] and references therein. A typical phenomenon one observes is the existence of solutions which are sharply concentrated near some subsets of their domain.

Concerning reaction–diffusion systems, this phenomenon is related to the so-called Turing’s instability, [50]. According to this principle, reaction–diffusion systems whose reactants have very different diffusivities might generate stable non-trivial patterns. This is indeed more likely to happen when more reactants are present since, as shown in [9,37], scalar reaction–diffusion equations in a convex domain admit only constant stable equilibria.

A well-known system is the following one

$$\begin{cases} \mathcal{U}_t = d_1 \Delta \mathcal{U} - \mathcal{U} + \frac{\mathcal{U}^p}{\mathcal{V}^q} & \text{in } \Omega \times (0, +\infty), \\ \mathcal{V}_t = d_2 \Delta \mathcal{V} - \mathcal{V} + \frac{\mathcal{U}^r}{\mathcal{V}^s} & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial \mathcal{U}}{\partial \nu} = \frac{\partial \mathcal{V}}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, +\infty), \end{cases} \quad (GM)$$

introduced in [20] to describe some biological experiment. The functions  $\mathcal{U}$  and  $\mathcal{V}$  represent the densities of some chemical substances, the numbers  $p, q, r, s$  are non-negative and such that  $0 < (p - 1)/q < r/(s + 1)$ , and it is assumed that the diffusivities  $d_1$  and  $d_2$  satisfy  $d_1 \ll 1 \ll d_2$ . In the stationary case of  $(GM)$ , as explained in [41,44], when  $d_2 \rightarrow +\infty$  the function  $\mathcal{V}$  is close to a constant (being nearly harmonic and with zero normal derivative at the boundary), and therefore the equation satisfied by  $\mathcal{U}$  is similar to  $(P_\varepsilon)$ , with  $\varepsilon^2 = d_1$ .

The typical concentration behavior of solutions  $u_\varepsilon$  to  $(P_\varepsilon)$  is via a scaling of the variables in the form  $u_\varepsilon(x) \sim u_0((x - Q)/\varepsilon)$ , where  $Q$  is some point of  $\overline{\Omega}$ , and where  $u_0$  is a solution of the problem

$$-\Delta u_0 + u_0 = u_0^p \quad \text{in } \mathbb{R}^N \quad (\text{or in } \mathbb{R}_+^N = \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_N > 0\}), \quad (1)$$

the domain depending on whether  $Q$  lies in the interior of  $\Omega$  or at the boundary; in the latter case Neumann conditions are imposed.

When  $p < (N + 2)/(N - 2)$  (and indeed only if this inequality is satisfied), problem (1) admits positive radial solutions which decay to zero at infinity. Solutions of  $(P_\varepsilon)$  with this profile are called *spike-layers*, since they are highly concentrated near some point of  $\overline{\Omega}$ . There is an extensive literature regarding this type of solutions, beginning from the papers [30,42,43]. Indeed their structure is very rich, and there are also solutions with multiple peaks, both at the boundary and at the interior of  $\Omega$ . We refer for example to the papers [11,14,21–24,28,29,51–53].

In recent years, some new types of solutions have been constructed: they indeed concentrate at sets of positive dimension and their profile consists of solutions of (1) which do not decay to zero at infinity. In [34,35] it has been shown that given any smooth bounded domain  $\Omega \subseteq \mathbb{R}^N$ ,  $N \geq 2$ , and any  $p > 1$ , there exists a sequence  $\varepsilon_j \rightarrow 0$  such that  $(P_{\varepsilon_j})$  possesses solutions concentrating at  $\partial\Omega$  along this sequence. Their profile is a solution of (1) (for  $N = 1$ ) on the half real line

which tends to zero at infinity and which satisfies the condition  $u'_0(0) = 0$ . This function can also be trivially extended as a cylindrical solution to (1) on the whole  $\mathbb{R}_+^N$ .

Later in [33] it has been proved that, if  $\Omega$  is a smooth bounded set of  $\mathbb{R}^3$ , if  $p > 1$  and if  $h$  is a closed, simple non-degenerate geodesic on  $\partial\Omega$ , then there exists again a sequence  $(\varepsilon_j)_j$  converging to zero such that  $(P_{\varepsilon_j})$  admits solutions  $u_{\varepsilon_j}$  concentrating along  $h$  as  $j$  tends to infinity. In this case the profile of  $u_{\varepsilon_j}$  is a decaying solution of (1) in  $\mathbb{R}_+^2$ , again extended to a cylindrical solution in higher dimension.

These are examples of a phenomenon which has been conjectured to hold in more general cases: in fact it is expected that, under generic assumptions, if  $\Omega \subseteq \mathbb{R}^N$  and if  $k$  is an integer between 1 and  $N - 1$ , there exist solutions of  $(P_\varepsilon)$  concentrating along  $k$ -dimensional sets when  $\varepsilon$  tends to zero. While the case  $k = N - 1$  has been tackled in [35], the goal of the present paper is to consider  $k \leq N - 2$ , and to prove this conjecture under rather mild assumptions on the limit set. Before stating our main theorem we introduce some preliminary notation.

Given a smooth  $k$ -dimensional submanifold  $K$  of  $\partial\Omega$ , and given any  $q \in K$  we can choose a system of coordinates  $(\bar{y}, \zeta)$  in  $\Omega$  orthonormal at  $q$  and such that  $(\bar{y}, 0)$  are coordinates on  $K$ , and with the property that

$$\frac{\partial}{\partial \bar{y}_a} \Big|_q \in T_q K, \quad a = 1, \dots, k; \quad \frac{\partial}{\partial \zeta_i} \Big|_q \in T_q \partial\Omega, \quad i = 1, \dots, n; \quad \frac{\partial}{\partial \zeta_{n+1}} \Big|_q = \nu(q), \quad (2)$$

where we have set  $n = N - k - 1$ . Our main theorem is the following: we refer to Section 2 for the geometric terminology.

**Theorem 1.1.** *Let  $\Omega \subseteq \mathbb{R}^N$ ,  $N \geq 3$ , be a smooth and bounded domain, and let  $K \subseteq \partial\Omega$  be a compact embedded non-degenerate minimal submanifold of dimension  $k \in \{1, \dots, N - 2\}$ . Then, if  $p \in (1, (N - k + 2)/(N - k - 2))$ , there exists a sequence  $\varepsilon_j \rightarrow 0$  such that  $(P_{\varepsilon_j})$  admits positive solutions  $u_{\varepsilon_j}$  concentrating along  $K$  as  $j \rightarrow \infty$ . Precisely there exists a positive constant  $C$ , depending on  $\Omega$ ,  $K$  and  $p$  such that for any  $x \in \Omega$*

$$u_{\varepsilon_j}(x) \leq C e^{-\frac{\text{dist}(x,K)}{C\varepsilon_j}};$$

moreover for any  $q \in K$ , in a system of coordinates  $(\bar{y}, \zeta)$  satisfying (2), for any integer  $m$  one has  $u_{\varepsilon_j}(0, \varepsilon_j \cdot) \xrightarrow{C_{\text{loc}}^m(\mathbb{R}_+^{n+1})} w_0(\cdot)$ , where  $w_0 : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}$  is the unique radial solution of

$$\begin{cases} -\Delta u + u = u^p & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathbb{R}_+^{n+1}, \\ u > 0, & u \in H^1(\mathbb{R}_+^{n+1}). \end{cases} \quad (3)$$

**Remarks 1.2.** (a) Differently from the previous papers concerning the case  $N = 3$  and  $k = 1$ , or concentration at the whole  $\partial\Omega$ , we require an upper bound on  $p$  depending on  $N$  and  $k$ . This condition is rather natural, since (3) is solvable if and only if  $p < (N - k + 2)/(N - k - 2)$ , see [8,46,49] and in this case the solution is radial and unique (up to a translation), see [18,26]. In any case, our assumptions allow supercritical exponents as well.

(b) As for the results in [33–35], existence is proved only along a sequence  $\varepsilon_j \rightarrow 0$  (actually with our proof it can be obtained for  $\varepsilon$  in a sequence of intervals  $(a_j, b_j)$  approaching zero, but

not for any small  $\varepsilon$ ). This is caused by a resonance phenomenon we are going to discuss below, explaining the ideas of the proof. This resonance is peculiar of multidimensional spike-layers, see also [15], and other geometric problems, see [32,38]. In some cases, when some symmetry is present, it is possible to get rid of this resonance phenomenon working in spaces of invariant functions. We refer for example to the papers [2,3,5–7,12,13,36,40,47].

We can describe the resonance phenomenon, which causes the main difficulty in proving Theorem 1.1, in the following way. By the change of variables  $x \mapsto \varepsilon x$ , we are reduced to consider the problem

$$\begin{cases} -\Delta u + u = u^p & \text{in } \Omega_\varepsilon, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ u > 0 & \text{in } \Omega_\varepsilon, \end{cases} \tag{P_\varepsilon}$$

where  $\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega$ . As for (2), given  $\hat{q} \in K_\varepsilon := \frac{1}{\varepsilon}K$ , we can choose scaled coordinates  $(y, \zeta)$  on  $\Omega_\varepsilon$  such that  $\partial_{y_a}|_{\hat{q}} \in T_{\hat{q}}K_\varepsilon$ ,  $\partial_{\zeta_i}|_{\hat{q}} \in T_{\hat{q}}\partial\Omega_\varepsilon$  and  $\partial_{\zeta_{n+1}}|_{\hat{q}} = \nu(\hat{q})$ . Then, letting  $\tilde{u}_\varepsilon$  denote the scaling of  $u_\varepsilon$  to  $\Omega_\varepsilon$ , we have that, in a plane through  $\hat{q}$  normal to  $K_\varepsilon$ ,  $\tilde{u}_\varepsilon$  behaves like  $\tilde{u}_\varepsilon(0, \zeta) = u_\varepsilon(0, \varepsilon\zeta) \simeq w_0(\zeta)$ . This amounts to the fact that  $\tilde{u}_\varepsilon(x) \simeq w_0(\text{dist}(x, K_\varepsilon))$ ,  $x \in \Omega_\varepsilon$ , and therefore  $\tilde{u}_\varepsilon$  has a fixed profile in the directions perpendicular to the expanding domain  $K_\varepsilon$ . Since the function  $w_0(\text{dist}(x, K_\varepsilon))$  can be considered as an *approximate solution* to  $(\tilde{P}_\varepsilon)$ , it is natural to use local inversion arguments near this function in order to find true solutions. For this purpose it is necessary to understand the spectrum of the linearization of  $(\tilde{P}_\varepsilon)$  at approximate solutions.

For simplicity, let us assume for the moment that  $K$  is  $(N - 2)$ -dimensional, namely that its codimension in  $\partial\Omega$  is equal to 1, as in [33]. Then, letting  $\tilde{\nu}$  denote the normal to  $K$  in  $\partial\Omega$ , we can parameterize naturally a neighborhood of  $K_\varepsilon$  as a product of the form  $K_\varepsilon \times (-\delta/\varepsilon, \delta/\varepsilon)$ , where  $\delta$  is a small positive number, via the exponential map in  $\partial\Omega_\varepsilon$

$$(y, s) \mapsto \exp_y^{\partial\Omega_\varepsilon}(s\tilde{\nu}); \quad (y, s) \in K_\varepsilon \times \left(-\frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon}\right). \tag{4}$$

Similarly, if  $\nu(y, s)$  is the inner unit normal to  $\partial\Omega_\varepsilon$  at the image of  $(y, s)$  under the above map, we can parameterize a neighborhood of  $K_\varepsilon$  in  $\Omega_\varepsilon$  with a product  $K_\varepsilon \times (-\delta/\varepsilon, \delta/\varepsilon) \times (0, \delta/\varepsilon)$  by

$$(y, s, t) \mapsto \exp_y^{\partial\Omega_\varepsilon}(s\tilde{\nu}) + t\nu(y, s); \quad (y, s, t) \in K_\varepsilon \times \left(-\frac{\delta}{\varepsilon}, \frac{\delta}{\varepsilon}\right) \times \left(0, \frac{\delta}{\varepsilon}\right).$$

When  $\varepsilon$  tends to zero, the standard Euclidean metric of  $\Omega_\varepsilon$  becomes closer and closer (on the above set) to the product of the metric of  $K_\varepsilon$  and that of  $\mathbb{R}^2$  (parameterized by the variables  $s$  and  $t$  as Cartesian coordinates). Therefore, since the set  $(-\delta/\varepsilon, \delta/\varepsilon) \times (0, \delta/\varepsilon)$  converges to  $\mathbb{R}_+^2 = \{(s, t) \in \mathbb{R}^2: t > 0\}$ , in a first approximation we get that the linearization of  $(\tilde{P}_\varepsilon)$  at  $\tilde{u}_\varepsilon$  is

$$\begin{cases} -\Delta_{K_\varepsilon} u - \partial_{ss}^2 u - \partial_{tt}^2 u + u - pw_0(\zeta)u = 0 & \text{in } K_\varepsilon \times \mathbb{R}_+^2, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } K_\varepsilon \times \partial\mathbb{R}_+^2. \end{cases} \tag{5}$$

The spectrum of this linear operator can be evaluated almost explicitly. Referring to Section 4 for details (see also [33, Proposition 2.9] for the case  $N = 3$ ), here we just give some qualitative description of its properties.

Given an arbitrary function  $u \in H^1(K_\varepsilon \times \mathbb{R}_+^2)$ , and letting  $\zeta = (s, t)$ , we can decompose it in Fourier modes in the variables  $y$  as

$$u(y, \zeta) = \sum_j \phi_j(\varepsilon y) u_j(\zeta).$$

Here  $\phi_j$  are the eigenfunctions of the Laplace–Beltrami operator on  $K$ , namely  $-\Delta_K \phi_j = \rho_j \phi_j$ ,  $j = 0, 1, 2, \dots$ , where the eigenvalues  $(\rho_j)_j$  are counted with their multiplicities.

If  $u$  is an eigenfunction (with respect to the duality induced by the space  $H^1(K_\varepsilon \times \mathbb{R}_+^2)$ ) of the linear operator in (5) with corresponding eigenvalue  $\lambda$ , then it can be shown (see Section 4 for details) that the functions  $u_j$  satisfy the equation

$$\begin{cases} (1 - \lambda)[-\Delta u_j + (1 + \alpha)u_j] - p w_0^{p-1} u_j = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial u_j}{\partial t} = 0 & \text{on } \partial \mathbb{R}_+^2, \end{cases} \tag{6}$$

where  $\alpha = \varepsilon^2 \rho_j$ . It is known that when  $\alpha = 0$  the latter problem admits a negative eigenvalue  $\eta_0$  (with eigenfunction  $w_0$ ), a zero eigenvalue  $\sigma_0$  (with eigenfunction  $\partial_s w_0$ ), while all the other eigenvalues are positive. This structure is due to the fact that  $w_0$  is a mountain-pass solution of (3) (so its Morse index is at most 1), and the presence of a kernel derives from the fact that this equation is invariant by translation in the  $s$  variable. When  $\alpha$  is positive instead, it turns out that the first eigenvalue  $\eta_\alpha$  of (6) and the second one  $\sigma_\alpha$  are strictly increasing functions of  $\alpha$  with positive derivative, and tend to 1 as  $\alpha \rightarrow +\infty$ ; moreover, the eigenfunctions corresponding to  $\eta_\alpha$  (respectively  $\sigma_\alpha$ ) are radial (respectively odd in  $s$ ) for every value of  $\alpha$ . In particular, there exists  $\bar{\alpha} > 0$  such that  $\eta_{\bar{\alpha}} = 0$ , so when  $\varepsilon^2 \rho_j$  is close to  $\bar{\alpha}$  we obtain some small eigenvalues of the original linearized problem (5).

From the monotonicity in  $\alpha$  and from the Weyl’s asymptotic formula for  $\rho_j$ , it follows that the eigenvalues of the operator in (5) are, roughly, either of the form  $\eta_0 + \varepsilon^2 j^{2/(N-2)}$  for some  $j \in \mathbb{N}$ , or of the form  $\varepsilon^2 l^{2/(N-2)}$  for some  $l \in \mathbb{N}$ , or have a uniform positive bound from below.

In the case of general codimension it is not possible to decompose a neighborhood of  $K$  (in  $\partial \Omega$ ) as for (4), but instead one has to model it on the *normal bundle* of  $K_\varepsilon$  in  $\Omega_\varepsilon$ , see Section 4.2 for details. Considering the corresponding approximate linearized operator, one can prove that its eigenvalues are now, roughly either of the form  $\eta_{\varepsilon^2 \rho_j} \simeq \eta_0 + \varepsilon^2 j^{2/k}$ , or of the form  $\sigma_{\varepsilon^2 \omega_l} \simeq \varepsilon^2 l^{2/k}$ ,  $j, l \in \mathbb{N}$ , or, again, have a uniform positive bound from below. Here  $(\rho_j)_j$  still represent the eigenvalues of the Laplace–Beltrami operator on  $K$ , while the numbers  $(\omega_l)_l$  stand for the eigenvalues of the *normal Laplacian* of  $K$  (considered as a submanifold of  $\partial \Omega$ ), see Section 2 for its definition and the corresponding Weyl’s asymptotic formula. We are interested in particular in the following two features of the spectrum:

- (1) *Resonances*: There are two kinds of eigenvalues which can approach zero. First of all, those of the form  $\eta_\alpha$  when  $\alpha$  is close to  $\bar{\alpha}$ . This happens when  $\varepsilon^2 j^{2/k} \simeq \bar{\alpha}$ , namely when  $j \simeq \varepsilon^{-k}$ ; furthermore, the average distance between two consecutive such eigenvalues is of order  $\varepsilon^2 j^{2/k-1} \simeq j^{-1} \simeq \varepsilon^k$ . The other resonant eigenvalues are of the form  $\sigma_\alpha \simeq \alpha$  for  $\alpha$  close to zero, namely when  $\alpha = \varepsilon^2 l^{2/k}$  and  $l$  is sufficiently small (compared to, say, some negative power of  $\varepsilon$ ). Hence the distance from zero of the smallest eigenvalues of this type is of order  $\varepsilon^2$ . Indeed, an accurate expansion in  $\varepsilon$ , see Section 5.2, yields that this distance is bounded from below by a multiple of  $\varepsilon^2$  when  $K$  is a non-degenerate minimal submanifold.

(2) *Eigenfunctions*: As for the case of codimension 1, it turns out that the eigenfunctions corresponding to the  $\eta_\alpha$ 's are of the form  $\phi_j(\varepsilon y)u_j(\zeta)$ , where  $u_j$  is radial in the variable  $\zeta$  ( $\zeta$  represent here some orthonormal coordinates in the normal bundle of  $K_\varepsilon$ ). The function  $\phi_j$  instead oscillates faster and faster as  $\varepsilon$  tends to zero, since  $j$  is of order  $\varepsilon^{-k}$ . On the other hand it is possible to show, see Section 4.2, that the eigenfunctions corresponding to the  $\sigma_\alpha$ 's are products  $v_l(|\zeta|)\langle \zeta, \varphi_l \rangle_N$ , where  $\langle \cdot, \cdot \rangle_N$  is the scalar product in  $NK$ , and where  $\varphi_l$  is a section of the normal bundle  $NK_\varepsilon$ , and precisely an eigenfunction (scaled in  $\varepsilon$ ) of the normal Laplacian of  $K$ . Since the resonant modes correspond to low indices  $l$ ,  $\varphi_l$  does not oscillate *as fast* as the resonant  $\phi_j$ 's.

So far we considered an approximate operator, because in (5) we assumed a splitting of the metric into a block form in the variables  $(y, \zeta)$ . Since we expect to deal with small eigenvalues, a careful analysis of the approximate solutions is needed (to apply local inversion arguments), and also a refined understanding of the small eigenvalues with the corresponding eigenfunctions.

Therefore we first try to obtain approximate solutions as accurate as possible. For doing this, as in [33–35], one can introduce suitable coordinates on  $\Omega_\varepsilon$  near  $K_\varepsilon$ , expand formally  $(\hat{P}_\varepsilon)$  in powers of  $\varepsilon$ , and solve it term by term using functions of the form

$$u_{l,\varepsilon}(y, \zeta) = [w_0 + \varepsilon w_1 + \dots + \varepsilon^l w_l](\varepsilon y, \zeta' + \Phi_0(\varepsilon y) + \dots + \varepsilon^{l-2} \Phi_{l-2}(\varepsilon y), \zeta_{n+1});$$

$$\zeta = (\zeta', \zeta_{n+1}). \tag{7}$$

Here  $\Phi_0, \dots, \Phi_{l-2}$  represent smooth sections of the normal bundle  $NK$ , and the functions  $(w_i)_i$  are determined implicitly via equations of the type

$$\begin{cases} -\Delta w_i + w_i - p w_0(\zeta) w_i = F_i(\varepsilon y, w_0, \dots, w_{i-1}, \Phi_0, \dots, \Phi_{i-2}) & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial w_i}{\partial \nu} = 0 & \text{on } \partial \mathbb{R}_+^{n+1}. \end{cases} \tag{8}$$

Notice that the operator acting on  $w_i$  is nothing but the linearization of (3) at  $w_0$  (shifted in  $\zeta'$  by  $\Phi_0 + \dots + \varepsilon^{l-2} \Phi_{l-2}$ ), which has an  $n$ -dimensional kernel due to the invariance by translation in  $\zeta'$ . The functions  $\Phi_i$  are chosen in order to obtain orthogonality of  $F_i$  to the kernel, and to guarantee solvability in  $w_i$ . In doing this, the non-degeneracy condition on  $K$  comes into play, since the  $\Phi_i$ 's solve equations of the form  $\mathfrak{J}\Phi_i = G_i(\bar{y})$ .  $\mathfrak{J}$  denotes the *Jacobi operator* of  $K$ , related to the second variation of the volume functional, which is invertible by the non-degeneracy assumption on the minimal submanifold. Notice also that we wrote the variable  $y$  with a factor  $\varepsilon$  on the front. This is in order to emphasize the slow dependence in  $y$  of these functions. In fact, recalling that (in the model problem described above) resonance occurs mostly when dealing with highly oscillating eigenfunctions, if we require slow dependence in  $y$  then there is no obstruction in solving  $(\hat{P}_\varepsilon)$  up to an arbitrary order  $\varepsilon^l$ .

Next one linearizes  $(\hat{P}_\varepsilon)$  near the approximate solutions just found. Compared to the above model problem, the eigenvalues will be perturbed by some amount, due to the presence of the corrections  $(w_i)_i$  and to the geometry of the problem. In fact the amount will be in general of order  $\varepsilon$ , since this is the size of the corrections (from the  $w_i$ 's and the expansions of the metric coefficients, see Lemma 3.2). This prevents a direct control of the small eigenvalues of the linearized operator (at  $u_{l,\varepsilon}$ ) since, as discussed above, the characteristic size of the spectral gaps at resonance are of order  $\varepsilon^2$  or  $\varepsilon^k$ .

To overcome this problem, we look at the eigenvalues as functions of  $\varepsilon$ . The counterparts of the numbers  $\sigma_{\varepsilon^2\omega_l}$  can be again obtained via a Taylor’s expansion in  $\varepsilon$ , and they turn out to be constant multiples of  $\varepsilon^2$  times the eigenvalues of  $\mathfrak{J}$  (up to an error of order  $o(\varepsilon^2)$ ), so they are never zero. On the other hand, the counterparts of the  $\eta_{\varepsilon^2\rho_j}$ ’s could vanish for some values of  $\varepsilon$  but, recalling the expansion  $\eta_{\varepsilon^2\rho_j} \simeq \eta_0 + \varepsilon^2 j^{2/k}$ , one can hope that generically in  $\varepsilon$  none of these eigenvalues will be zero.

This is indeed shown using a classical theorem due to T. Kato, see [25, p. 445], which allows us to estimate the derivatives of the eigenvalues with respect to  $\varepsilon$ . To apply this result one needs some control not only on the initial eigenvalues but also on the corresponding eigenfunctions, and this is what basically the last sections are devoted to. There we prove that if  $\lambda = o(\varepsilon^2)$  is an eigenvalue of the linearized operator, the eigenfunctions (up to a small error) are linear combinations of products like  $\phi_j(\varepsilon y)u_j(\zeta)$ , for  $j \simeq \varepsilon^{-k}$  and for suitable functions  $u_j$  radial in  $\zeta$ . Then we deduce that  $\partial\lambda/\partial\varepsilon$  is close to a number depending on  $\varepsilon, N, p$  and  $K$  only. As a consequence, the spectral gaps near zero will *shift*, as  $\varepsilon$  varies, almost without squeezing, yielding invertibility for suitable values of the parameter. This method also provides estimates on the norm of the inverse operator, which blows-up with rate  $\max\{\varepsilon^{-k}, \varepsilon^{-2}\}$  when  $\varepsilon$  tends to zero, see Remark 6.8.

Finally, a straightforward application of the implicit function theorem gives the desired result. To fix the ideas, when  $p \leq (N + 2)/(N - 2)$ , solutions of  $(\tilde{P}_\varepsilon)$  can be found as critical points of the following functional

$$J_\varepsilon(u) = \frac{1}{2} \int_{\Omega_\varepsilon} (|\nabla u|^2 + u^2) - \frac{1}{p+1} \int_{\Omega_\varepsilon} |u|^{p+1}, \quad u \in H^1(\Omega_\varepsilon). \tag{9}$$

One proves that  $\|J'_\varepsilon(u_{I,\varepsilon})\|_{H^1(\Omega_\varepsilon)} \leq C_{I,k} \varepsilon^{I+1-k/2}$  for  $\varepsilon$  small. Even if the norm of the inverse linear operator blows-up when  $\varepsilon$  tends to zero, choosing  $I$  sufficiently large (depending only on  $k$  and  $p$ ), one can find a solution using the contraction mapping theorem near  $u_{I,\varepsilon}$ .

The general strategy of this proof, and especially Kato’s theorem, has been used in [33–35], so throughout the paper we will be sketchy in the parts where simple adaptations apply. However the present setting requires some new ingredients: we are going to explain next what are the differences with respect to these and to some other related papers. First of all, compared to [34,35], where the case  $k = N - 1$  was treated, here we need to characterize the limit set among all the possible ones, since the codimension is higher, and this reflects in the fact that the limit problem (3) is degenerate. This requires to introduce the normal sections  $\Phi_0, \dots, \Phi_{I-2}$  in (7), and to use the non-degeneracy condition on  $K$ .

The localization of the limit set has been indeed also faced in [33]. Here, apart from including that result as a particular case, allowing higher dimensions and codimensions, we need a more geometric approach. The main issue, as we already remarked, is that we cannot use parameterizations with product sets as in (4), since the normal bundle of  $K$  is not trivial in general. At this point some interplay between the analytic and geometric features of the problem is needed. In particular the first and second eigenfunctions of the linearization of (3) (the profile of  $\tilde{u}_\varepsilon$  at every point  $q$  of  $K$ ) can be viewed of *scalar* or *vectorial* nature. More precisely, the eigenfunction corresponding to the first eigenvalue is radial and unique up to a scalar multiple. On the other hand the eigenfunctions corresponding to the second eigenvalue have the symmetry of the first spherical harmonics in the unit sphere of  $N_q K$ , and they are in one-to-one correspondence with the vectors of  $N_q K$ . The same holds true for the eigenfunctions of problem (6) when  $\alpha \geq 0$ . When  $q$

varies over the limit set, these eigenfunctions (which are the resonant ones), depending on their symmetry determine respectively a scalar function on  $K$  or a section of the normal bundle  $NK$ , on which the Laplace–Beltrami operator or the normal Laplacian act naturally, see in particular Section 4. Apart from these considerations some other difficulties arise, more technical in nature, due to the more general character of the present result compared to that in [33]. Heavier computations are involved, especially since the curvature tensors have more components, and some extra terms appear. Anyway, some of the arguments have been simplified.

Finally, we should point out the differences with respect to the papers [15,32,38], where also special solutions of the nonlinear Schrödinger equation or constant mean curvature surfaces are found. In [15,38] the *spectral gaps* are relatively big, and the eigenvalues can be located using direct comparison arguments, so there is no need to invoke Kato’s theorem. In [32] arbitrarily small spectral gaps are allowed, but while there one has to study a partial differential equation on a surface only, here we need to analyze the equation on the whole space, which takes some extra work. Also, the Riemannian manifold we consider here,  $\partial\Omega$ , has an *extrinsic* curvature as a subset of  $\mathbb{R}^N$ , and therefore some error terms turn out to be of order  $\varepsilon$ , and not  $\varepsilon^2$ , see Remark 3.4 (a). Nevertheless, we take great advantage of the geometric construction in [32], especially in their choice of coordinates near the limit set. We believe that our method could adapt to study concentration at general manifolds for the nonlinear Schrödinger equation as well, as conjectured in [2].

The paper is organized in the following way. We first introduce some notations and conventions. In Section 2 we collect some notions in differential geometry, like the Fermi coordinates near a minimal submanifold, the normal Laplacian, the Laplace–Beltrami and the Jacobi operators as well as the asymptotics of their eigenvalues. In Section 3 we construct the approximate solution  $u_{I,\varepsilon}$ . In Section 4 we study some spectral properties for the limit problem (3) (with some extension) and we then derive a model for the linearized operator at  $u_{I,\varepsilon}$ . In Section 5 we turn then to the real linearized operator: we construct some approximate eigenfunctions which allow us to split our functional space as direct sum of subspaces for which the linearized operator is almost diagonal. In Section 6, using this splitting we characterize the eigenfunctions corresponding to resonant eigenvalues. From these estimates we can obtain invertibility, via Kato’s theorem, and prove our main result Theorem 1.1.

*Notation and conventions*

- Dealing with coordinates, Greek letters like  $\alpha, \beta, \dots$ , will denote indices varying between 1 and  $N - 1$ , while capital letters like  $A, B, \dots$  will vary between 1 and  $N$ ; Roman letters like  $a$  or  $b$  will run from 1 to  $k$ , while indices like  $i, j, \dots$  will run between 1 and  $n := N - k - 1$ .
- $\zeta_1, \dots, \zeta_n, \zeta_{n+1}$  will denote coordinates in  $\mathbb{R}^{n+1} = \mathbb{R}^{N-k}$ , and they will also be written as  $\zeta' = (\zeta_1, \dots, \zeta_n)$ ,  $\zeta = (\zeta', \zeta_{n+1})$ .
- The manifold  $K$  will be parameterized with coordinates  $\bar{y} = (\bar{y}_1, \dots, \bar{y}_k)$ . Its dilation  $K_\varepsilon := \frac{1}{\varepsilon}K$  will be parameterized by coordinates  $(y_1, \dots, y_k)$  related to the  $\bar{y}$ ’s simply by  $\bar{y} = \varepsilon y$ .
- Derivatives with respect to the variables  $\bar{y}$ ,  $y$  or  $\zeta$  will be denoted by  $\partial_{\bar{y}}$ ,  $\partial_y$ ,  $\partial_\zeta$ , and for brevity sometimes we might use the symbols  $\partial_{\bar{a}}$  and  $\partial_i$  for  $\partial_{\bar{y}_a}$  and  $\partial_{\zeta_i}$  respectively.
- In a local system of coordinates,  $(\bar{g}_{\alpha\beta})_{\alpha\beta}$  are the components of the metric on  $\partial\Omega$  naturally induced by  $\mathbb{R}^N$ . Similarly,  $(\bar{g}_{AB})_{AB}$  are the entries of the metric on  $\Omega$  in a neighborhood of the boundary.  $(H_{\alpha\beta})_{\alpha\beta}$  will denote the components of the mean curvature operator of  $\partial\Omega$  into  $\mathbb{R}^N$ .



Below, for simplicity, the constant  $C$  is allowed to vary from one formula to another, also within the same line, and will assume larger and larger values. It is always understood that  $C$  depends on  $\Omega$ , the dimension  $N$  and the exponent  $p$ . It will be explicitly written  $C_l, C_\delta, \dots$ , if the constant  $C$  depends also on other quantities, like an integer  $l$ , a parameter  $\delta$ , etc. Similarly, the positive constant  $\gamma$  will assume smaller and smaller values.

For a real positive variable  $r$  and an integer  $m$ ,  $O(r^m)$  (respectively  $o(r^m)$ ) will denote a function for which  $|O(r^m)/r^m|$  remains bounded (respectively  $|o(r^m)/r^m|$  tends to zero) when  $r$  tends to zero. We might also write  $o_\varepsilon(1)$  for a quantity which tends to zero as  $\varepsilon$  tends to zero. With  $\mathcal{O}(r^m)$  we denote functions which depend on the above variables  $(\bar{y}, \zeta)$ , which are of order  $r^m$ , and whose partial derivatives of any order, with respect to the vector fields  $\partial_\alpha, r \partial_i$ , are bounded by a constant times  $r^m$ .

$L_i$  will stand in general for a differential operator of order at most  $i$  in both the variables  $\bar{y}$  and  $\zeta$  (unless differently specified), whose coefficients are assumed to be smooth in  $\bar{y}$ .

For summations, we might use the notation  $\sum_c^d$  to indicate that the sum is taken over an integer index varying from  $[c]$  to  $[d]$  (the integer parts of  $c$  and  $d$  respectively). We might use the same convention when we make an integer index vary between  $c$  and  $d$ . We also use the standard convention of summing terms where repeated indices appear.

We will assume throughout the paper that the exponent  $p$  is at most critical, namely that  $p \leq (N + 2)/(N - 2)$ , so that problem  $(P_\varepsilon)$  is variational in  $H^1(\Omega)$ . We will indicate at the end what are the arguments necessary to deal with the general case.

The results of this paper are illustrated in the preliminary note [31].

## 2. Geometric background

In this section we list some preliminary notions in differential geometry. First of all we introduce Fermi coordinates near a submanifold of  $\partial\Omega$ , recall the definition of minimal submanifold, and introduce the Laplace–Beltrami and the Jacobi operators, together with some of their spectral properties. We refer for example to [4,48] as basic references in differential geometry.

### 2.1. Fermi coordinates on $\partial\Omega$ near $K$

Let  $K$  be a  $k$ -dimensional submanifold of  $(\partial\Omega, \bar{g})$  ( $1 \leq k \leq N - 1$ ) and set  $n = N - k - 1$  (see our notation). We choose along  $K$  a local orthonormal frame field  $((E_a)_{a=1,\dots,k}, (E_i)_{i=1,\dots,n})$  which is oriented. At points of  $K$ ,  $T\partial\Omega$  splits naturally as  $TK \oplus NK$ , where  $TK$  is the tangent space to  $K$  and  $NK$  represents the normal bundle, which are spanned respectively by  $(E_a)_a$  and  $(E_j)_j$ .

Denote by  $\nabla$  the connection induced by the metric  $\bar{g}$  and by  $\nabla^N$  the corresponding normal connection on the normal bundle. Given  $q \in K$ , we use some geodesic coordinates  $\bar{y}$  centered at  $q$ . We also assume that at  $q$  the normal vectors  $(E_i)_i, i = 1, \dots, n$ , are transported parallelly (with respect to  $\nabla^N$ ) through geodesics from  $q$ , so in particular

$$\bar{g}(\nabla_{E_a} E_j, E_i) = 0 \quad \text{at } q, \quad i, j = 1, \dots, n, \quad a = 1, \dots, k. \tag{10}$$

In a neighborhood of  $q$ , we choose *Fermi coordinates*  $(\bar{y}, \zeta)$  on  $\partial\Omega$  defined by

$$(\bar{y}, \zeta) \rightarrow \exp_{\bar{y}}^{\partial\Omega} \left( \sum_{i=1}^n \zeta_i E_i \right); \quad (\bar{y}, \zeta) = ((\bar{y}_a)_a, (\zeta_i)_i), \tag{11}$$

where  $\exp_{\bar{y}}^{\partial\Omega}$  is the exponential map at  $\bar{y}$  in  $\partial\Omega$ .

By our choice of coordinates, on  $K$  the metric  $\bar{g}$  splits in the following way

$$\bar{g}(q) = \bar{g}_{ab}(q) d\bar{y}_a \otimes d\bar{y}_b + \bar{g}_{ij}(q) d\zeta_i \otimes d\zeta_j; \quad q \in K. \tag{12}$$

We denote by  $\Gamma_a^b(\cdot)$  the 1-forms defined on the normal bundle of  $K$  by

$$\Gamma_a^b(E_i) = \bar{g}(\nabla_{E_a} E_b, E_i). \tag{13}$$

We will also denote by  $R_{\alpha\beta\gamma\delta}$  the components of the curvature tensor with lowered indices, which are obtained by means of the usual ones  $R_{\beta\gamma\delta}^\sigma$  by

$$R_{\alpha\beta\gamma\delta} = \bar{g}_{\alpha\sigma} R_{\beta\gamma\delta}^\sigma.$$

When we consider the metric coefficients in a neighborhood of  $K$ , we obtain a deviation from formula (12), which is expressed by the next lemma, see Proposition 2.1 in [32] for the proof. Denote by  $r$  the distance function from  $K$ .

**Lemma 2.1.** *In the above coordinates  $(\bar{y}, \zeta)$ , for any  $a = 1, \dots, k$  and any  $i, j = 1, \dots, n$ , we have*

$$\begin{aligned} \bar{g}_{ij}(0, \zeta) &= \delta_{ij} + \frac{1}{3} R_{istj} \zeta_s \zeta_t + \mathcal{O}(r^3); \\ \bar{g}_{aj}(0, \zeta) &= \mathcal{O}(r^2); \\ \bar{g}_{ab}(0, \zeta) &= \delta_{ab} - 2\Gamma_a^b(E_i) \zeta_i + [R_{sabl} + \Gamma_a^c(E_s) \Gamma_c^b(E_l)] \zeta_s \zeta_l + \mathcal{O}(r^3). \end{aligned}$$

Here  $R_{istj}$  are computed at the point  $q$  of  $K$  parameterized by  $(0, 0)$ .

### 2.2. Laplace–Beltrami, normal Laplacian and Jacobi operators

In this subsection we recall some basic definitions and spectral properties of differential operators associated to minimal submanifolds. We first recall some notions about the *Laplace–Beltrami* operator, the normal connection and the *normal Laplacian*.

If  $(M, g)$  is an  $m$ -dimensional Riemannian manifold, the Laplace–Beltrami operator on  $M$  is defined in local coordinates by

$$\Delta_g = \frac{1}{\sqrt{\det g}} \partial_A (\sqrt{\det g} g^{AB} \partial_B), \tag{14}$$

where the indices  $A$  and  $B$  run in  $1, \dots, m$ , and where  $g^{AB}$  denote the components of the inverse of the matrix  $g_{AB}$ .

Let  $K \subseteq M$  be a  $k$ -dimensional submanifold,  $k \leq m - 1$ . The normal connection  $\nabla^N$  on a normal vector field  $V$  is defined as the projection of the connection  $\nabla V$  onto  $NK$ . Moreover, one has the following formula regarding the horizontal derivative of the product  $\langle \cdot, \cdot \rangle_N$  in the normal bundle (see [48, vol. 4, Chapter 7.C] for further details)

$$X \langle V, W \rangle_N = \langle \nabla_X^N V, W \rangle + \langle V, \nabla_X^N W \rangle,$$

for any smooth sections  $V$  and  $W$  in  $NK$ . If we choose an orthonormal frame  $(E_i)_i$  for  $NK$  along  $K$ , we can write

$$\nabla_{\partial_{\bar{a}}}^N E_j = \beta_j^l(\partial_{\bar{a}}) E_l,$$

for some differential forms  $\beta_j^l$  (we recall our notation  $\partial_{\bar{a}} = \partial/\partial \bar{y}_a$ ). Since the normal fields  $(E_i)_i$  are chosen to be orthonormal, it follows that for any horizontal vector field  $X$  there holds  $X\langle E_i, E_j \rangle_N = 0$ , and hence one has

$$\beta_j^l(\partial_{\bar{a}}) = -\beta_l^j(\partial_{\bar{a}}) \quad \forall l, j = 1, \dots, n. \tag{15}$$

This holds true, in particular, if we choose Fermi coordinates. Since indeed the normal fields are extended via (normal) parallel transport from  $q$  to some neighborhood through the exponential map, it follows that  $\beta_j^l(\partial_{\bar{a}})(0, 0, \dots, \bar{y}_a, 0, \dots, 0) = 0$ , and hence

$$\beta_j^l(\partial_{\bar{a}}) = 0 \quad \text{at } q \quad \forall a = 1, \dots, k, \text{ and } \forall l, j = 1, \dots, n; \tag{16}$$

$$\partial_{\bar{a}}(\beta_j^l(\partial_{\bar{a}})) = 0 \quad \text{at } q \quad \forall a = 1, \dots, k, \text{ and } \forall l, j = 1, \dots, n. \tag{17}$$

Recalling these facts, we can derive the expression of the *normal Laplacian* in Fermi coordinates in the following way: given a normal vector field  $V = V^j E_j$ , there holds

$$\nabla_{\partial_{\bar{a}}}^N V = \partial_{\bar{a}} V^j E_j + V^j \beta_j^l(\partial_{\bar{a}}) E_l.$$

For any two normal vector fields  $V$  and  $W$  we have, by the definition of  $\Delta_K^N$

$$\int_K \langle \nabla^N V, \nabla^N W \rangle_N dV_{\bar{g}} = - \int_K \langle \Delta_K^N V, W \rangle_N dV_{\bar{g}}.$$

We compute now the expression of  $\Delta_K^N$  evaluating the left-hand side and integrating by parts

$$\begin{aligned} \int_K \langle \nabla^N V, \nabla^N W \rangle_N dV_{\bar{g}} &= \int_K \langle \partial_{\bar{a}} V^j E_j + V^j \beta_j^l(\partial_{\bar{a}}) E_l, \partial_{\bar{b}} W^i E_i + W^i \beta_i^h(\partial_{\bar{b}}) \rangle_N \bar{g}^{ab} \sqrt{\det \bar{g}} \\ &= \int_K [\partial_{\bar{a}} V^i \partial_{\bar{b}} W^i + \partial_{\bar{a}} V^j W^i \beta_i^j(\partial_{\bar{b}}) + V^j \beta_j^i(\partial_{\bar{a}}) \partial_{\bar{b}} W^i \\ &\quad + V^j W^i \beta_j^l(\partial_{\bar{a}}) \beta_l^i(\partial_{\bar{b}})] \bar{g}^{ab} \sqrt{\det \bar{g}}. \end{aligned}$$

This quantity, for any  $V$  and  $W$ , has to coincide with  $-\int_K (\Delta_K^N V)^i W^i \sqrt{\det \bar{g}}$ , so we deduce that

$$\begin{aligned} (\Delta_K^N V)^i &= \Delta_K(V^i) + \frac{1}{\sqrt{\det \bar{g}}} \partial_{\bar{b}} (V^j \beta_j^i(\partial_{\bar{a}}) \bar{g}^{ab} \sqrt{\det \bar{g}}) \\ &\quad - \bar{g}^{ab} (\partial_{\bar{a}} V^j \beta_i^j(\partial_{\bar{b}}) + W^j \beta_j^l(\partial_{\bar{a}}) \beta_l^i(\partial_{\bar{b}})) \sqrt{\det \bar{g}}. \end{aligned} \tag{18}$$

In Fermi coordinates at  $q$ , which is parameterized by  $(0, 0)$ , we have that

$$\bar{g}_{ab} = \delta_{ab}, \quad \partial_{\bar{c}} \bar{g}_{ab} = 0 \quad \text{and} \quad \partial_{\bar{c}} \sqrt{\det \bar{g}} = 0, \tag{19}$$

and we also have (16)–(17). Hence (18) simplifies in the following way

$$(\Delta_K^N V)^i = \Delta_K(V^i) \quad \text{at } q. \tag{20}$$

Let  $C^\infty(NK)$  be the space of smooth normal vector fields on  $K$ . For  $\Phi \in C^\infty(NK)$ , we can define the one-parameter family of submanifolds  $t \mapsto K_{t,\Phi}$  by

$$K_{t,\Phi} := \{ \exp_{\bar{y}}^{\partial\Omega} (t\Phi(\bar{y})) : \bar{y} \in K \}. \tag{21}$$

The first variation formula of the volume is the equation

$$\left. \frac{d}{dt} \right|_{t=0} \text{Vol}(K_{t,\Phi}) = \int_K \langle \Phi, \mathbf{H} \rangle_N dV_K, \tag{22}$$

where  $\mathbf{H}$  is the mean curvature (vector) of  $K$  in  $\partial\Omega$ ,  $\langle \cdot, \cdot \rangle_N$  denotes the restriction of  $\bar{g}$  to  $NK$ , and  $dV_K$  the volume element of  $K$ .

The submanifold  $K$  is said to be *minimal* if it is a critical point for the volume functional, namely if

$$\left. \frac{d}{dt} \right|_{t=0} \text{Vol}(K_{t,\Phi}) = 0 \quad \text{for any } \Phi \in C^\infty(NK) \tag{23}$$

or equivalently, by (22), if the mean curvature  $\mathbf{H}$  is identically zero on  $K$ . It is possible to prove that, if  $\Gamma_a^b(E_i)$  is as in (13), then

$$K \text{ is minimal} \quad \Leftrightarrow \quad \Gamma_a^a(E_i) = 0 \quad \text{for any } i = 1, \dots, n. \tag{24}$$

We point out that in the last formula we are summing over the index  $a$ , which is repeated.

The *Jacobi operator*  $\mathfrak{J}$  appears in the expression of the second variation of the volume functional for a minimal submanifold  $K$

$$\left. \frac{d^2}{dt^2} \right|_{t=0} \text{Vol}(K_{t,\Phi}) = - \int_K \langle \mathfrak{J}\Phi, \Phi \rangle_N dV_K; \quad \Phi \in C^\infty(NK), \tag{25}$$

and is given by

$$\mathfrak{J}\Phi := -\Delta_K^N \Phi + \mathfrak{R}^N \Phi - \mathfrak{B}^N \Phi, \tag{26}$$

where  $\mathfrak{R}^N, \mathfrak{B}^N : NK \rightarrow NK$  are defined as

$$\mathfrak{R}^N \Phi = (R(E_a, \Phi)E_a)^N; \quad \bar{g}(\mathfrak{B}^N \Phi, n_K) := \Gamma_b^a(\Phi)\Gamma_a^b(n_K),$$

for any unit normal vector  $n_K$  to  $K$ . The operator  $\Delta_K^N$  is the *normal Laplacian* on  $K$  defined in (20).

A submanifold  $K$  is said to be *non-degenerate* if the Jacobi operator  $\mathfrak{J}$  is invertible, or equivalently if the equation  $\mathfrak{J}\Phi = 0$  has only the trivial solution among the sections in  $NK$ .

We recall now some Weyl asymptotic formulas, referring for example to [10], or to [27,39] for further details. Let  $(M, g)$  be a compact closed Riemannian manifold of dimension  $m$ , and let  $\Delta_g$  be the Laplace–Beltrami operator. Letting  $(\rho_i)_i, i = 0, 1, \dots$ , denote the eigenvalues of  $-\Delta_g$  (ordered to be non-decreasing in  $i$  and counted with their multiplicity), we have that

$$\rho_i \sim C_m \left( \frac{i}{\text{Vol}(M)} \right)^{\frac{2}{m}} \quad \text{as } i \rightarrow \infty, \tag{27}$$

where  $\text{Vol}(M)$  is the volume of  $(M, g)$  and  $C_m$  is a constant depending only on the dimension  $m$  (the Weyl constant). A similar estimate, which can be proved using (18) and (27), holds for the normal Laplacian  $\Delta_K^N$  on a  $k$ -dimensional submanifold  $K \subseteq M$ . In fact, letting  $(\omega_j)_j, j = 0, 1, \dots$ , denote the eigenvalues of  $-\Delta_K^N$  (still chosen to be non-decreasing in  $j$  and counted with multiplicity), one has

$$\omega_j \sim C_{m,k} \left( \frac{j}{\text{Vol}(K)} \right)^{\frac{2}{k}} \quad \text{as } j \rightarrow \infty, \tag{28}$$

where  $C_{m,k}$  depends on the dimensions  $m$  and  $k$  only.

Considering the Jacobi operator  $\mathfrak{J}$  for a minimal submanifold  $K$ , it is easy to see from (26) that, since  $\mathfrak{J}$  differs from  $-\Delta_K^N$  only by a bounded quantity, we have the same asymptotic formula for its eigenvalues  $(\mu_l)_l$ , and thereby

$$\mu_l \sim C_{m,k} \left( \frac{l}{\text{Vol}(K)} \right)^{\frac{2}{k}} \quad \text{as } l \rightarrow \infty. \tag{29}$$

In the following, we let  $(\phi_i)_i$  (respectively  $(\varphi_j)_j, (\psi_l)_l$ ) denote a base of eigenfunctions of  $-\Delta_K$  (respectively of  $-\Delta_K^N, \mathfrak{J}$ ), normalized in  $L^2(K)$  (respectively in  $L^2(K; NK)$ ), namely the set of functions (respectively normal sections of  $K$ ) satisfying

$$-\Delta_K \phi_i = \rho_i \phi_i; \quad -\Delta_K^N \varphi_j = \omega_j \varphi_j; \quad \mathfrak{J} \psi_l = \mu_l \psi_l, \quad i, j, l = 1, 2, \dots$$

Finally, using the eigenvalues  $(\rho_j)_j$  and  $(\mu_l)_l$ , one can express the  $L^2$  norms, or the Sobolev norms of linear combinations of the  $\phi_j$ 's and the  $\psi_l$ 's. In particular, if  $f = \sum_j \alpha_j \phi_j$ , and if  $g = \sum_l \beta_l \psi_l$  are an  $L^2$  function and an  $L^2$  normal section of  $K$ , and if  $L_1 = \sum_\alpha c_\alpha(\bar{y}) \partial_{\bar{y}}^\alpha, L_2 = \sum_\alpha \tilde{c}_\alpha(\bar{y}) (\nabla_{\bar{y}}^N)^\alpha$  are differential operators of order  $d$  with smooth coefficients acting on functions and normal sections respectively, then one has

$$\|L_1 f\|_{L^2(K)}^2 \leq C_{L_1} \sum_j (1 + \rho_j^d) \alpha_j^2; \quad \|L_2 g\|_{L^2(K; NK)}^2 \leq C_{L_2} \sum_l (1 + |\mu_l|^d) \beta_l^2. \tag{30}$$

An estimate similar to the latter one in (30) holds by replacing the  $\mu_l$ 's by the  $\omega_j$ 's, namely if  $g' = \sum_j \beta'_j \varphi_j$ , then  $\|L_2 g'\|_{L^2(K; NK)}^2 \leq C_{L_2} \sum_j (1 + |\omega_j|^d) (\beta'_j)^2$ .

### 3. Approximate solutions to $(\tilde{P}_\varepsilon)$

In this section, given any positive integer  $I$ , we construct functions  $u_{I,\varepsilon}$  which solve  $(\tilde{P}_\varepsilon)$  up to an error of order  $\varepsilon^{I+1}$ . We will find approximate solutions of  $(\tilde{P}_\varepsilon)$  in the following form

$$\begin{aligned} &\chi_\varepsilon(|\zeta|)(w_0(\zeta' + \Phi(\varepsilon y), \zeta_{n+1}) + \varepsilon w_1(\varepsilon y, \zeta' + \Phi(\varepsilon y), \zeta_{n+1}) + \dots \\ &+ \varepsilon^I w_I(\varepsilon y, \zeta' + \Phi(\varepsilon y), \zeta_{n+1})), \end{aligned} \tag{31}$$

where  $\Phi(\varepsilon y) = \Phi_0(\varepsilon y) + \dots + \varepsilon^{I-2}\Phi_{I-2}(\varepsilon y)$  and where the cutoff function  $\chi_\varepsilon$  satisfies the properties

$$\begin{cases} \chi_\varepsilon(t) = 1 & \text{for } t \in [0, \frac{1}{2}\varepsilon^{-\gamma}], \\ \chi_\varepsilon(t) = 0 & \text{for } t \in [\frac{3}{4}\varepsilon^{-\gamma}, \varepsilon^{-\gamma}], \\ |\chi_\varepsilon^{(l)}(t)| \leq C_l \varepsilon^{l\gamma}, \quad l \in \mathbb{N}. \end{cases} \tag{32}$$

Here  $\Phi_0, \dots, \Phi_{I-2}$  are smooth vector fields from  $K$  into  $NK$ , while  $w_1, \dots, w_I$  are suitable functions determined recursively by an iteration procedure. For doing this we choose a system of coordinates in a neighborhood of  $\partial\Omega_\varepsilon$  for which the new metric coefficients can be expanded in powers of  $\varepsilon$ , see Lemma 3.2. In this way we can also expand  $(\tilde{P}_\varepsilon)$  formally in powers of  $\varepsilon$  and solve it term by term. The functions  $(w_i)_i$  will be obtained as solutions of an equation arising from the linearization of (3) at  $w_0$ , while the normal sections  $(\Phi_i)_i$  will be determined using the invertibility of the Jacobi operator. Notice that, by the translation invariance of (3), the linearized operator possesses a non-trivial kernel, which turns out to be spanned by  $\{\partial_{\zeta_1} w_0, \dots, \partial_{\zeta_n} w_0\}$ . The role of  $\Phi_0, \dots, \Phi_{I-2}$  is to obtain at every step orthogonality to this kernel and to solve the equation using Fredholm’s alternative.

The method here is similar in spirit to the one used in [33] except for the fact that, working in higher dimensions and codimensions, more geometric tools are needed. Therefore, we will mainly focus on the new and geometric aspects of the construction, omitting some details about the rigorous estimates on the error terms, which can be handled as in [33].

#### 3.1. Choice of coordinates near $\partial\Omega_\varepsilon$ and properties of approximate solutions

Let  $\mathcal{Y}_0: \mathcal{U} \rightarrow \partial\Omega$ , where  $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \subseteq \mathbb{R}^k \times \mathbb{R}^n$  is a neighborhood of 0 in  $\mathbb{R}^{N-1}$ , be a parameterization of  $\partial\Omega$  near some point  $q \in K$  through the Fermi coordinates  $(\bar{y}, \zeta)$  described before.

Let  $\gamma \in (0, 1)$  be a small number which, we recall, is allowed to assume smaller and smaller values throughout the paper. Then for  $\varepsilon > 0$  we set

$$B_{\varepsilon,\gamma} = \{x \in \mathbb{R}_+^{n+1} : |x| < \varepsilon^{-\gamma}\}.$$

Next we introduce a parameterization of a neighborhood (in  $\Omega_\varepsilon$ ) of  $q/\varepsilon \in \partial\Omega_\varepsilon$  though the map  $\mathcal{Y}_\varepsilon$  given by

$$\mathcal{Y}_\varepsilon(y, \zeta', \zeta_{n+1}) = \frac{1}{\varepsilon}\mathcal{Y}_0(\varepsilon y, \varepsilon\zeta') + \zeta_{n+1}v(\varepsilon y, \varepsilon\zeta'), \quad x = (y, \zeta', \zeta_{n+1}) \in \frac{1}{\varepsilon}\mathcal{U}_1 \times B_{\varepsilon,\gamma}, \tag{33}$$

where  $\varepsilon y = \bar{y}$  and where  $\nu(\varepsilon y, \varepsilon \zeta')$  is the inner unit normal to  $\partial\Omega$  at  $\Upsilon_0(\varepsilon y, \varepsilon \zeta')$ . We have

$$\frac{\partial \Upsilon_\varepsilon}{\partial y_a} = \frac{\partial \Upsilon_0}{\partial y_a}(\varepsilon y, \varepsilon \zeta') + \varepsilon \zeta_{n+1} \frac{\partial \nu}{\partial \bar{y}_a}(\varepsilon y, \varepsilon \zeta'); \quad \frac{\partial \Upsilon_\varepsilon}{\partial \zeta_i} = \frac{\partial \Upsilon_0}{\partial \zeta_i}(\varepsilon y, \varepsilon \zeta') + \varepsilon \zeta_{n+1} \frac{\partial \nu}{\partial \zeta_i}(\varepsilon y, \varepsilon \zeta').$$

Using the equation

$$dv_x[v] = \mathbf{H}(x)[v], \tag{34}$$

we find

$$\begin{aligned} \frac{\partial \Upsilon_\varepsilon}{\partial y_a} &= [Id + \varepsilon \zeta_{n+1} \mathbf{H}(\varepsilon y, \varepsilon \zeta')] \frac{\partial \Upsilon_0}{\partial \bar{y}_a}(\varepsilon y, \varepsilon \zeta'); \\ \frac{\partial \Upsilon_\varepsilon}{\partial \zeta_i} &= [Id + \varepsilon \zeta_{n+1} \mathbf{H}(\varepsilon y, \varepsilon \zeta')] \frac{\partial \Upsilon_0}{\partial \zeta_i}(\varepsilon y, \varepsilon \zeta'). \end{aligned} \tag{35}$$

Differentiating  $\Upsilon_\varepsilon$  with respect to  $\zeta_{n+1}$  we also get

$$\frac{\partial \Upsilon_\varepsilon}{\partial \zeta_{n+1}} = \nu(\varepsilon y, \varepsilon \zeta'). \tag{36}$$

Hence, letting  $g_{AB}$  be the coefficients of the flat metric  $g = g_\varepsilon$  (we are emphasizing the role of the parameter  $\varepsilon$  in the entries, which is due to the dependence in  $\varepsilon$  of the map  $\Upsilon_\varepsilon$ ) of  $\mathbb{R}^N$  in the coordinates  $(y, \zeta', \zeta_{n+1})$ , with easy computations we deduce that

$$\begin{aligned} g_{\alpha\beta}(\tilde{y}, \zeta_{n+1}) &= \bar{g}_{\alpha\beta}(\varepsilon \tilde{y}) + \varepsilon \zeta_{n+1} (H_{\alpha\delta} \bar{g}_{\delta\beta} + H_{\beta\delta} \bar{g}_{\delta\alpha})(\varepsilon \tilde{y}) \\ &\quad + \varepsilon^2 \zeta_{n+1}^2 H_{\alpha\delta} H_{\sigma\beta} \bar{g}_{\delta\sigma}(\varepsilon \tilde{y}), \quad \tilde{y} = (y, \zeta'); \end{aligned} \tag{37}$$

$$g_{\alpha N} \equiv 0; \quad g_{NN} \equiv 1. \tag{38}$$

Using the parameterization in (33), a solution  $u$  of  $(\tilde{P}_\varepsilon)$  satisfies the equation

$$-\frac{1}{\sqrt{\det g}} [\partial_B (g^{AB} \sqrt{\det g})] \partial_A u - g^{AB} \partial_{AB}^2 u + u - u^p = 0 \quad \text{in } \frac{1}{\varepsilon} \mathcal{U}_1 \times B_{\varepsilon, \gamma} \tag{39}$$

with Neumann boundary conditions on  $\{\zeta_{n+1} = 0\}$ . Looking at the term of order  $\varepsilon^i$  in this equation, we will determine recursively the functions  $(w_i)_i$  and  $(\Phi_{i-2})_i$  (defined in (31)) for  $i = 1, \dots, I$ . The specific choice of the integer  $I$ , which will be determined later, will depend on the dimension  $N$  of  $\Omega$ , the dimension  $k$  of  $K$ , and the exponent  $p$ . For the moment we let it denote just an arbitrary integer. The main result of this section is the following one.

**Proposition 3.1.** *Consider the Euler functional  $J_\varepsilon$  defined in (9) and associated to problem  $(\tilde{P}_\varepsilon)$  (for  $p \leq \frac{N+2}{N-2}$ ). Then for any  $I \in \mathbb{N}$  there exists a function  $u_{I,\varepsilon} : \Omega_\varepsilon \rightarrow \mathbb{R}$  with the following properties*

$$\|J'_\varepsilon(u_{I,\varepsilon})\|_{H^1(\Omega_\varepsilon)} \leq C_I \varepsilon^{I+1-\frac{k}{2}}; \quad u_{I,\varepsilon} \geq 0 \quad \text{in } \Omega_\varepsilon; \quad \frac{\partial u_{I,\varepsilon}}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{40}$$

where  $C_I$  depends only on  $\Omega$ ,  $K$ ,  $p$  and  $I$ . Moreover in the above coordinates there holds

$$\begin{cases} |\nabla_y^{(m)} u_{I,\varepsilon}(y, \zeta)| \leq C_{m,I} \varepsilon^m e^{-|\zeta|} P_I(\zeta), \\ \|\nabla_y^{(m)} \nabla_\zeta u_{I,\varepsilon}(y, \zeta)\| \leq C_{m,I} \varepsilon^m e^{-|\zeta|} P_I(\zeta), \\ \|\nabla_y^{(m)} \nabla_\zeta^2 u_{I,\varepsilon}(y, \zeta)\| \leq C_{m,I} \varepsilon^m e^{-|\zeta|} P_I(\zeta); \end{cases} \quad y \in \frac{1}{\varepsilon} \mathcal{U}_1, \zeta \in B_{\gamma,\varepsilon}, m = 0, 1, \dots, \quad (41)$$

where  $\nabla_y^{(m)}$  (respectively  $\nabla_\zeta^{(i)}$ ) is any derivative of order  $m$  with respect to the  $y$  variables (respectively of order  $i$  with respect to the  $\zeta$  variables), where  $C_{m,I}$  is a constant depending only on  $\Omega$ ,  $K$ ,  $p$ ,  $I$  and  $m$ , and where  $P_I(\zeta)$  are suitable polynomials in  $\zeta$ .

In the next subsection we show how to construct the approximate solution  $u_{I,\varepsilon}$  and we give some general ideas for the derivation of the estimates in (41). We refer to [33] for rigorous and detailed proofs.

### 3.2. Proof of Proposition 3.1

This subsection is devoted to the explicit construction of  $u_{I,\varepsilon}$ . First of all we expand the Laplace–Beltrami operator (applied to an arbitrary function  $u$ ) in Fermi coordinates, and then by means of this expansion we define implicitly and recursively the functions  $(w_i)_i$  and the normal sections  $(\Phi_i)_i$ .

#### 3.2.1. Expansion of $\Delta_{g_\varepsilon} u$ in Fermi coordinates

We first provide a Taylor expansion of the coefficients of the metric  $g = g_\varepsilon$ . From Lemma 2.1 and formula (37) we have immediately the following result.

**Lemma 3.2.** *For the (Euclidean) metric  $g_\varepsilon$  in the above coordinates we have the expansions*

$$\begin{aligned} g_{ij} &= \delta_{ij} + 2\varepsilon \zeta_{n+1} H_{ij} + \frac{1}{3} \varepsilon^2 R_{istj} \zeta_s \zeta_t + \varepsilon^2 \zeta_{n+1}^2 (H^2)_{ij} + \mathcal{O}(\varepsilon^3 |\zeta|^3); \\ g_{aj} &= 2\varepsilon \zeta_{n+1} H_{aj} + \mathcal{O}(\varepsilon^2 |\zeta|^2); \\ g_{ab} &= \delta_{ab} - 2\varepsilon \Gamma_a^b(E_i) \zeta_i + 2\varepsilon \zeta_{n+1} H_{ab} + \varepsilon^2 [R_{sabl} + \Gamma_a^c(E_s) \Gamma_c^b(E_l)] \zeta_s \zeta_l \\ &\quad + \varepsilon^2 \zeta_{n+1}^2 (H^2)_{ab} + \mathcal{O}(\varepsilon^3 |\zeta|^3); \\ g_{\alpha N} &\equiv 0; \quad g_{NN} \equiv 1. \end{aligned}$$

Using these formulas, we are interested in expanding  $\Delta_{g_\varepsilon} u$  in powers of  $\varepsilon$  for a function  $u$  of the form

$$u(\bar{y}, \zeta) = u(\varepsilon y, \zeta).$$

Such a function represents indeed an *ansatz* for each term of the sum in (31).

We recall that, when differentiating functions with respect to the variables  $y, \zeta$ , we will mean that  $\partial_a = \partial_{y_a}$  and  $\partial_i = \partial_{\zeta_i}$ . When dealing with the scaled variables  $\bar{y}$  we will write explicitly  $\partial_{\bar{y}_a}$ , so that, if  $u$  is as above, we have  $\partial_a u(\varepsilon y, \zeta) = \varepsilon \partial_{\bar{y}_a} u(\bar{y}, \zeta)$ .



**Lemma 3.3.** *Given any positive integer  $I$  and a function  $u : \frac{1}{\varepsilon}\mathcal{U}_1 \times B_{\varepsilon,\gamma} \rightarrow \mathbb{R}$  of the form  $u(\varepsilon y, \zeta)$ , we have*

$$\begin{aligned} \Delta_{g_\varepsilon} u &= \partial_{\bar{i}i}^2 u + \partial_{\zeta_{n+1}\bar{\zeta}_{n+1}}^2 u + \varepsilon [H_\alpha^\alpha \partial_{\zeta_{n+1}} u - 2\zeta_{n+1} H_{ij} \partial_{\bar{i}j}^2 u] \\ &+ \varepsilon^2 [L_{2,1}u + L_{2,2}u + L_{2,3}u] + \sum_{i=3}^I \varepsilon^i L_i u + \varepsilon^{I+1} \tilde{L}_{I+1} u, \end{aligned} \tag{42}$$

where

$$\begin{aligned} L_{2,1}u &= \partial_{\bar{y}_a \bar{y}_a}^2 u - 4\zeta_{n+1} H_{ia} \partial_{\zeta_i \bar{y}_a}^2 u; \\ L_{2,2}u &= 3\zeta_{n+1}^2 (H^2)_{ij} \partial_{\zeta_i \zeta_j}^2 u + 2\zeta_{n+1} H_{ab} \Gamma_b^a(E_i) \partial_i u - 2\zeta_{n+1} \text{tr}(H^2) \partial_{\zeta_{n+1}} u; \\ L_{2,3}u &= \left( R_{iaal} + \frac{1}{3} R_{ihhl} \right) \zeta_l \partial_i u - \frac{1}{3} R_{mijl} \zeta_m \zeta_l \partial_{\zeta_i \zeta_j}^2 u - \frac{1}{3} R_{miji} \zeta_m \partial_{\zeta_j} u \\ &- \zeta_j \Gamma_a^b(E_i) \Gamma_b^a(E_j) \partial_{\zeta_i} u + 2\zeta_i H_{ab} \Gamma_b^a(E_i) \partial_{\zeta_{n+1}} u, \end{aligned}$$

and where the  $L_i$ 's are linear operators of order 1 and 2 acting on the variables  $\bar{y}$  and  $\zeta$  whose coefficients are polynomials (of order at most  $i$ ) in  $\zeta$  uniformly bounded (and smooth) in  $\bar{y}$ . The operator  $\tilde{L}_{I+1}$  is still linear and satisfies the same properties of the  $L_i$ 's, except that its coefficients are not polynomials in  $\zeta$ , although they are bounded by polynomials in  $\zeta$ .

**Proof.** The proof is simply based on a Taylor expansion of the metric coefficients in terms of the geometric properties of  $\partial\Omega$  and  $K$ , as in Lemma 3.2. Recall that the Laplace–Beltrami operator is given by

$$\Delta_{g_\varepsilon} = \frac{1}{\sqrt{\det g_\varepsilon}} \partial_A (\sqrt{\det g_\varepsilon} g_\varepsilon^{AB} \partial_B),$$

where indices  $A$  and  $B$  run between 1 and  $N$ . We can write

$$\Delta_{g_\varepsilon} = g_\varepsilon^{AB} \partial_{AB}^2 + (\partial_A g_\varepsilon^{AB}) \partial_B + \frac{1}{2} \partial_A (\log \det g_\varepsilon) g_\varepsilon^{AB} \partial_B.$$

Using the expansions of Lemma 3.3, we easily see that

$$\begin{aligned} g_\varepsilon^{AB} \partial_{AB}^2 u &= \partial_{\zeta_i \zeta_i}^2 u + \partial_{\zeta_{n+1} \bar{\zeta}_{n+1}}^2 u - 2\varepsilon \zeta_{n+1} H_{ij} \partial_{\zeta_i \zeta_j}^2 u \\ &+ \varepsilon^2 \left\{ \partial_{\bar{y}_a \bar{y}_a}^2 + \left( 3\zeta_{n+1}^2 (H^2)_{ij} - \frac{1}{3} R_{mijl} \zeta_m \zeta_l \right) \partial_{\zeta_i \zeta_i}^2 u - 4\zeta_{n+1} H_{ia} \partial_{\zeta_i \bar{y}_a}^2 u \right\} \\ &+ \mathcal{O}(\varepsilon^3 |\zeta|^3). \end{aligned}$$

We can also prove

$$\begin{aligned} \sqrt{\det g_\varepsilon} &= 1 + \varepsilon \zeta_{n+1} H_\alpha^\alpha + \frac{1}{6} \varepsilon^2 R_{mii l} \zeta_m \zeta_l + \frac{1}{2} \varepsilon^2 (R_{maal} + \Gamma_a^c(E_m) \Gamma_c^a(E_l)) \zeta_m \zeta_l \\ &+ \varepsilon^2 \left\{ \frac{1}{2} \zeta_{n+1}^2 (H_\alpha^\alpha)^2 - \zeta_{n+1} \operatorname{tr}(H^2) + 2 \zeta_{n+1} \zeta_i H_{ab} \Gamma_b^a(E_i) - \zeta_i \zeta_j \Gamma_a^b(E_i) \Gamma_b^a(E_j) \right\} \\ &+ \mathcal{O}(\varepsilon^3 |\zeta|^3), \end{aligned}$$

which gives

$$\begin{aligned} \log \sqrt{\det g_\varepsilon} &= \varepsilon \zeta_{n+1} H_\alpha^\alpha + \varepsilon^2 \{ 2 \zeta_{n+1} \zeta_i H_{ab} \Gamma_b^a(E_i) - \zeta_{n+1}^2 \operatorname{tr}(H^2) - \zeta_i \zeta_j \Gamma_a^b(E_i) \Gamma_b^a(E_j) \} \\ &+ \frac{1}{6} \varepsilon^2 R_{mii l} \zeta_m \zeta_l + \frac{1}{2} \varepsilon^2 (R_{maal} + \Gamma_a^c(E_m) \Gamma_c^a(E_l)) \zeta_m \zeta_l + \mathcal{O}(\varepsilon^3 |\zeta|^3). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \partial_A (\log \sqrt{\det g_\varepsilon}) g^{AB} \partial_B &= \varepsilon^2 \left\{ 2 \zeta_{n+1} H_{ab} \Gamma_b^a(E_i) - \zeta_j \Gamma_a^b(E_i) \Gamma_b^a(E_j) + \frac{1}{3} R_{mhh l} \zeta_l + R_{iaal} \zeta_l \right\} \partial_i u \\ &+ \varepsilon H_\alpha^\alpha \partial_{\zeta_{n+1}} u + \varepsilon^2 \{ 2 \zeta_l H_{ab} \Gamma_b^a(E_l) - 2 \zeta_{n+1} \operatorname{tr}(H^2) \} \partial_{\zeta_{n+1}} u \\ &+ \mathcal{O}(\varepsilon^3 |\zeta|^3). \end{aligned}$$

Collecting these formulas together, we obtain the desired result.  $\square$

**Remarks 3.4.** (a) The term of order  $\varepsilon$  in the expansion of  $\Delta_g u$  in (42) depends on the fact that  $\partial \Omega$  has an extrinsic curvature in  $\mathbb{R}^N$ . Such a term does not appear in the analogous expansion for the mean curvature of tubes condensing on minimal subvarieties of an abstract manifold, see Proposition 4.1 in [32] (where the small parameter  $\rho$  is the counterpart of our parameter  $\varepsilon$ ).

(b) For later purposes, see for example Lemma 6.1, it is convenient to analyze in further detail the operator  $L_3$  in (42), and in particular the coefficients of the second derivatives in the  $\bar{y}$  variables. It follows from the above expansions that the coefficient of  $\partial_{\bar{y}_a \bar{y}_b}^2$  in  $L_3$  is given by

$$2(\zeta_i \Gamma_a^b(E_i) - \zeta_{n+1} H_{ab}).$$

### 3.2.2. Construction of the approximate solution

We show now how to construct the approximate solutions of  $(\tilde{P}_\varepsilon)$  via an iterative method. Given  $l - 2$  smooth vector fields  $\Phi_0, \dots, \Phi_{l-2}$  we define first the following function  $\hat{u}_{l,\varepsilon}$  on  $K \times \mathbb{R}^{n+1}$ , see (31)

$$\begin{aligned} \hat{u}_{l,\varepsilon}(\bar{y}, \zeta) &= w_0(\zeta' + \Phi(\bar{y}), \zeta_{n+1}) + \varepsilon w_1(\bar{y}, \zeta' + \Phi(\bar{y}), \zeta_{n+1}) + \dots \\ &+ \varepsilon^l w_l(\bar{y}, \zeta' + \Phi(\bar{y}), \zeta_{n+1}), \end{aligned}$$

where  $\Phi = \Phi_0 + \varepsilon \Phi_1 + \dots + \varepsilon^{l-2} \Phi_{l-2}$ . In the following, with an abuse of notation, we will consider  $\hat{u}_{l,\varepsilon}$  (and  $w_0, \dots, w_l$ ) as functions of the variables  $y$  and  $\zeta$  through the change of coordinates  $\bar{y} = \varepsilon y$ .

To define the functions  $(w_j)_j$  and  $(\Phi_j)_j$  we expand Eq. (39) formally in powers of  $\varepsilon$  for  $u = \hat{u}_{l,\varepsilon}$  (using mostly Lemma 3.3) and we analyze each term separately. Looking at the coefficient of  $\varepsilon$  in the expansion we will determine  $w_1$ , while looking at the coefficient of  $\varepsilon^j$  we will

determine  $w_j$  and  $\Phi_{j-2}$ , for  $j = 2, \dots, I$ . In this procedure we use crucially the invertibility of the Jacobi operator (recall that we are assuming  $K$  to be non-degenerate) and the spectral properties of the linearization of (3) at  $w_0$ .

- *Step 1: Construction of  $w_1$*

We begin by taking  $I = 1$  and  $\Phi = 0$ . From Lemma 3.3 we get *formally*

$$-\Delta_{g_\varepsilon} \hat{u}_{1,\varepsilon} + \hat{u}_{1,\varepsilon} - \hat{u}_{1,\varepsilon}^p = -\Delta_{\mathbb{R}^{n+1}} w_0 + w_0 - w_0^p + \varepsilon(-\Delta_{\mathbb{R}^{n+1}} w_1 + w_1 - p w_0^{p-1} w_1) - \varepsilon[H_\alpha^\alpha \partial_{\zeta_{n+1}} w_0 - 2\zeta_{n+1} H_{ij} \partial_{ij}^2 w_0] + O(\varepsilon^2).$$

The term of order 1 (in the power expansion in  $\varepsilon$ ) vanishes trivially since  $w_0$  solves (3), and in order to make the coefficient of  $\varepsilon$  vanish,  $w_1$  must satisfy the following equation

$$\mathcal{L}_0 w_1 = H_\alpha^\alpha \partial_{\zeta_{n+1}} w_0 - 2\zeta_{n+1} H_{ij} \partial_{ij}^2 w_0, \tag{43}$$

where  $\mathcal{L}_0$  is the linearization of (3) at  $w_0$ , namely

$$\begin{cases} -\Delta w_1 + w_1 - p w_0^{p-1} w_1 = H_\alpha^\alpha \partial_{\zeta_{n+1}} w_0 - 2\zeta_{n+1} H_{ij} \partial_{ij}^2 w_0 & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial w_1}{\partial \zeta_{n+1}} = 0 & \text{on } \{\zeta_{n+1} = 0\}. \end{cases}$$

Since  $\mathcal{L}_0$  is self-adjoint and Fredholm on  $H^1(\mathbb{R}_+^{n+1})$ , the equation is solvable if and only if the right-hand side is orthogonal to the kernel of  $\mathcal{L}_0$ , namely if and only if the  $L^2$  product of the right-hand side with  $\partial w_0 / \partial \zeta_i$  vanishes for  $i = 1, \dots, n$ , see Proposition 4.1 below. This is clearly satisfied in our case since both  $\partial_{\zeta_{n+1}} w_0$  and  $\partial_{ij}^2 w_0$  are even in  $\zeta'$ , while the  $\partial w_0 / \partial \zeta_i$ 's are odd in  $\zeta'$  for every  $i$ . Besides the existence of  $w_1$ , from elliptic regularity estimates we can prove its exponential decay in  $\zeta$  and its smoothness in  $\bar{y}$  (see for example Lemma 3.4 in [33]). Precisely, there exists a positive constant  $C_1$  (depending only on  $\Omega, K$  and  $p$ ) such that for any integer  $\ell$  there holds

$$|\nabla_{\bar{y}}^{(\ell)} w_1(\bar{y}, \zeta)| \leq C_1 C_l (1 + |\zeta|)^{C_l} e^{-|\zeta|}; \quad (\bar{y}, \zeta) \in K \times \mathbb{R}^{n+1}, \tag{44}$$

where  $C_l$  depends only on  $l, p, K$  and  $\Omega$ .

- *Step 2: Expansion at an arbitrary order*

We consider next the coefficient of  $\varepsilon^{\tilde{I}}$  for an integer  $\tilde{I}$  between 2 and  $I$ , and we assume that the functions  $w_1, \dots, w_{\tilde{I}-1}$  and the vector fields  $\Phi_0, \dots, \Phi_{\tilde{I}-3}$  have been determined by induction in  $\tilde{I}$ . The couple  $(w_{\tilde{I}}, \Phi_{\tilde{I}-2})$  will be found reasoning as for  $w_1$ : in particular an equation for  $\Phi_{\tilde{I}-2}$  (solvable by the invertibility of  $\mathfrak{J}$ ) is obtained by imposing orthogonality of some expression to the kernel of  $\mathcal{L}_0$ , and then  $w_{\tilde{I}}$  is found again with Fredholm's alternative.

Expanding (39) with  $u = \hat{u}_{I,\varepsilon}$ , we easily see that (formally), in the coefficient of  $\varepsilon^{\tilde{I}}$ , the function  $w_{\tilde{I}}$  appears as solution of the equation

$$\begin{cases} \mathcal{L}_\Phi w_{\tilde{I}} = F_{\tilde{I}}(\bar{y}, \zeta, w_0, w_1, \dots, w_{\tilde{I}-1}, \Phi_0, \dots, \Phi_{\tilde{I}-2}) & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial w_{\tilde{I}}}{\partial \zeta_{n+1}} = 0 & \text{on } \{\zeta_{n+1} = 0\}, \end{cases} \tag{45}$$

where  $\mathcal{L}_\Phi$  is defined by

$$\mathcal{L}_\Phi u = -\Delta u + u - p w_0^{p-1} (\zeta' + \Phi(\bar{y}), \zeta_{n+1}) u,$$

and where  $F_{\tilde{I}}$  is some smooth function of its arguments (which we are assuming determined by induction). Our next goal is to understand the role of  $\Phi_{\tilde{I}-2}$  in the orthogonality condition on  $F_{\tilde{I}}$  (to the kernel of  $\mathcal{L}_\Phi$ ). In order to do this, we notice that, using Lemma 3.3 for  $u = \hat{u}_{I,\varepsilon}$ , the function  $\Phi$  (precisely its derivatives in  $\bar{y}$ ) appears through the chain rule when we differentiate  $u$  with respect to the  $\bar{y}$  variables. Moreover, for testing the orthogonality of the right-hand side in (45) to the kernel of  $\mathcal{L}_\Phi$ , we have to multiply it by the functions  $\frac{\partial w_0}{\partial \zeta_i} (\zeta' + \Phi(\bar{y}), \zeta_{n+1})$ ,  $i = 1, \dots, n$ , so this condition will yield an equation for  $\Phi$  (and in particular for  $\Phi_{\tilde{I}-2}$ ) through a change of variables of the form  $\zeta' \mapsto \zeta' + \Phi(\bar{y})$ .

Therefore, in the expansion of  $\Delta_g \hat{u}_{I,\varepsilon}$ , we focus only on the terms (of order  $\varepsilon^{\tilde{I}}$ ) containing either derivatives with respect to the  $\bar{y}$  variables, which we collected in  $L_{2,1}$ , or containing explicitly the variables  $\zeta'$ , which are listed in  $L_{2,3}$ . In particular, none of these terms appear in the first line of (42).

Denoting the components of  $\Phi$  by  $(\Phi^j)_j$  (in the basis  $(E_j)_j$  of  $NK$ ), there holds

$$\begin{aligned} \partial_{\bar{y}_a} (u(\bar{y}, \zeta' + \Phi(\bar{y}), \zeta_{n+1})) &= \partial_{\bar{y}_a} u(\bar{y}, \zeta' + \Phi, \zeta_{n+1}) + \frac{\partial \Phi^j}{\partial \bar{y}_a} \frac{\partial u}{\partial \zeta_j} (\bar{y}, \zeta' + \Phi(\bar{y}), \zeta_{n+1}); \\ \partial_{\bar{y}_a \bar{y}_a}^2 (u(\bar{y}, \zeta' + \Phi(\bar{y}), \zeta_{n+1})) &= \partial_{\bar{y}_a \bar{y}_a}^2 u(\bar{y}, \zeta' + \Phi, \zeta_{n+1}) + 2 \frac{\partial \Phi^j}{\partial \bar{y}_a} \partial_{\bar{y}_a \zeta_j}^2 u(\bar{y}, \zeta' + \Phi, \zeta_{n+1}) \\ &\quad + \frac{\partial^2 \Phi^j}{\partial \bar{y}_a \partial \bar{y}_a} \frac{\partial u}{\partial \zeta_j} (\bar{y}, \zeta' + \Phi(\bar{y}), \zeta_{n+1}) \\ &\quad + \frac{\partial \Phi^j}{\partial \bar{y}_a} \frac{\partial \Phi^l}{\partial \bar{y}_a} \frac{\partial^2 u}{\partial \zeta_j \partial \zeta_l} (\bar{y}, \zeta' + \Phi(\bar{y}), \zeta_{n+1}); \\ \frac{\partial^2}{\partial \zeta_l \partial \bar{y}_a} (u(\bar{y}, \zeta' + \Phi(\bar{y}), \zeta_{n+1})) &= \partial_{\zeta_l \bar{y}_a}^2 u(\bar{y}, \zeta' + \Phi, \zeta_{n+1}) \\ &\quad + \frac{\partial \Phi^j}{\partial \bar{y}_a} \frac{\partial^2 u}{\partial \zeta_j \partial \zeta_l} (\bar{y}, \zeta' + \Phi(\bar{y}), \zeta_{n+1}). \end{aligned}$$

Therefore, recalling the definition of  $\hat{u}_{I,\varepsilon}$ , since  $\partial_{\bar{y}_a} w_0 = 0$  we find that

$$L_{2,1} \hat{u}_{I,\varepsilon} = \frac{\partial^2 \Phi^j}{\partial \bar{y}_a \partial \bar{y}_a} \frac{\partial w_0}{\partial \zeta_j} + \frac{\partial \Phi^j}{\partial \bar{y}_a} \frac{\partial \Phi^l}{\partial \bar{y}_a} \frac{\partial^2 w_0}{\partial \zeta_j \partial \zeta_l} - 4 \zeta_{n+1} H_{Ia} \frac{\partial \Phi^j}{\partial \bar{y}_a} \frac{\partial^2 w_0}{\partial \zeta_j \partial \zeta_l}$$

$$\begin{aligned}
 & + \sum_{i=1}^{\tilde{I}} \varepsilon^i \left\{ \partial_{\bar{y}_a \bar{y}_a}^2 w_i + 2 \frac{\partial \Phi^j}{\partial \bar{y}_a} \partial_{\bar{y}_a \zeta_j}^2 w_i + \frac{\partial^2 \Phi^j}{\partial \bar{y}_a \partial \bar{y}_a} \frac{\partial w_i}{\partial \zeta_j} + \frac{\partial \Phi^j}{\partial \bar{y}_a} \frac{\partial \Phi^l}{\partial \bar{y}_a} \frac{\partial^2 w_i}{\partial \zeta_j \partial \zeta_l} \right. \\
 & \left. - 4 \zeta_{n+1} H_{la} \left( \partial_{\zeta_l \bar{y}_a}^2 w_i + \frac{\partial \Phi^j}{\partial \bar{y}_a} \frac{\partial^2 w_i}{\partial \zeta_j \partial \zeta_l} \right) \right\}.
 \end{aligned}$$

- Step 3: Determining  $w_{\tilde{I}}$  and  $\Phi_{\tilde{I}-2}$  for  $\tilde{I} \geq 2$

When we look at the coefficient of  $\varepsilon^{\tilde{I}}$  in  $\varepsilon^2 L_{2,1} \hat{u}_{I,\varepsilon}$ , the terms containing  $\Phi_{\tilde{I}-2}$  are given by

$$\frac{\partial^2 \Phi^j}{\partial \bar{y}_a \partial \bar{y}_a} \frac{\partial w_0}{\partial \zeta_j} - 4 \zeta_{n+1} H_{la} \frac{\partial \Phi^j}{\partial \bar{y}_a} \frac{\partial^2 w_0}{\partial \zeta_j \partial \zeta_l} \left( + \frac{\partial \Phi^j}{\partial \bar{y}_a} \frac{\partial \Phi^l}{\partial \bar{y}_a} \frac{\partial^2 w_0}{\partial \zeta_j \partial \zeta_l} \text{ if } \tilde{I} = 2 \right).$$

When we project  $\Delta_g \hat{u}_{I,\varepsilon} - \hat{u}_{I,\varepsilon} + \hat{u}_{I,\varepsilon}^p$  onto the kernel of  $\mathcal{L}_\Phi$ , namely when we multiply this expression by  $\frac{\partial w_0}{\partial \zeta_s}(\zeta' + \Phi(\bar{y}), \zeta_{n+1})$ ,  $s = 1, \dots, n$ , considering the terms of order  $\varepsilon^{\tilde{I}}$  involving  $\Phi_{\tilde{I}-2}$ , we have no contribution from the first line and from  $L_{2,2}$  in (42) (with  $u = \hat{u}_{I,\varepsilon}$ ), as explained in Step 2. Also, in (42), the factors of  $\varepsilon^i$  for  $i \geq 3$ , multiplied by  $\varepsilon^{\tilde{I}-2} \Phi_{\tilde{I}-2}$  will give higher order terms. In conclusion, we only need to pay attention to  $L_{2,1}$  and  $L_{2,3}$ .

When we multiply  $\varepsilon^2 L_{2,3} w_0(\zeta' + \Phi, \zeta_{n+1})$  by  $\frac{\partial w_0}{\partial \zeta_s}(\zeta' + \Phi, \zeta_{n+1})$ ,  $s = 1, \dots, n$ , we can obtain the coefficient of  $\varepsilon^{\tilde{I}} \Phi_{\tilde{I}-2}^h$  in the following way.

Looking for example at the first term in  $\varepsilon^2 L_{2,3}$  we get

$$\begin{aligned}
 & \varepsilon^2 \int_{\mathbb{R}_+^{n+1}} \left( R_{iaal} + \frac{1}{3} R_{ihhl} \right) \zeta_l \partial_i w_0(\zeta' + \Phi, \zeta_{n+1}) \partial_s w_0(\zeta' + \Phi, \zeta_{n+1}) d\zeta \\
 & = \varepsilon^2 \int_{\mathbb{R}_+^{n+1}} \left( R_{iaal} + \frac{1}{3} R_{ihhl} \right) (\zeta_l - \Phi^l) \partial_i w_0(\zeta', \zeta_{n+1}) \partial_s w_0(\zeta', \zeta_{n+1}) d\zeta \\
 & = \varepsilon^2 \int_{\mathbb{R}_+^{n+1}} \left( R_{iaal} + \frac{1}{3} R_{ihhl} \right) \zeta_l \partial_i w_0(\zeta', \zeta_{n+1}) \partial_s w_0(\zeta', \zeta_{n+1}) d\zeta \\
 & \quad - \varepsilon^2 \sum_{j=0}^{\tilde{I}-2} \varepsilon^j \Phi_j^l \int_{\mathbb{R}_+^{n+1}} \left( R_{iaal} + \frac{1}{3} R_{ihhl} \right) \partial_i w_0(\zeta', \zeta_{n+1}) \partial_s w_0(\zeta', \zeta_{n+1}) d\zeta.
 \end{aligned}$$

Since  $w_0$  is even in  $\zeta'$ , it follows by symmetry that the term of order  $\varepsilon^{\tilde{I}}$  containing  $\Phi_{\tilde{I}-2}$  in the last expression is given by

$$-C_0 \left( R_{saal} + \frac{1}{3} R_{shhl} \right) \Phi_{\tilde{I}-2}^l, \tag{46}$$

where we have set

$$C_0 = \int_{\mathbb{R}_+^{n+1}} (\partial_1 w_0)^2. \tag{47}$$

From similar arguments, the third and the fourth terms in  $L_{2,3} w_0$  give respectively

$$\frac{1}{3} R_{lisi} C_0 \Phi_{\tilde{l}-2}^l, \tag{48}$$

and

$$C_0 \Gamma_a^b(E_s) \Gamma_b^a(E_l) \Phi_{\tilde{l}-2}^l.$$

The last term in  $L_{2,3} w_0$  gives no contribution since the coefficient of  $\Phi_{\tilde{l}-2}$  vanishes by oddness, so it remains to consider the second term. Integrating by parts we find

$$\frac{2}{3} R_{mijl} \Phi_{\tilde{l}-2}^l \int_{\mathbb{R}_+^{n+1}} \zeta_m \partial_{\zeta_s} w_0 \partial_{\zeta_i \zeta_j}^2 w_0 d\zeta \left( + \partial_{\zeta_i \bar{y}_a}^2 \Phi_{\tilde{l}-2}^m \partial_{\zeta_j \bar{y}_a}^2 \Phi_{\tilde{l}-2}^l \int_{\mathbb{R}_+^{n+1}} \partial_{\zeta_i \zeta_j}^2 w_0 \partial_{\zeta_s} w_0 d\zeta \text{ if } \tilde{l} = 2 \right).$$

In case  $\tilde{l} = 2$  the quantity within round brackets cancels by oddness, therefore in any case we only need to estimate the first one. Still by oddness in  $\zeta'$ , the first integral is non-zero only if, either  $i = j$  and  $m = s$ , or  $i = s$  and  $j = m$ , or  $i = m$  and  $j = s$ .

In the latter case we have vanishing by the antisymmetry of the curvature tensor in the first two indices. Therefore the only terms left to consider are

$$\sum_i \frac{2}{3} R_{siii} \Phi_{\tilde{l}-2}^l \int_{\mathbb{R}_+^{n+1}} \zeta_s \partial_{\zeta_s} w_0 \partial_{\zeta_i \zeta_i}^2 w_0 d\zeta + \sum_i \frac{2}{3} R_{isil} \Phi_{\tilde{l}-2}^l \int_{\mathbb{R}_+^{n+1}} \zeta_i \partial_{\zeta_s} w_0 \partial_{\zeta_s \zeta_i}^2 w_0 d\zeta.$$

Observe that, integrating by parts, when  $s \neq i$  there holds

$$\int_{\mathbb{R}_+^{n+1}} \zeta_s \partial_{\zeta_s} w_0 \partial_{\zeta_i \zeta_i}^2 w_0 d\zeta = - \int_{\mathbb{R}_+^{n+1}} \zeta_i \partial_{\zeta_s} w_0 \partial_{\zeta_s \zeta_i}^2 w_0 d\zeta.$$

Hence, still by the antisymmetry of the curvature tensor we are left with

$$- \sum_i \frac{4}{3} R_{siii} \Phi_{\tilde{l}-2}^l \int_{\mathbb{R}_+^{n+1}} \zeta_i \partial_{\zeta_s} w_0 \partial_{\zeta_s \zeta_i}^2 w_0 d\zeta.$$

The last integral can be computed with a further integration by parts and is equal to  $-\frac{1}{2} C_0$ , so we get

$$\frac{2}{3} R_{siii} C_0 \Phi_{\tilde{l}-2}^l.$$

This quantity cancels exactly with the second term in (46) and with (48).

When we multiply  $\varepsilon^2 L_{2,1} w_0(\zeta' + \Phi, \zeta_{n+1})$  by  $\frac{\partial w_0}{\partial \zeta_s}(\zeta' + \Phi, \zeta_{n+1})$ ,  $s = 1, \dots, n$ , the terms containing  $\varepsilon^{\tilde{l}} \Phi_{\tilde{l}-2}^h$  are given by

$$\int_{\mathbb{R}_+^{n+1}} \frac{\partial^2 \Phi_{\tilde{l}-2}^j}{\partial \bar{y}_a \partial \bar{y}_a} \partial_{\zeta_j} w_0 \partial_{\zeta_s} w_0 d\zeta - 4 \int_{\mathbb{R}_+^{n+1}} \zeta_{n+1} H_{la} \frac{\partial \Phi_{\tilde{l}-2}^j}{\partial \bar{y}_a} \partial_{\zeta_j \zeta_l}^2 w_0 \partial_{\zeta_s} w_0 d\zeta$$

$$\left( + \int_{\mathbb{R}_+^{n+1}} \frac{\partial \Phi_{\tilde{l}-2}^j}{\partial \bar{y}_a} \frac{\partial \Phi_{\tilde{l}-2}^l}{\partial \bar{y}_a} \partial_{\zeta_j \zeta_l}^2 w_0 \partial_{\zeta_s} w_0 d\zeta \text{ if } \tilde{l} = 2 \right),$$

which give by oddness

$$C_0 \frac{\partial^2 \Phi_{\tilde{l}-2}^j}{\partial \bar{y}_a \partial \bar{y}_a}.$$

Collecting the above computations, we conclude that  $F_{\tilde{l}}(\bar{y}, \zeta, w_0, w_1, \dots, w_{\tilde{l}-1}, \Phi_0, \dots, \Phi_{\tilde{l}-2})$ , the right-hand side of (45), is  $L^2$ -orthogonal to the kernel of  $\mathcal{L}_\Phi$  if and only if  $\Phi_{\tilde{l}-2}$  satisfies an equation of the form

$$C_0 \left( \frac{\partial^2 \Phi_{\tilde{l}-2}^s}{\partial \bar{y}_a \partial \bar{y}_a} - R_{saal} \Phi_{\tilde{l}-2}^l + \Gamma_a^b(E_s) \Gamma_b^a(E_l) \Phi_{\tilde{l}-2}^l \right)$$

$$= G_{\tilde{l}-2}(\bar{y}, \zeta, w_0, w_1, \dots, w_{\tilde{l}-1}, \Phi_0, \dots, \Phi_{\tilde{l}-3}),$$

for some expression  $G_{\tilde{l}-2}$ . This equation can indeed be solved in  $\Phi_{\tilde{l}-2}$ . In fact, observe that the operator acting on  $\Phi_{\tilde{l}-2}$  in the left-hand side is nothing but the Jacobi operator, which is invertible by the non-degeneracy condition on  $K$ .

Having defined  $\Phi_{\tilde{l}-2}$  in this way, we turn to the construction of  $w_{\tilde{l}}$  which, we recall, satisfies Eq. (45). Having imposed the orthogonality condition, we get again solvability and, as for  $w_1$ , one can prove the following estimates

$$|\nabla_{\bar{y}}^{(\ell)} w_{\tilde{l}}(\bar{y}, \zeta)| \leq C_{\tilde{l}} C_l (1 + |\zeta|)^{C_{\tilde{l}}} e^{-|\zeta|}; \quad (\bar{y}, \zeta) \in K \times \mathbb{R}^{n+1}, \tag{49}$$

where  $C_l$  depends only on  $l, p, K$  and  $\Omega$ .

As already mentioned, we limit ourselves to the formal construction of the functions  $u_{l,\varepsilon}$ , omitting the details about the rigorous estimates of the error terms, which can be obtained reasoning as in [33]. We only mention that the number  $\gamma$  has to be chosen sufficiently small to obtain the positivity of  $u_{l,\varepsilon}$ , after we multiply  $\hat{u}_{l,\varepsilon}$  by the cutoff function  $\chi_\varepsilon$ , see (31) and (32).

#### 4. A model linear problem

In this section we consider a model for the linearized equation at approximate solutions which, for  $p \leq (N + 2)/(N - 2)$  (as we are assuming until the last subsection), corresponds to  $J_\varepsilon''(u_{l,\varepsilon})$ . We first study a one-parameter family of eigenvalue problems, which include the linearization

at  $w_0$  of (3). Then we turn to the model for  $J''_\varepsilon(u_{I,\varepsilon})$ , which can be studied, roughly, using separation of variables.

4.1. *Some spectral analysis in  $\mathbb{R}_+^{n+1}$*

In this subsection we consider a class of eigenvalue problems, being mainly interested in the symmetries of the corresponding eigenfunctions. We denote points of  $\mathbb{R}^{n+1}$  by  $(n + 1)$ -tuples  $\zeta_1, \zeta_2, \dots, \zeta_n, \zeta_{n+1} = (\zeta', \zeta_{n+1})$ , and we let

$$\mathbb{R}_+^{n+1} = \{(\zeta_1, \zeta_2, \dots, \zeta_n, \zeta_{n+1}) \in \mathbb{R}^{n+1}: \zeta_{n+1} > 0\}.$$

For  $p \in (1, (n + 3)/(n - 1))$  ( $(n + 3)/(n - 1)$  is the critical exponent in  $\mathbb{R}^{n+1}$ ) we consider problem (3) which, we recall, is

$$\begin{cases} -\Delta u + u = u^p & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathbb{R}_+^{n+1}, \\ u > 0, & u \in H^1(\mathbb{R}_+^{n+1}). \end{cases}$$

It is well known, see e.g. [26], that this problem possesses a radial solution  $w_0(r), r^2 = \sum_{i=1}^{n+1} \zeta_i^2$ , which satisfies the properties

$$\begin{cases} w'_0(r) < 0 & \text{for every } r > 0, \\ \lim_{r \rightarrow \infty} e^r r^{\frac{n}{2}} w_0(r) = \alpha_{n,p} > 0, & \lim_{r \rightarrow \infty} \frac{w'_0(r)}{w_0(r)} = -1, \end{cases} \tag{50}$$

where  $\alpha_{n,p}$  is a positive constant depending only on  $n$  and  $p$ . Moreover, it turns out that all the solutions of (3) coincide with  $w_0$  up to a translation in the  $\zeta'$  variables, see [18,19].

Solutions of (3) can be found as critical points of the functional  $\bar{J}$  defined by

$$\bar{J}(u) = \frac{1}{2} \int_{\mathbb{R}_+^{n+1}} (|\nabla u|^2 + u^2) - \frac{1}{p+1} \int_{\mathbb{R}_+^{n+1}} |u|^{p+1}; \quad u \in H^1(\mathbb{R}_+^{n+1}). \tag{51}$$

We have the following non-degeneracy result, see e.g. [45].

**Proposition 4.1.** *The kernel of  $\bar{J}''(w_0)$  is generated by the functions  $\partial w_0/\partial \zeta_1, \dots, \partial w_0/\partial \zeta_n$ . More precisely, there holds*

$$\bar{J}''(w_0)[w_0, w_0] = -(p - 1) \|w_0\|_{H^1(\mathbb{R}_+^{n+1})}^2,$$

and

$$\bar{J}''(w_0)[v, v] \geq C^{-1} \|v\|_{H^1(\mathbb{R}_+^{n+1})}^2, \quad \forall v \in H^1(\mathbb{R}_+^{n+1}), \quad v \perp w_0, \partial_{\zeta_1} w_0, \dots, \partial_{\zeta_n} w_0,$$

for some positive constant  $C$ . In particular, we have  $\eta < 0, \sigma = 0$  and  $\tau > 0$ , where  $\eta, \sigma$  and  $\tau$  are respectively the first, the second and the third eigenvalue of  $\bar{J}''(w_0)$ . Furthermore the eigenvalue  $\eta$  is simple while  $\sigma$  has multiplicity  $n$ .



Notice that, writing the eigenvalue equation  $\bar{J}''(w_0)[u] = \lambda u$  in  $H^1(\mathbb{R}_+^{n+1})$ , taking the scalar product with an arbitrary test function and integrating by parts one finds that  $u$  satisfies

$$\begin{cases} -\Delta u + u - pw_0^{p-1}u = \lambda(-\Delta u + u) & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial u}{\partial \zeta_{n+1}} = 0 & \text{on } \partial\mathbb{R}_+^{n+1}. \end{cases}$$

The goal of this subsection (the motivation will become clear in the next one) is to study a more general version of this eigenvalue problem, namely

$$\begin{cases} -\Delta u + (1 + \alpha)u - pw_0^{p-1}u = \lambda(-\Delta u + (1 + \alpha)u) & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial u}{\partial v} = 0 & \text{on } \partial\mathbb{R}_+^{n+1}, \end{cases} \tag{52}$$

where  $\alpha \geq 0$ . It is convenient to introduce the Hilbert space (which coincides with  $H^1(\mathbb{R}_+^{n+1})$ , but endowed with an equivalent norm)

$$H_\alpha = \left\{ u \in H^1(\mathbb{R}_+^{n+1}) : \|u\|_\alpha^2 = \int_{\mathbb{R}_+^{n+1}} (|\nabla u|^2 + (1 + \alpha)u^2) \right\},$$

with corresponding scalar product  $(\cdot, \cdot)_\alpha$ . We also let  $T_\alpha : H_\alpha \rightarrow H_\alpha$  be defined by duality in the following way

$$(T_\alpha u, v)_{H_\alpha} = \int_{\mathbb{R}_+^{n+1}} ((\nabla u \cdot \nabla v) + (1 + \alpha)uv) - p \int_{\mathbb{R}_+^{n+1}} w_0^{p-1}uv; \quad u, v \in H_\alpha. \tag{53}$$

When  $\alpha = 0$ , the operator  $T_0$  is nothing but  $\bar{J}''(w_0)$ . For  $\alpha \geq 0$ , the eigenfunctions of  $T_\alpha$  satisfy (52). We want to study the first three eigenvalues of  $T_\alpha$  depending on the parameter  $\alpha$ .

**Proposition 4.2.** *Let  $\eta_\alpha, \sigma_\alpha$  and  $\tau_\alpha$  denote the first three eigenvalues of  $T_\alpha$ . Then  $\eta_\alpha, \sigma_\alpha$  and  $\tau_\alpha$  are non-decreasing in  $\alpha$ . For every value of  $\alpha$ ,  $\eta_\alpha$  is simple and there holds*

$$\frac{\partial \eta_\alpha}{\partial \alpha} > 0; \quad \lim_{\alpha \rightarrow +\infty} \eta_\alpha = 1.$$

*The eigenvalue  $\sigma_\alpha$  has multiplicity  $n$  and for  $\alpha$  small it satisfies  $\partial \sigma_\alpha / \partial \alpha > 0$ . The eigenfunction  $u_\alpha$  corresponding to  $\eta_\alpha$  is radial in  $\zeta$  and radially decreasing, while the eigenfunctions corresponding to  $\sigma_\alpha$  are spanned by functions of the form  $v_{\alpha,i}(\zeta) = \hat{v}_\alpha(|\zeta|)\zeta_i/|\zeta|$ ,  $i = 1, \dots, n$ , for some radial function  $\hat{v}_\alpha(|\zeta|)$ . If  $u_\alpha$  and  $v_\alpha$  are normalized so that  $\|u_\alpha\|_\alpha = \|v_{\alpha,i}\|_\alpha = 1$ , then they depend smoothly on  $\alpha$ . Moreover we have*

$$|\nabla^{(l)} u_\alpha(x)| + |\nabla^{(l)} (v_{\alpha,i})(x)| \leq C_l e^{-\frac{|x|}{C_l}},$$

*provided  $\alpha$  stays in a fixed bounded set of  $\mathbb{R}$ .*

Before proving the proposition we state a preliminary lemma.

**Lemma 4.3.** *Let  $\tau$  denote the third eigenvalue of  $\bar{J}''(w_0)$ . Then, for  $\alpha \geq 0$ , every eigenfunction corresponding to an eigenvalue  $\lambda \leq \tau/2$  of (52) is either radial and corresponds to the least eigenvalue, or is a radial function times a first-order spherical harmonic (in the angular variable  $\theta = \zeta/|\zeta|$ ) with zero coefficient in  $\zeta'$ , and correspond to the second eigenvalue.*

**Proof.** First of all we notice that, extending evenly across  $\partial\mathbb{R}_+^{n+1}$  any function  $u \in H^1(\mathbb{R}_+^{n+1})$  which is a solution of (52), we obtain a smooth entire solution of

$$-\Delta u + (1 + \alpha)u - pw_0^{p-1}u = \lambda(-\Delta u + (1 + \alpha)u).$$

Next, we decompose  $u$  in spherical harmonics in the angular variable  $\theta$  (we are using only spherical harmonics which are even in  $\zeta_{n+1}$ )

$$u = \sum_{i=0}^{\infty} u_i(|\zeta|)Y_{i,e}(\theta); \quad \zeta \in \mathbb{R}^{n+1}, \theta = \frac{\zeta}{|\zeta|} \in S^n.$$

Here  $Y_{i,e}$  is the  $j$ th eigenfunction of  $-\Delta_{S^n}$  (which is even in  $\zeta_{n+1}$ ), namely it satisfies  $\Delta_{S^n} Y_{i,e} = \lambda_{i,e}^{S^n} Y_{i,e}$ , where we have denoted by  $\lambda_{i,e}^{S^n}$  the  $i$ th eigenvalue of  $-\Delta_{S^n}$  on the space of even functions in  $\zeta_{n+1}$ . In particular, the function  $Y_{0,e}$  is constant on  $S^n$  and correspond to  $\lambda_{1,e}^{S^n} = 0$ , while  $\lambda_{2,e}^{S^n} = n$  has multiplicity  $n$ . The eigenfunctions corresponding to  $\lambda_{2,e}^{S^n}$  are (up to a constant multiple) the restrictions, from  $\mathbb{R}^{n+1}$  to  $S^n$ , of the linear functions in  $\zeta'$ .

The Laplace equation in polar coordinates writes as

$$\Delta_{\mathbb{R}^{n+1}} u = \Delta_r u + \frac{1}{r^2} \Delta_{S^n} u,$$

where  $\Delta_r = \frac{d^2}{dr^2} + \frac{n}{r} \frac{d}{dr}$ . Therefore, if  $u = \sum_{i=0}^{\infty} u_i(|\zeta|)Y_{i,e}(\theta)$  is a solution of (52), then every radial component  $u_i$  satisfies the equation

$$\begin{cases} (1 - \lambda)(-v'' - \frac{n}{r}v' + (1 + \alpha + \frac{\lambda_{i,e}^{S^n}}{r^2})v) - pw_0^{p-1}v = 0 & \text{in } \mathbb{R}_+ \\ v'(0) = 0. \end{cases} \tag{54}$$

We also notice that, since the space of functions  $\{v(r)Y_{i,e}(\theta)\}$  (for a fixed  $i$ ) is sent into itself by the Laplace operator, every Fourier component (in the angular variables) of an eigenfunction of (52) is still an eigenfunction.

We call  $\lambda_{\alpha,i,j}$  the  $j$ th eigenvalue of (54). From Proposition 4.1 it follows that  $\lambda_{0,1,1} = -(p - 1) < 0$  and that  $\lambda_{0,1,j} > \tau$  for  $j \geq 2$ . In fact, a radial eigenfunction of  $\bar{J}''(w_0)$  which is not (a multiple of)  $w_0$  itself must correspond to an eigenvalue greater or equal than  $\tau$ , which is positive. On the other hand, it follows from Proposition 4.1 that  $\lambda_{0,2,1} = 0$ , and also that  $\lambda_{0,2,j} \geq \tau > 0$  for  $j \geq 2$ . Finally, since  $\lambda_{0,i,1} \geq \tau > 0$  for  $i \geq 3$ , we have in addition  $\lambda_{0,i,j} \geq \tau$  for every  $i \geq 3$  and for every  $j \geq 1$ .

After these considerations, we turn to the case  $\alpha > 0$ , for which similar arguments will apply. Solutions of (54) can be found as extrema (minima, for example) of the Rayleigh quotient

$$\frac{\int_{\mathbb{R}_+} r^n [(v')^2 + (1 + \alpha + \frac{\lambda_{i,e}^{S^n}}{r^2})v^2] - p \int_{\mathbb{R}_+} r^n w_0^{p-1} v^2}{\int_{\mathbb{R}_+} r^n [(v')^2 + (1 + \alpha + \frac{\lambda_{i,e}^{S^n}}{r^2})v^2]} \tag{55}$$

from a standard min–max procedure. Using elementary inequalities it is easy to see that the above quotient is non-decreasing in  $\alpha$ . Therefore it follows that  $\lambda_{\alpha,1,j} > 0$  for  $j \geq 2$ , that  $\lambda_{\alpha,2,j} \geq \tau > 0$  for  $j \geq 2$  and that  $\lambda_{\alpha,i,j} \geq \tau$  for every  $i \geq 3$  and for every  $j \geq 1$ . This concludes the proof.  $\square$

**Proof of Proposition 4.2.** The simplicity of  $\eta_\alpha$  can be proved as in [34, Section 3], using spherical rearrangements and the maximum principle. The weak monotonicity in  $\alpha$  of the eigenvalues can be easily shown using the Rayleigh quotient in the space  $H_\alpha$ , as for (55).

The smoothness of  $\alpha \mapsto \eta_\alpha$  and of  $\alpha \mapsto u_\alpha$  can be deduced in the following way. Since the two spaces  $H^1(\mathbb{R}_+^{n+1})$  and  $H_\alpha$  coincide, and since the eigenvalues of an operator do not depend on the choice of the (equivalent) norms, we can consider  $T_\alpha$  acting on  $H^1(\mathbb{R}_+^{n+1})$  endowed with its standard norm (independent of  $\alpha$ ). Having fixed the space, we notice that the explicit expression of  $T_\alpha$  is given by

$$T_\alpha u = [-\Delta + 1]^{-1}(-\Delta u + (1 + \alpha)u - p w_0^{p-1} u). \tag{56}$$

In fact, letting  $T_\alpha u = q \in H^1(\mathbb{R}_+^{n+1})$ , taking the scalar product with any  $v \in H^1(\mathbb{R}_+^{n+1})$  and using (53) we find

$$\int_{\mathbb{R}_+^{n+1}} [(\nabla q \cdot \nabla v) + qv] = \int_{\mathbb{R}_+^{n+1}} [(\nabla u \cdot \nabla v) + (1 + \alpha)uv] - p \int_{\mathbb{R}_+^{n+1}} w_0^{p-1} uv,$$

which leads to (56) by the arbitrariness of  $v$ . It is clear that the operator in (56) depends smoothly on  $\alpha$  and therefore, being  $\eta_\alpha$  simple, the smooth dependence on  $\alpha$  of  $\eta_\alpha$  and  $u_\alpha$  follows.

We now compute the derivative of  $\eta_\alpha$  with respect to  $\alpha$ . The function  $u_\alpha$  satisfies

$$\begin{cases} (1 - \eta_\alpha)(-\Delta u_\alpha + (1 + \alpha)u_\alpha) = p w_0^{p-1} u_\alpha & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial u_\alpha}{\partial \nu} = 0 & \text{on } \partial \mathbb{R}_+^{n+1}. \end{cases} \tag{57}$$

Differentiating with respect to  $\alpha$  the equation  $\|u_\alpha\|_\alpha^2 = 1$ , we find

$$\frac{d}{d\alpha} \|u_\alpha\|_\alpha^2 = 0 \Rightarrow \left( \frac{du_\alpha}{d\alpha}, u_\alpha \right)_\alpha = - \int_{\mathbb{R}_+^{n+1}} u_\alpha^2. \tag{58}$$

On the other hand, differentiating (57), we obtain

$$\begin{cases} -\frac{d\eta_\alpha}{d\alpha}(-\Delta u_\alpha + (1 + \alpha)u_\alpha) + (1 - \eta_\alpha)(-\Delta(\frac{du_\alpha}{d\alpha}) + (1 + \alpha)\frac{du_\alpha}{d\alpha} + u_\alpha) = p w_0^{p-1} \frac{du_\alpha}{d\alpha} & \text{in } \mathbb{R}_+^{n+1}, \\ \frac{\partial}{\partial \nu}(\frac{du_\alpha}{d\alpha}) = 0 & \text{on } \partial \mathbb{R}_+^{n+1}. \end{cases} \tag{59}$$

Multiplying (59) by  $u_\alpha$ , integrating by parts and using (58), one gets

$$\frac{d\eta_\alpha}{d\alpha} = (1 - \eta_\alpha) \int_{\mathbb{R}_+^{n+1}} u_\alpha^2 > 0. \tag{60}$$

Indeed, since  $T_\alpha \leq Id_{H^1(\mathbb{R}_+^{n+1})}$ , every eigenvalue of  $T_\alpha$  is strictly less than 1, and in particular  $(1 - \eta_\alpha) > 0$ . We now consider the second eigenvalue  $\sigma_\alpha$ . For any  $\alpha \geq 0$  it is possible to make a separation of variables, finding eigenfunctions of (52) of the form  $Y_{i,e} \hat{v}_{\alpha,i}$ , where  $Y_{i,e} = \zeta_i / |\zeta|$ ,  $i = 1, \dots, n$ , correspond to  $\lambda_{2,e}^{\sigma_\alpha}$ . Also, from Lemma 4.3 we know that for  $\alpha$  close to 0 (indeed, as long as  $\sigma_\alpha < \tau$ ) every eigenfunction corresponding to  $\sigma_\alpha$  is of this form, for some  $i \in \{1, \dots, n\}$ . Therefore, if we restrict ourselves to the space of functions of the form  $\hat{v}(|\zeta|)\zeta_i/|z|$  for a fixed  $i \in \{1, \dots, n\}$ , the first eigenvalue for (52) becomes simple, so we can reason as before, obtaining smoothness in  $\alpha$  and the strict monotonicity of  $\sigma_\alpha$ .

We prove next that the eigenvalue  $\eta_\alpha$  converges to 1 as  $\alpha \rightarrow +\infty$ . There holds

$$\eta_\alpha = \inf_{u \in H_\alpha} \frac{\int_{\mathbb{R}_+^{n+1}} [|\nabla u|^2 + (1 + \alpha)u^2 - pw_0^{p-1}u^2]}{\int_{\mathbb{R}_+^{n+1}} [|\nabla u|^2 + (1 + \alpha)u^2]}.$$

Fixing any  $\delta > 0$ , it is sufficient to notice that

$$|\nabla u|^2 + ((1 + \alpha) - pw_0^{p-1})u^2 \geq (1 - \delta)[|\nabla u|^2 + (1 + \alpha)u^2] \quad \text{for every } u,$$

provided  $\alpha$  is sufficiently large. This concludes the proof of the claim.

The decay on  $u_\alpha$ ,  $v_{\alpha,i}$  and their derivatives is standard and can be shown as in [34], so we do not give details here.  $\square$

**Remark 4.4.** Proposition 4.2 implies in particular that there is a unique  $\bar{\alpha} > 0$  such that  $\eta_{\bar{\alpha}} = 0$ . Moreover, we have also

$$u_0 = \tilde{C}_0 w_0; \quad v_0^h = \bar{C}_0 \partial_h w_0,$$

for some positive constants  $\tilde{C}_0$  and  $\bar{C}_0$ .

We also need to introduce a variant of the eigenvalue problem (52), for which we impose vanishing of the eigenfunctions outside a certain set. For  $\varepsilon > 0$  and for  $\gamma \in (0, 1)$  we define

$$B_{\varepsilon,\gamma} = \{x \in \mathbb{R}_+^{n+1} : |x| < \varepsilon^{-\gamma}\}, \tag{61}$$

and let

$$H_\varepsilon^1 = \{u \in H^1(B_{\varepsilon,\gamma}) : u(x) = 0 \text{ for } |x| = \varepsilon^{-\gamma}\}.$$

We let  $H_{\alpha,\varepsilon}$  denote the space  $H_\varepsilon^1$  endowed with the norm

$$\|u\|_{\alpha,\varepsilon}^2 = \int_{B_{\varepsilon,\gamma}} [|\nabla u|^2 + (1 + \alpha)u^2]; \quad u \in H_\varepsilon^1,$$

and the corresponding scalar product  $(\cdot, \cdot)_{\alpha, \varepsilon}$ . Similarly, we define  $T_{\alpha, \varepsilon}$  by

$$(T_{\alpha, \varepsilon} u, v)_{\alpha, \varepsilon} = \int_{B_{\varepsilon, \gamma}} [(\nabla u \cdot \nabla v) + (1 + \alpha)uv - pw_0^{p-1}uv]; \quad u, v \in H_{\alpha, \varepsilon}.$$

The operator  $T_{\alpha, \varepsilon}$  satisfies properties analogous to  $T_\alpha$ . We list them in the next proposition, which also gives a comparison between the first eigenvalues and eigenfunctions of  $T_\alpha$  and  $T_{\alpha, \varepsilon}$ .

**Proposition 4.5.** *There exists  $\varepsilon_0 > 0$  such that for  $\varepsilon \in (0, \varepsilon_0)$  the following properties hold true. Let  $\eta_{\alpha, \varepsilon}$ ,  $\sigma_{\alpha, \varepsilon}$  and  $\tau_{\alpha, \varepsilon}$  denote the first three eigenvalues of  $T_{\alpha, \varepsilon}$ . Then  $\eta_{\alpha, \varepsilon}$ ,  $\sigma_{\alpha, \varepsilon}$  and  $\tau_{\alpha, \varepsilon}$  are non-decreasing in  $\alpha$ . For every value of  $\alpha$ ,  $\eta_{\alpha, \varepsilon}$  is simple and  $\partial \eta_{\alpha, \varepsilon} / \partial \alpha > 0$ . For  $\alpha$  sufficiently small,  $\sigma_{\alpha, \varepsilon}$  has multiplicity  $n$  and  $\partial \sigma_{\alpha, \varepsilon} / \partial \alpha > 0$ . The eigenfunction  $u_{\alpha, \varepsilon}$  corresponding to  $\eta_{\alpha, \varepsilon}$  is radial in  $\zeta$  and radially decreasing, while the eigenfunctions corresponding to  $\sigma_{\alpha, \varepsilon}$  are spanned by functions  $v_{\alpha, \varepsilon, i}$  of the form  $v_{\alpha, \varepsilon, i}(\zeta) = \hat{v}_{\alpha, \varepsilon}(|\zeta|)\zeta_i / |\zeta|$ ,  $i = 1, \dots, n$ , for some radial function  $\hat{v}_{\alpha, \varepsilon}(|\zeta|)$ . The eigenvector  $u_{\alpha, \varepsilon}$  (respectively  $v_{\alpha, \varepsilon, i}$ ), normalized with  $\|u_{\alpha, \varepsilon}\|_{H_{\alpha, \varepsilon}} = 1$  (respectively  $\|v_{\alpha, \varepsilon, i}\|_{H_{\alpha, \varepsilon, i}} = 1$ ) corresponding to  $\eta_{\alpha, \varepsilon}$  (respectively  $\sigma_{\alpha, \varepsilon}$  for  $\alpha$  small) depend smoothly on  $\alpha$ . Moreover for some fixed  $C_1 > 0$  there holds*

$$|\nabla^{(l)} u_{\alpha, \varepsilon}(\zeta)| + |\nabla^{(l)} v_{\alpha, \varepsilon, i}(\zeta)| \leq C_1 e^{-\frac{|\zeta|}{C_1}}, \quad \text{for } i = 0, \dots, n; \tag{62}$$

$$|\eta_\alpha - \eta_{\alpha, \varepsilon}| + \|u_\alpha - u_{\alpha, \varepsilon}\|_{H^1(\mathbb{R}_+^{n+1})} + |\sigma_\alpha - \sigma_{\alpha, \varepsilon}| + \|v_{\alpha, i} - v_{\alpha, \varepsilon, i}\|_{H^1(\mathbb{R}_+^{n+1})} \leq C e^{-\frac{\varepsilon^{-\gamma}}{C}}, \tag{63}$$

provided  $\alpha$  stays in a fixed bounded set of  $\mathbb{R}$ . The functions  $u_{\alpha, \varepsilon}$  and  $v_{\alpha, \varepsilon, i}$  in this formula have been set identically 0 outside  $B_{\varepsilon, \gamma}$ . Furthermore,  $\tau_{\alpha, \varepsilon} \geq \tau_\alpha \geq \tau$  for every value of  $\alpha$  and  $\varepsilon$ .

The proof is very similar that of Proposition 2.3 in [35], and hence we omit it here. It is still based on some elementary inequalities and on the Rayleigh quotient. The quantitative estimates in (63) can be deduced using cutoff functions and the Green’s representation formula for the operator  $-\Delta + (1 + \alpha)$  in  $\mathbb{R}_+^{n+1}$ .

As a consequence of this proposition (taking  $\alpha = 0$ ) we obtain that, if (for  $\varepsilon$  small)  $u \in H_\varepsilon^1$  has no Fourier components (in  $\theta$ ) with indices less or equal to  $n$ , then  $(T_{0, \varepsilon} u, u)_{0, \varepsilon} \geq \frac{\tau}{2}(u, u)_{0, \varepsilon}$ . Equivalently, there holds

$$p \int_{B_{\varepsilon, \gamma}} w_0^{p-1}(|\zeta|)u^2 \leq \left(1 - \frac{\tau}{2}\right) \int_{B_{\varepsilon, \gamma}} (-\Delta u + u)u d\zeta$$

$$\text{for any } u = \sum_{i=n+1}^\infty u_j(|\zeta|)Y_{j, e}(\theta), \quad u \in H_\varepsilon^1. \tag{64}$$

#### 4.2. A model for $J_\varepsilon''(u_{1, \varepsilon})$

In this subsection, using the analysis of the previous one, we construct a model operator which, up to some extent, mimics the properties of  $J_\varepsilon''(u_{1, \varepsilon})$ , and for which we can give an

explicit description of the spectrum. Although the related construction in [33] is a particular case of the one made here, the general spirit is quite different, and is more geometric in nature.

First of all, we choose an orthonormal frame  $(E_i)_i$  as before, and we define a metric  $\hat{g}$  on  $NK$  as follows. For  $v \in NK$ , a tangent vector  $V \in T_v NK$  can be identified with the velocity of a curve  $v(t)$  in  $NK$  which is equal to  $v$  at time  $t = 0$ . The same holds true for another tangent vector  $W \in T_v NK$ . Then the metric  $\hat{g}$  on  $NK$  is defined on the couple  $(V, W)$  in the following way (see [16, p. 79])

$$\hat{g}(V, W) = \bar{g}(\pi_* V, \pi_* W) + \left\langle \frac{D^N v}{dt} \Big|_{t=0}, \frac{D^N w}{dt} \Big|_{t=0} \right\rangle_N.$$

In this formula  $\pi$  denotes the natural projection from  $NK$  onto  $K$ , and  $\frac{D^N v}{dt}$  denotes the (normal) covariant derivative of the vector field  $v(t)$  along the curve  $\pi v(t)$ . In the notation of Section 2.2 we have that, if  $v(t) = v^j(t)E_j(t)$ , then

$$\frac{D^N v}{dt} = \frac{dv^j(t)}{dt} E_j(t) + v^j(t) \beta_j^l \left( \pi_* \frac{dv(t)}{dt} \right) E_l.$$

Therefore, if we choose a system of coordinates  $\bar{y}$  on  $K$  and then a system of coordinates on  $NK$  defined by

$$(\bar{y}, \bar{\zeta}) \in \mathbb{R}^k \times \mathbb{R}^n \mapsto \bar{\zeta}^j E_j(\bar{y}),$$

we get that

$$\hat{g}_{\bar{a}\bar{b}}(\bar{y}, \bar{\zeta}) = \bar{g}_{\bar{a}\bar{b}}(\bar{y}) + \bar{\zeta}^i \bar{\zeta}^j \langle \nabla_{\partial_{\bar{a}}}^N E_i, \nabla_{\partial_{\bar{b}}}^N E_j \rangle_N = \bar{g}_{\bar{a}\bar{b}}(\bar{y}) + \bar{\zeta}^i \bar{\zeta}^j \beta_i^l(\partial_{\bar{a}}) \beta_j^l(\partial_{\bar{b}}),$$

and

$$\hat{g}_{\bar{a}\bar{i}}(\bar{y}, \bar{\zeta}) = \bar{\zeta}^j \beta_j^i(\partial_{\bar{a}}); \quad \hat{g}_{\bar{i}\bar{j}}(\bar{y}, \bar{\zeta}) = \delta_{\bar{i}\bar{j}},$$

where we have set  $\partial_{\bar{i}} = \partial/\partial \bar{\zeta}_i$ . We notice also that the following co-area type formula holds, for any smooth compactly supported function  $f : NK \rightarrow \mathbb{R}$

$$\int_{NK} f dV_{\hat{g}} = \int_K \left( \int_{N_{\bar{y}}K} f(\bar{\zeta}) d\bar{\zeta} \right) dV_{\bar{g}}(\bar{y}). \tag{65}$$

This follows immediately from the fact that  $\det \hat{g} = \det \bar{g}$ , which in turn can be verified by expressing  $\hat{g}$  as a product of three matrices like

$$\begin{pmatrix} Id & \zeta\beta \\ 0 & Id \end{pmatrix} \begin{pmatrix} \bar{g} & 0 \\ 0 & Id \end{pmatrix} \begin{pmatrix} Id & 0 \\ \zeta\beta & Id \end{pmatrix},$$

the first and the third having determinant equal to 1.

Having defined the metric  $\hat{g}$ , we express the Laplacian of a function  $u$  defined on  $NK$  with respect to this metric. In Fermi coordinates centered at some point  $q \in K$ , using (16), (17) and (19), it turns out that (for  $\bar{y} = 0$ )

$$\Delta_{\hat{g}}u = \partial_{\bar{a}\bar{a}}^2u + \partial_{\bar{i}\bar{i}}^2u. \tag{66}$$

Next we define the set  $S_\varepsilon$  as

$$S_\varepsilon = \{(v, \zeta_{n+1}) \in NK_\varepsilon \times \mathbb{R}_+ : (|v|^2 + \zeta_{n+1}^2)^{\frac{1}{2}} \leq \varepsilon^{-\nu}\}, \quad \mathbb{R}_+ = \{\zeta_{n+1} : \zeta_{n+1} > 0\},$$

where  $NK_\varepsilon$  stands for the normal bundle of  $K_\varepsilon$  (in  $\Omega_\varepsilon$ ). We next endow  $S_\varepsilon$  with a natural metric, inherited by  $\hat{g}$  through a scaling. If  $R_\varepsilon$  denotes the dilation  $x \mapsto \varepsilon x$  in  $\mathbb{R}^N$  (extended naturally to its subsets), we define a metric  $\tilde{g}_\varepsilon$  on  $S_\varepsilon$  by

$$\tilde{g}_\varepsilon = \frac{1}{\varepsilon^2}[(R_\varepsilon)_*\hat{g}] \otimes d\zeta_{n+1}^2.$$

In particular, choosing coordinates  $(y, \zeta')$  on  $NK_\varepsilon$  via the scaling  $(\bar{y}, \bar{\zeta}) = \varepsilon(y, \zeta')$ , one easily checks that the components of  $\tilde{g}_\varepsilon$  are given by

$$\begin{aligned} (\tilde{g}_\varepsilon)_{ab}(y, v) &= (\bar{g})_{\bar{a}\bar{b}}(\varepsilon y) + \varepsilon^2 v^i v^j \beta_i^l(\partial_{\bar{a}})(\varepsilon y) \beta_j^l(\partial_{\bar{b}})(\varepsilon y), \\ (\tilde{g}_\varepsilon)_{ai}(y, v) &= \varepsilon v^j \beta_j^i(\partial_{\bar{a}})(\varepsilon y); \quad (\tilde{g}_\varepsilon)_{ij}(\bar{y}, v) = \delta_{ij}, \end{aligned}$$

and also

$$(\tilde{g}_\varepsilon)_{NN} \equiv 1; \quad (\tilde{g}_\varepsilon)_{N\alpha} \equiv 0.$$

Therefore, if  $u$  is a smooth function in  $S_\varepsilon$ , it follows that in the above coordinates  $(y, \zeta', \zeta_{n+1})$  (at  $y = 0$ )

$$\Delta_{\tilde{g}_\varepsilon}u = \partial_{aa}^2u + \partial_{ii}^2u + \partial_{\zeta_{n+1}\zeta_{n+1}}^2u. \tag{67}$$

In the following, to emphasize a slow dependence of a function  $u$  in the variables  $y$ , we will often write  $u(y, \zeta) = u(\varepsilon y, \zeta)$  (where, we recall,  $\zeta = (\zeta', \zeta_{n+1})$ ), identifying with an abuse of notation the variable  $y$  parameterizing  $K_\varepsilon$  with  $\bar{y}$ , parameterizing  $K$ . In this case we have that (at the origin of the Fermi coordinates)

$$\Delta_{\tilde{g}_\varepsilon}u = \varepsilon^2 \partial_{\bar{a}\bar{a}}^2u + \partial_{\bar{i}\bar{i}}^2u + \partial_{\zeta_{n+1}\zeta_{n+1}}^2u. \tag{68}$$

For later purposes, we evaluate  $\Delta_{\tilde{g}_\varepsilon}$  on functions with a special structure. In particular, if we deal with a function  $u$  of the form  $u(\bar{y}, \zeta) = \phi(\bar{y})v(|\zeta|)$ , we have that

$$\Delta_{\tilde{g}_\varepsilon}u = \varepsilon^2(\Delta_K\phi(\bar{y}))v(|\zeta|) + \phi(\bar{y})\Delta_\zeta v, \tag{69}$$

and if instead  $u(\bar{y}, \zeta) = v(|\zeta|)\psi^h(\bar{y})\zeta_h/|\zeta|$  for some smooth normal section  $\psi = \psi^h E_h$ , then we find

$$\Delta_{\tilde{g}_\varepsilon}u = \varepsilon^2(\Delta_K^N\psi)^h(\bar{y})\frac{\zeta_h}{|\zeta|}v(|\zeta|) + \psi^h(\bar{y})\Delta_\zeta\left(v(|\zeta|)\frac{\zeta_h}{|\zeta|}\right). \tag{70}$$

Now we introduce the function space  $H_{S_\varepsilon}$  defined as the family of functions in  $H^1(S_\varepsilon)$  which vanish on  $\{|v|^2 + \zeta_{n+1}^2 = \varepsilon^{-2\gamma}\}$ , endowed with the scalar product

$$(u, v)_{H_{S_\varepsilon}} = \int_{S_\varepsilon} (\nabla_{\tilde{g}_\varepsilon} u \cdot \nabla_{\tilde{g}_\varepsilon} v + uv) dV_{\tilde{g}_\varepsilon}. \tag{71}$$

We consider next the operator  $T_{S_\varepsilon} : H_{S_\varepsilon} \rightarrow H_{S_\varepsilon}$  defined by duality as

$$(T_{S_\varepsilon} u, v)_{H_{S_\varepsilon}} = \int_{S_\varepsilon} (\nabla_{\tilde{g}_\varepsilon} u \cdot \nabla_{\tilde{g}_\varepsilon} v + uv - pw_0^{p-1}(|\zeta|)uv) dV_{\tilde{g}_\varepsilon}, \tag{72}$$

for arbitrary  $u, v \in H_{S_\varepsilon}$ . Our goal is to characterize some of the eigenvalues of  $T_{S_\varepsilon}$ , with the corresponding eigenfunctions.

For simplicity, if  $u_{\alpha,\varepsilon}, v_{\alpha,\varepsilon,i}, \eta_{\alpha,\varepsilon}$  and  $\sigma_{\alpha,\varepsilon}$  are given by Proposition 4.5, recalling our notation from Section 2.2, we also set

$$u_{j,\varepsilon} = u_{\varepsilon^2\rho_{j,\varepsilon}}; \quad v_{l,\varepsilon,i} = v_{\varepsilon^2\omega_{l,\varepsilon,i}}; \quad \eta_{j,\varepsilon} = \eta_{\varepsilon^2\rho_{j,\varepsilon}}; \quad \sigma_{l,\varepsilon} = \sigma_{\varepsilon^2\omega_{l,\varepsilon,i}}. \tag{73}$$

We also assume that these functions are normalized so that

$$\begin{cases} \|u_{j,\varepsilon}\|_{\varepsilon^2\rho_{j,\varepsilon}}^2 = \int_{B_{\gamma,\varepsilon}} (|\nabla u_{j,\varepsilon}|^2 + (1 + \varepsilon^2\rho_j)u_{j,\varepsilon}^2) = 1; \\ \|v_{l,\varepsilon,i}\|_{\varepsilon^2\omega_{l,\varepsilon,i}}^2 = \int_{B_{\gamma,\varepsilon}} (|\nabla v_{l,\varepsilon,i}|^2 + (1 + \varepsilon^2\omega_l)v_{l,\varepsilon,i}^2) = 1. \end{cases} \tag{74}$$

After these preliminaries, we can state our result.

**Proposition 4.6.** *Let  $\varepsilon_0, \varepsilon$  be as in Proposition 4.5. Let  $\lambda < \tau/4$  be an eigenvalue of  $T_{S_\varepsilon}$ . Then either  $\lambda = \eta_{j,\varepsilon}$  for some  $j$ , or  $\lambda = \sigma_{l,\varepsilon}$  for some index  $l$ . The corresponding eigenfunctions  $u$  are of the form*

$$u(y, \zeta) = \sum_{\{j: \eta_{j,\varepsilon}=\lambda\}} \alpha_j \phi_j(\varepsilon y) u_{j,\varepsilon}(\zeta) + \sum_{\{l: \sigma_{l,\varepsilon}=\lambda\}} \beta_l \varphi_l^i(\varepsilon y) v_{l,\varepsilon,i}(\zeta), \tag{75}$$

where  $(y, \zeta)$  denote the above coordinates on  $S_\varepsilon$ , and where  $(\alpha_j)_j, (\beta_l)_l$  are arbitrary constants. Vice versa, every function of the form (75) is an eigenfunction of  $T_{S_\varepsilon}$  with eigenvalue  $\lambda$ . In particular the eigenvalues of  $T_{S_\varepsilon}$  which are smaller than  $\tau/4$  coincide with the numbers  $(\eta_{j,\varepsilon})_j$  or  $(\sigma_{l,\varepsilon})_l$  which are smaller than  $\tau/4$ .

**Proof.** The proof is based on separation of variables and the spectral analysis of Proposition 4.5. Integrating by parts, one can check that the eigenfunction  $u$  of  $T_{S_\varepsilon}$  satisfies the following equation

$$\begin{cases} (1 - \lambda)(-\Delta_{\tilde{g}_\varepsilon} u + u) - pw_0^{p-1}(\zeta)u = 0 & \text{in } S_\varepsilon, \\ \frac{\partial u}{\partial \zeta_{n+1}} = 0 & \text{on } \{\zeta_{n+1} = 0\}. \end{cases} \tag{76}$$



As before, we can extend  $u$  evenly in  $\zeta_{n+1}$ , to obtain a smooth solution of the differential equation in (76) in the set  $\{(v, \zeta_{n+1}) \in NK_\varepsilon \times \mathbb{R} : (|v|^2 + \zeta_{n+1}^2)^{1/2} \leq \varepsilon^{-\gamma}\}$ . Hence, fixing  $y \in K_\varepsilon$ , we can use Fourier decomposition in the angular variable of  $\zeta$ , and we can write

$$u(y, \zeta) = \sum_{l=0}^{\infty} u_l(y, |\zeta|)Y_{l,e}(\theta),$$

where  $\theta = \zeta/|\zeta| \in S^n$ , and where  $Y_{l,e}$  is the  $l$ th spherical harmonic function which is even in  $\zeta_{n+1}$ . We now decompose  $u$  further in a convenient way as

$$u = \underline{u}_0 + \underline{u}_1 + \underline{u}_2, \tag{77}$$

where

$$\begin{aligned} \underline{u}_0 &= \frac{1}{\sqrt{|S^n|}}u_0(y, |\zeta|); & \underline{u}_1 &= \sum_{l=1, \dots, n} u_l(y, |\zeta|)Y_{l,e}(\theta); \\ \underline{u}_2 &= \sum_{l \geq n+1} u_l(y, |\zeta|)Y_{l,e}(\theta). \end{aligned}$$

Integrating by parts, the last formula, together with (65), (69) and (70) (recall that  $Y_{l,e}$  for  $l = 1, \dots, n$  are linear combinations of  $\zeta_h/|\zeta|$  on  $S^n$ ,  $h = 1, \dots, n$ ) easily implies that  $(\underline{u}_i, \underline{u}_j)_{H_{S_\varepsilon}} = 0$  for  $i \neq j$  and that  $(T_{S_\varepsilon} \underline{u}_i, \underline{u}_j)_{H_{S_\varepsilon}} = 0$  for  $i \neq j$ , namely that  $T_{S_\varepsilon}$  diagonalizes with respect to the above decomposition (77).

We begin by considering the action of  $T_{S_\varepsilon}$  on  $\underline{u}_0$ . Using a Fourier decomposition of  $\underline{u}_0(y, |\zeta|)$  through the eigenfunctions  $(\phi_j)_j$  of the Laplace–Beltrami operator on  $K_\varepsilon$  we set

$$\underline{u}_0(y, |\zeta|) = \sum_{j=0}^{\infty} \phi_j(\varepsilon y) \tilde{u}_j(|\zeta|).$$

By (69) we get immediately that for any  $j$

$$\Delta_{\tilde{g}_\varepsilon}(\phi_j(\varepsilon y) \tilde{u}_j(|\zeta|)) = (\varepsilon^2 \Delta_{\tilde{g}} + \Delta_\zeta)(\phi_j(\varepsilon y) \tilde{u}_j(|\zeta|)) = (\Delta_\zeta - \varepsilon^2 \rho_j) \phi_j(\varepsilon y) \tilde{u}_j(|\zeta|).$$

As a consequence we find that  $\underline{u}_0 \in H_\varepsilon^1$  satisfies the following partial differential equation in  $B_{\varepsilon, \gamma}$ , with Neumann boundary conditions on  $\{\zeta_{n+1} = 0\}$

$$\begin{aligned} -\Delta_{\tilde{g}_\varepsilon} \underline{u}_0 + \underline{u}_0 - pw_0^{p-1}(|\zeta|) \underline{u}_0 \\ = \sum_{j=0}^{\infty} \phi_j(\varepsilon y) (-\Delta_\zeta \tilde{u}_j(|\zeta|) + (1 + \varepsilon^2 \rho_j) \tilde{u}_j(|\zeta|) - pw_0^{p-1}(|\zeta|) \tilde{u}_j(|\zeta|)). \end{aligned}$$

From this formula it follows that if  $T_{S_\varepsilon} u = \lambda u$  for some  $\lambda$ , then by the orthogonality to  $\underline{u}_1, \underline{u}_2$  we have also  $T_{S_\varepsilon} \underline{u}_0 = \lambda \underline{u}_0$ , and each of the components  $\tilde{u}_j$  (which are radial in  $\zeta$ ) satisfies the

eigenvalue equation  $T_{\varepsilon^2 \rho_j, \varepsilon} \tilde{u}_j = \lambda \tilde{u}_j$  in  $H_{\varepsilon^2 \rho_j, \varepsilon}$  with the same value of  $\lambda$ , where we are using the notation of Section 4.1. Using the same terminology, we can further decompose  $\tilde{u}_j$  as

$$\tilde{u}_j(|\zeta|) = \alpha_j u_{j, \varepsilon} + \bar{u}_{j, \varepsilon} \quad \text{with } \alpha_j \in \mathbb{R} \text{ and with } (u_{j, \varepsilon}, \bar{u}_{j, \varepsilon})_{\varepsilon^2 \rho_j, \varepsilon} = 0.$$

From the spectral analysis carried out in the previous subsection it follows that if  $\lambda < \tau/4$  (and  $\varepsilon$  is sufficiently small), then  $\bar{u}_{j, \varepsilon} = 0$  for every  $j$ , and  $\lambda = \eta_{j, \varepsilon}$  for some set of indices  $j$ .

We now turn to the evaluation of  $T_{S_\varepsilon}$  on  $\underline{u}_1$ . Similarly as before, expanding with respect to the eigenfunctions of the normal Laplacian we can decompose  $\underline{u}_1$  in the following way

$$\underline{u}_1(y, \zeta) = \sum_{l \geq 0} \sum_{i=1}^n \tilde{v}_l(|\zeta|) \varphi_{l,i}(\varepsilon y) \frac{\zeta_i}{|\zeta|},$$

and from (70) we deduce that

$$\begin{aligned} \Delta_{\tilde{g}_\varepsilon} \left( \sum_{i=1}^n \tilde{v}_l(|\zeta|) \varphi_{l,i}(\varepsilon y) \frac{\zeta_i}{|\zeta|} \right) &= \sum_{i=1}^n (\varepsilon^2 \Delta_K^N \varphi_l)^i \frac{\zeta_i}{|\zeta|} v_l(|\zeta|) + \sum_{i=1}^n \varphi_{l,i}(\varepsilon y) \Delta_\zeta \left( \tilde{v}_l(|\zeta|) \frac{\zeta_i}{|\zeta|} \right) \\ &= (\Delta_\zeta - \varepsilon^2 \omega_l) \left( \sum_{i=1}^n \tilde{v}_l(|\zeta|) \varphi_{l,i}(\varepsilon y) \frac{\zeta_i}{|\zeta|} \right). \end{aligned}$$

As a consequence we find that also

$$\begin{aligned} -\Delta_{\tilde{g}_\varepsilon} \underline{u}_1 + \underline{u}_1 - p w_0^{p-1}(|\zeta|) \underline{u}_1 \\ = \sum_{l \geq 0} \sum_{i=1}^n \varphi_{l,i}(\varepsilon y) \left[ -\Delta_\zeta \left( \tilde{v}_l(|\zeta|) \frac{\zeta_i}{|\zeta|} \right) + (1 + \varepsilon^2 \omega_l) \tilde{v}_l(|\zeta|) \frac{\zeta_i}{|\zeta|} - p w_0^{p-1}(|\zeta|) \tilde{v}_l(|\zeta|) \frac{\zeta_i}{|\zeta|} \right]. \end{aligned}$$

Hence, by the spectral analysis of the previous subsection, reasoning as for  $\underline{u}_0$  we deduce that if  $\underline{u}_1$  satisfies  $T_{S_\varepsilon} \underline{u}_1 = \lambda \underline{u}_1$  with  $\lambda < \tau/4$ , then  $\tilde{v}_l(|\zeta|) \zeta_i / |\zeta| = v_{l, \varepsilon, i}$ , and hence it follows that  $\lambda = \sigma_{l, \varepsilon}$  for some set of indices  $l$ .

Finally, we turn to  $\underline{u}_2$ . Proceeding as for the definition of the metric  $\hat{g}$  (and using the same notation), we can introduce a bilinear form  $\mathfrak{g}$  (semi-positive definite) on  $TNK$  defined by

$$\mathfrak{g}(V, W) = \left\langle \left. \frac{D^N v}{dt} \right|_{t=0}, \left. \frac{D^N w}{dt} \right|_{t=0} \right\rangle_N.$$

Using again a scaling in  $\varepsilon$ , we can also introduce the following bilinear form on  $S_\varepsilon$

$$\mathfrak{g}_\varepsilon = \frac{1}{\varepsilon^2} (R_\varepsilon)_* \mathfrak{g} \otimes d\zeta_{n+1}^2.$$

The components of this form in the above coordinates  $(y, \zeta)$  are given by

$$\begin{aligned} (\mathfrak{g}_\varepsilon)_{ab}(y, v) &= \varepsilon^2 v^i v^j \beta_i^l(\partial_{\bar{a}})(\varepsilon y) \beta_j^l(\partial_{\bar{b}})(\varepsilon y); & (\mathfrak{g}_\varepsilon)_{ai}(y, v) &= \varepsilon v^j \beta_j^i(\partial_{\bar{a}})(\varepsilon y); \\ (\mathfrak{g}_\varepsilon)_{ij}(\bar{y}, v) &= \delta_{ij}; & (\mathfrak{g}_\varepsilon)_{NN} &\equiv 1; & (\mathfrak{g}_\varepsilon)_{N\alpha} &\equiv 0. \end{aligned}$$

We then define by duality the operator  $\mathfrak{T}_\varepsilon$  through the formula

$$(\mathfrak{T}_\varepsilon u, u)_{H_{S_\varepsilon}} := \int_{S_\varepsilon} [\mathfrak{g}_\varepsilon(\nabla_{\tilde{g}_\varepsilon} u, \nabla_{\tilde{g}_\varepsilon} u) + u^2 - pw_0^{p-1}(|\zeta|)u^2] dV_{\tilde{g}_\varepsilon}.$$

Moreover, computing the pointwise action of  $\mathfrak{T}_\varepsilon$ , integrating by parts, reasoning as for the derivation of (68), and using (65), one finds that

$$(\mathfrak{T}_\varepsilon u, u)_{H_{S_\varepsilon}} = \int_{K_\varepsilon} \left[ \int_{S_{y,\varepsilon}} (-u\Delta_\zeta u + u^2 - pw_0(|\zeta|)^{p-1}) d\zeta \right] dV_{\tilde{g}_\varepsilon}(y), \quad u \in H_{S_\varepsilon}, \quad (78)$$

where we have set  $\tilde{g}_\varepsilon = \frac{1}{\varepsilon^2}(R_\varepsilon)_*\bar{g}$  and  $S_{y,\varepsilon} = \{(v, \zeta_{n+1}) \in N_y K_\varepsilon \times \mathbb{R}_+ : (|v|^2 + \zeta_{n+1}^2)^{\frac{1}{2}} \leq \varepsilon^{-\nu}\}$ .

Hence, using (65) (with the scaled metric  $\tilde{g}_\varepsilon$ ), (64) with  $u = \underline{u}_2$  and (78) we find

$$\begin{aligned} p \int_{S_\varepsilon} w_0^{p-1} \underline{u}_2^2 dV_{\tilde{g}_\varepsilon} &= p \int_{K_\varepsilon} \left( \int_{S_{y,\varepsilon}} w_0^{p-1} \underline{u}_2^2 \right) dV_{\tilde{g}_\varepsilon}(y) \\ &\leq \left(1 - \frac{\tau}{2}\right) \int_{K_\varepsilon} \left[ \int_{S_{y,\varepsilon}} (-\underline{u}_2 \Delta_\zeta \underline{u}_2 + \underline{u}_2^2) \right] dV_{\tilde{g}_\varepsilon}(y). \end{aligned}$$

Since  $\tau < 1$  (being an eigenvalue of  $\bar{J}''(w_0) \leq Id_{H^1(\mathbb{R}_+^{n+1})}$ ), we deduce that

$$\begin{aligned} (T_{S_\varepsilon} u, u)_{H_{S_\varepsilon}} &= (\mathfrak{T}_\varepsilon u, u)_{H_{S_\varepsilon}} + \int_{S_\varepsilon} [(\hat{g}_\varepsilon - \mathfrak{g}_\varepsilon)(\nabla_{\tilde{g}_\varepsilon} u, \nabla_{\tilde{g}_\varepsilon} u) + u^2] dV_{\tilde{g}_\varepsilon} \\ &\geq \frac{\tau}{2} \int_{S_\varepsilon} [\mathfrak{g}_\varepsilon(\nabla_{\tilde{g}_\varepsilon} u, \nabla_{\tilde{g}_\varepsilon} u) + u^2] dV_{\tilde{g}_\varepsilon} + \int_{S_\varepsilon} [(\hat{g}_\varepsilon - \mathfrak{g}_\varepsilon)(\nabla_{\tilde{g}_\varepsilon} u, \nabla_{\tilde{g}_\varepsilon} u) + u^2] dV_{\tilde{g}_\varepsilon} \\ &\geq \frac{\tau}{2} \|u\|_{H_{S_\varepsilon}}^2. \end{aligned}$$

It follows that there are no eigenvectors of the form  $\underline{u}_2$  corresponding to eigenvalues smaller than  $\tau/2$ . This concludes the proof.  $\square$

**Remark 4.7.** For later purposes, it is convenient to consider a splitting of the functions in  $H_{S_\varepsilon}$  which is slightly different from the one in (77). If  $\underline{u}_0, \underline{u}_1$  and  $\underline{u}_2$  are as above, with

$$\underline{u}_0 = \sum_{j \geq 0} \phi_j(\varepsilon y) \tilde{u}_j(|\zeta|); \quad \underline{u}_1 = \sum_{l \geq 0} \sum_{i=1}^n \tilde{v}_l(|\zeta|) \varphi_{l,i}(\varepsilon y) \frac{\zeta_i}{|\zeta|},$$

for some real sequences  $(\alpha_j)_j, (\beta_l)_l$ , we can write

$$\tilde{u}_j(|\zeta|) = \alpha_j u_{j,\varepsilon}(|\zeta|) + \bar{u}_{j,\varepsilon}(|\zeta|), \quad \text{with } (u_{j,\varepsilon}, \bar{u}_{j,\varepsilon})_{\varepsilon^2 \rho_j, \varepsilon} = 0;$$

$$\tilde{v}_l(|\zeta|) \frac{\zeta_i}{|\zeta|} = \beta_l v_{l,\varepsilon,i}(\zeta) + \bar{v}_{l,\varepsilon}(|\zeta|) \frac{\zeta_i}{|\zeta|} := \beta_l v_{l,\varepsilon,i}(\zeta) + \bar{v}_{l,\varepsilon,i}(\zeta), \quad \text{with } (v_{l,\varepsilon,i}, \bar{v}_{l,\varepsilon,i})_{\varepsilon^2 \omega_l, \varepsilon} = 0.$$

Now we set  $u = u_0 + u_1 + u_2$ , where

$$u_0 = \sum_{j=0}^{\infty} \alpha_j u_{j,\varepsilon}(|\zeta|) \phi_j(\varepsilon y); \quad u_1 = \sum_{l=0}^{\infty} \beta_l v_{l,\varepsilon,i}(\zeta) \phi_l^i(\varepsilon y);$$

$$u_2 = \sum_{j=0}^{\infty} \bar{u}_{j,\varepsilon}(|\zeta|) \phi_j(\varepsilon y) + \sum_{l=0}^{\infty} \bar{v}_{l,\varepsilon,i}(\zeta) \phi_l^i(\varepsilon y) + \underline{u}_2.$$

Then by (74) one can check that  $(u_i, u_j)_{H_{S_\varepsilon}} = 0$  for  $i \neq j$ , and that

$$\|u\|_{H_{S_\varepsilon}}^2 = \|u_0\|_{H_{S_\varepsilon}}^2 + \|u_1\|_{H_{S_\varepsilon}}^2 + \|u_2\|_{H_{S_\varepsilon}}^2 = \frac{1}{\varepsilon^k} \sum_{j=0}^{\infty} \alpha_j^2 + \frac{1}{\varepsilon^k} \sum_{l=0}^{\infty} \beta_l^2 + \|u_2\|_{H_{S_\varepsilon}}^2; \quad (79)$$

$$(T_{S_\varepsilon} u, u)_{H_{S_\varepsilon}} = \sum_{j=0}^{\infty} \eta_{j,\varepsilon} \alpha_j^2 + \sum_{l=0}^{\infty} \sigma_{l,\varepsilon} \beta_l^2 + (T_{S_\varepsilon} u_2, u_2)_{H_{S_\varepsilon}};$$

$$(T_{S_\varepsilon} u_2, u_2)_{H_{S_\varepsilon}} \geq C \|u_2\|_{H_{S_\varepsilon}}^2, \quad (80)$$

for some fixed positive constant  $C$ .

From the last proposition we deduce the following corollary, regarding the Morse index of  $T_{S_\varepsilon}$ .

**Corollary 4.8.** *Let  $\gamma \in (0, 1)$ , and let  $T_{S_\varepsilon} : H_{S_\varepsilon} \rightarrow H_{S_\varepsilon}$  be defined as before. Then, as  $\varepsilon$  tends to zero, the Morse index of  $T_{S_\varepsilon}$  satisfies the estimate*

$$M.I.(T_{S_\varepsilon}) \simeq \left( \frac{\bar{\alpha}}{C_k} \right)^{\frac{k}{2}} \text{Vol}(K) \varepsilon^{-k},$$

where  $\bar{\alpha}$  is the unique real number for which  $\eta_{\bar{\alpha}} = 0$  (see Remark 4.4).

**Proof.** From Proposition 4.6 we have that the Morse index of  $T_{S_\varepsilon}$  is equal to the number of negative  $\eta_{j,\varepsilon}$ 's. By the estimate in (63), this number is asymptotic to the number of  $j$ 's for which  $\eta_{\varepsilon^2 \rho_j}$  is negative. Therefore it is sufficient to count the number of eigenvalues  $\rho_j$  for which  $\varepsilon^2 \rho_j$  is less than  $\bar{\alpha}$ . By the Weyl's asymptotic formula, see [27], we have that  $\rho_j \simeq C_k (j/\text{Vol}(K))^{k/2}$  so the conclusion follows immediately.  $\square$

### 5. Accurate analysis of the linearized operator

In this section we first compare  $J''_\varepsilon(u_{I,\varepsilon})$  to the model operator introduced in the previous one. A naive direct comparison will give errors of order  $\varepsilon$ , see Lemma 5.1 and Corollary 5.3, but sometimes we will need estimates of order  $\varepsilon^2$ . Therefore we will expand at a higher order the eigenvalues (of the linearized operator at  $u_{I,\varepsilon}$ ) close to zero with the corresponding eigenfunctions, to get sufficient control on the errors. Finally, using these expansions, we will define a suitable decomposition of the functional space for which the linearized operator is almost *diagonal*.

#### 5.1. Comparison of $J''_\varepsilon(u_{I,\varepsilon})$ and $T_{S_\varepsilon}$

We define first a bijection  $\tilde{\Upsilon}_\varepsilon$  from  $S_\varepsilon$  into a neighborhood of  $K_\varepsilon$  in  $\Omega_\varepsilon$  in the following way. Given the section  $\Phi = \Phi_0 + \varepsilon\Phi_1 + \dots + \varepsilon^{l-2}\Phi_{l-2}$  in  $NK$  constructed in Section 3, for any  $(v, \zeta_{n+1}) \in S_\varepsilon$ ,  $v \in N_y K_\varepsilon$ ,  $\zeta_{n+1} \in \mathbb{R}_+$ , we set

$$\tilde{\Upsilon}_\varepsilon(v, \zeta_{n+1}) = \exp_y^{\partial\Omega_\varepsilon}(v + \Phi(\varepsilon y)) + \zeta_{n+1}v(\exp_y^{\partial\Omega_\varepsilon}(v + \Phi(\varepsilon y))).$$

Then we define the set  $\Sigma_\varepsilon \subseteq \Omega_\varepsilon$  to be

$$\Sigma_\varepsilon = \tilde{\Upsilon}_\varepsilon(S_\varepsilon),$$

endowed with the standard Euclidean metric induced from  $\mathbb{R}^N$ . For  $u \in H_{S_\varepsilon}$ , we define the function  $\tilde{u} : \Sigma_\varepsilon \rightarrow \mathbb{R}$  by

$$\tilde{u}(z) = u(\tilde{\Upsilon}_\varepsilon^{-1}(z)), \quad z \in \Sigma_\varepsilon,$$

and letting  $\Lambda_\varepsilon$  to be the map  $u \mapsto \tilde{u}$ , we define

$$H_{\Sigma_\varepsilon} = \Lambda_\varepsilon(H_{S_\varepsilon}).$$

$H_{\Sigma_\varepsilon}$  has a natural structure of Hilbert (Sobolev) space inherited by  $H^1(\Omega_\varepsilon)$ , and we denote by  $(\cdot, \cdot)_{H_{\Sigma_\varepsilon}}$ ,  $\|\cdot\|_{H_{\Sigma_\varepsilon}}$  the corresponding scalar product and norm. More precisely, we can identify the space  $H_{\Sigma_\varepsilon}$  with the family of functions in  $H^1(\Omega_\varepsilon)$  which vanish identically in  $\Omega_\varepsilon \setminus \Sigma_\varepsilon$ .

We introduce next the operator  $T_{\Sigma_\varepsilon} : H_{\Sigma_\varepsilon} \rightarrow H_{\Sigma_\varepsilon}$  defined as the restriction to  $H_{\Sigma_\varepsilon}$  of  $J''_\varepsilon(u_{I,\varepsilon})$  which, using the duality in  $H_{\Sigma_\varepsilon}$ , has the following expression

$$(T_{\Sigma_\varepsilon}u, v)_{H_{\Sigma_\varepsilon}} = \int_{\Sigma_\varepsilon} (\nabla u \cdot \nabla v + uv) - p \int_{\Sigma_\varepsilon} u_{I,\varepsilon}^{p-1} uv = (u, v)_{H_{\Sigma_\varepsilon}} - p \int_{\Sigma_\varepsilon} u_{I,\varepsilon}^{p-1} uv. \quad (81)$$

Fixing these notations and definitions, following the arguments at the beginning of Section 4 in [33] one can easily prove the following result.

**Lemma 5.1.** *Identifying the functions in  $H_{S_\varepsilon}$  with the corresponding ones in  $H_{\Sigma_\varepsilon}$  via the map  $\Lambda_\varepsilon$ , for  $\varepsilon$  sufficiently small one has*

$$(u, v)_{H_{\Sigma_\varepsilon}} = (u, v)_{H_{S_\varepsilon}} + O(\varepsilon^{1-\gamma}) \|u\|_{H_{S_\varepsilon}} \|v\|_{H_{S_\varepsilon}} ;$$

$$(T_{\Sigma_\varepsilon} u, v)_{H_{\Sigma_\varepsilon}} = (T_{S_\varepsilon} u, v)_{H_{S_\varepsilon}} + O(\varepsilon^{1-\gamma}) \|u\|_{H_{S_\varepsilon}} \|v\|_{H_{S_\varepsilon}}$$

with error  $O(\varepsilon^{1-\gamma})$  independent of  $u, v \in H_{\Sigma_\varepsilon}$ .

We introduced the operator  $T_{\Sigma_\varepsilon}$  because it represents an accurate model for  $J''_\varepsilon(u_{I,\varepsilon})$ . In fact, since most of the functions we consider have an exponential decay away from  $K_\varepsilon$ , it is reasonable to expect that the spectrum of  $J''_\varepsilon(u_{I,\varepsilon})$  will be affected only by negligible quantities if we work in  $H_{\Sigma_\varepsilon}$  instead of  $H^1(\Omega_\varepsilon)$ . More precisely, one has the following result (we recall the definition of  $\tau$  from the previous section).

**Lemma 5.2.** *There exists a fixed constant  $C$ , depending on  $\Omega, K, I$  and  $p$  such that the eigenvalues of  $J''_\varepsilon(u_{I,\varepsilon})$  and  $T_{\Sigma_\varepsilon}$  satisfy*

$$|\lambda_j(J''_\varepsilon(u_{I,\varepsilon})) - \lambda_j(T_{\Sigma_\varepsilon})| \leq C e^{-\frac{1}{c\varepsilon^{-\gamma}}}, \quad \text{provided } \lambda_j(J''_\varepsilon(u_{I,\varepsilon})) \leq \frac{\tau}{2}.$$

Here we are indexing the eigenvalues in non-decreasing order, counted with multiplicity.

We omit the proof of this result because it is very similar in spirit to that of Lemma 5.5 in [34]. This is based on the fact that the number of the eigenvalues of  $T_{S_\varepsilon}$  which are less or equal than  $\frac{3}{4}\tau$  is bounded by  $\varepsilon^{-D}$  for some  $D > 0$  (see Proposition 4.6 and the Weyl’s asymptotic formulas in Section 2.2), together with the exponential decay of the eigenfunctions of  $J''_\varepsilon(u_{I,\varepsilon})$ , which can be shown as in [34, Lemma 5.1].

As a consequence of Lemmas 5.1 and 5.2 we obtain the following result.

**Corollary 5.3.** *In the above notation, for  $\varepsilon$  small one has that*

$$|\lambda_j(J''_\varepsilon(u_{I,\varepsilon})) - \lambda_j(T_{S_\varepsilon})| \leq C\varepsilon^{1-\gamma}, \quad \text{provided } \lambda_j(J''_\varepsilon(u_{I,\varepsilon})) \leq \frac{\tau}{2}. \tag{82}$$

Using Proposition 4.6 and Corollary 5.3, it is possible to obtain some qualitative information about the spectrum of the linearized operator  $J''_\varepsilon(u_{I,\varepsilon})$ . However, this kind of estimate is not sufficiently precise by the following considerations. First of all, since the eigenvalues of  $T_{S_\varepsilon}$  can approach zero at a rate  $\min\{\varepsilon^2, \varepsilon^k\}$ , the estimate (82) need to be improved if we want to guarantee the invertibility of  $J''_\varepsilon(u_{I,\varepsilon})$ . Furthermore, it would be natural to expect that the Jacobi operator (and its invertibility) plays some role in the expansion of the eigenvalues, and this is not apparent here.

On the other hand, Lemma 5.2 gives an accurate estimate on the eigenvalues of  $J''_\varepsilon(u_{I,\varepsilon})$  in terms of those of  $T_{\Sigma_\varepsilon}$ , so it will be convenient to analyze  $T_{\Sigma_\varepsilon}$  directly.

### 5.2. Approximate eigenfunctions of $T_{\Sigma_\varepsilon}$

In this subsection we construct approximate eigenfunctions to the linearized operator at the approximate solutions  $u_{I,\varepsilon}$ . By the reasons explained at the end of the previous subsection, we need a refined expansion of the small eigenvalues of  $T_{\Sigma_\varepsilon}$ , and in particular here we want to understand how the  $\sigma_{l,\varepsilon}$ ’s change when we pass from  $T_{S_\varepsilon}$  to  $T_{\Sigma_\varepsilon}$ .

It is sufficient here to take  $I = 2$ , because the terms of order higher than  $\varepsilon^2$  do not affect the expansions below. As for the construction of the approximate solutions  $u_{I,\varepsilon}$ , we proceed by expanding the eigenvalue equation formally in powers of  $\varepsilon$ . By the construction of  $u_{2,\varepsilon}$ , *formally* the following equation holds

$$-\Delta_{g_\varepsilon} u_{2,\varepsilon} + u_{2,\varepsilon} - u_{2,\varepsilon}^p = O(\varepsilon^3).$$

Using Fermi coordinates as in Section 3 and differentiating with respect to  $\zeta_h$ , we get

$$-\partial_h(\Delta_{g_\varepsilon} u_{2,\varepsilon}) + \partial_h u_{2,\varepsilon} - p u_{2,\varepsilon}^{p-1} \partial_h u_{2,\varepsilon} = O(\varepsilon^3). \tag{83}$$

From the general expression of the Laplace–Beltrami operator, see formula (14), we can easily see that

$$\begin{aligned} \partial_h(\Delta_{g_\varepsilon} u) &= \Delta_{g_\varepsilon}(\partial_h u) + \partial_h g_\varepsilon^{AB} \partial_{AB} u + \partial_h(\partial_A g_\varepsilon^{AB}) \partial_B u \\ &+ \frac{1}{2} g_\varepsilon^{AB} \partial_{hA}^2(\log(\det g_\varepsilon)) \partial_B u + \frac{1}{2} \partial_A(\log(\det g_\varepsilon)) (\partial_h g_\varepsilon^{AB}) \partial_B u. \end{aligned} \tag{84}$$

Let us now consider the second term on the right-hand side of (84): dividing the indices this is equivalent to

$$\partial_h g_\varepsilon^{ij} \partial_{ij}^2 u + 2 \partial_h g_\varepsilon^{ib} \partial_{ib}^2 u + \partial_h g_\varepsilon^{ab} \partial_{ab} u + 2 \partial_h g_\varepsilon^{AN} \partial_A \partial_{\zeta_{n+1}} u.$$

From Lemma 3.2, and using the fact that we get an  $\varepsilon$  factor each time we differentiate  $u$  with respect to  $y_a, y_b, \dots$ , we find that

$$\partial_h g_\varepsilon^{AB} \partial_{AB}^2 u = -\frac{2}{3} \varepsilon^2 R_{ihtj} \zeta_t \partial_{ij}^2 u + O(\varepsilon^3).$$

Similarly we get

$$\begin{aligned} \partial_h \partial_A g_\varepsilon^{AB} \partial_B u &= \frac{1}{3} \varepsilon^2 R_{hij} \partial_j u + O(\varepsilon^3); \\ \frac{1}{2} g_\varepsilon^{AB} \partial_{hA}^2(\log(\det g_\varepsilon)) \partial_B u &= \varepsilon^2 \left( \frac{1}{3} R_{illh} + R_{iaah} - \Gamma_a^b(E_i) \Gamma_b^a(E_h) \right) \partial_i u \\ &+ 2 H_{ab} \Gamma_a^b(E_h) \partial_{\zeta_{n+1}} u + O(\varepsilon^3), \end{aligned}$$

and

$$\frac{1}{2} \partial_A(\log(\det g_\varepsilon)) (\partial_h g_\varepsilon^{AB}) \partial_B u = O(\varepsilon^3).$$

Putting together all these terms we deduce that

$$\begin{aligned} \partial_h(\Delta_{g_\varepsilon} u) &= \Delta_{g_\varepsilon}(\partial_h u) - \frac{2}{3} \varepsilon^2 R_{ihtj} \zeta_t \partial_{ij} u + \varepsilon^2 \left( \frac{2}{3} R_{illh} + R_{iaah} - \Gamma_a^b(E_i) \Gamma_b^a(E_h) \right) \partial_i u \\ &+ O(\varepsilon^3). \end{aligned} \tag{85}$$

To construct the approximate eigenfunctions  $v_\varepsilon$  and the approximate eigenvalues  $\mu$ , we make an *ansatz* of the type

$$v_\varepsilon = (\psi^h(\bar{y})\partial_h u_{2,\varepsilon}(\bar{y}, \zeta' + \Phi(\bar{y}), \zeta_{n+1}) + \varepsilon^2 z_2(\bar{y}, \zeta)) + O(\varepsilon^3); \quad \mu = \varepsilon^2 \bar{\mu} + O(\varepsilon^3),$$

where the normal section  $\psi = (\psi^h)_h$ , the function  $z_2$  and the real number  $\bar{\mu}$  have to be determined.

We notice that the eigenvalue equation  $J''_\varepsilon(u_{2,\varepsilon})v = \lambda v$  in  $H^1(\Omega_\varepsilon)$ , with an integration by parts becomes

$$-\Delta_{g_\varepsilon} v + v - p(u_{2,\varepsilon})^{p-1} v = \lambda(-\Delta_{g_\varepsilon} v + v),$$

see also the derivation of (57).

For  $v = v_\varepsilon$  and  $\lambda = \mu$ , we have the following expansion

$$\begin{aligned} &-\Delta_{g_\varepsilon}(\psi^h(\bar{y})\partial_h u_{2,\varepsilon} + \varepsilon^2 z_2(\bar{y}, \zeta)) \\ &+ \psi^h(\bar{y})\partial_h u_{2,\varepsilon} + \varepsilon^2 z_2(\bar{y}, \zeta) - p(u_{2,\varepsilon})^{p-1}(\psi^h(\bar{y})\partial_h u_{2,\varepsilon} + \varepsilon^2 z_2(\bar{y}, \zeta)) \\ &= \varepsilon^2 \bar{\mu}[-\Delta_{g_\varepsilon}(\psi^h(\bar{y})\partial_h u_{2,\varepsilon} + \varepsilon^2 z_2(\bar{y}, \zeta)) + (\psi^h(\bar{y})\partial_h u_{2,\varepsilon} + \varepsilon^2 z_2(\bar{y}, \zeta))] \\ &= \varepsilon^2 \bar{\mu}[\psi^h(\bar{y})(-\Delta_{g_\varepsilon} \partial_h w_0 + \partial_h w_0)] + O(\varepsilon^3) \\ &= \varepsilon^2 \bar{\mu} p \psi^h(\bar{y}) w_0^{p-1} \partial_h w_0 + O(\varepsilon^3). \end{aligned}$$

From (85) we can expand the Laplacian in the last formula as

$$\begin{aligned} -\Delta_{g_\varepsilon}(\psi^h(\bar{y})\partial_h u_{2,\varepsilon}) &= -\varepsilon^2 \partial_{\bar{y}_a \bar{y}_a}^2 \psi^h \partial_h w_0 - 2\varepsilon^2 \partial_a \psi^h \partial_{j_h}^2 w_0 \partial_{\bar{y}_a} \Phi_0^j - \psi^h \Delta_{g_\varepsilon}(\partial_h u_{2,\varepsilon}) \\ &+ 4\varepsilon^2 \zeta_{n+1} H_{aj} \partial_{\bar{y}_a} \psi^h \partial_{j_h}^2 w_0 + O(\varepsilon^3) \\ &= -\varepsilon^2 \partial_{\bar{y}_a \bar{y}_a}^2 \psi^h \partial_h w_0 - 2\varepsilon^2 \partial_a \psi^h \partial_{j_h}^2 w_0 \partial_{\bar{y}_a} \Phi_0^j - \psi^h \partial_h (\Delta_{g_\varepsilon} u_{2,\varepsilon}) \\ &+ 4\varepsilon^2 \zeta_{n+1} H_{aj} \partial_{\bar{y}_a} \psi^h \partial_{j_h}^2 w_0 + \frac{2}{3} \varepsilon^2 \psi^h R_{ihtj} \zeta_t \partial_{ij} w_0 \\ &- \varepsilon^2 \psi^h \left( \frac{2}{3} R_{illh} + R_{iaah} - \Gamma_a^b(E_i) \Gamma_b^a(E_h) \right) \partial_i w_0 + O(\varepsilon^3). \end{aligned}$$

Using (83) jointly with the last equality, and recalling our previous notation (from Section 3)

$$\mathcal{L}_\Phi u = -\Delta u + u - p w_0^{p-1} (\zeta' + \Phi(\varepsilon y), \zeta_{n+1}),$$

we obtain the following condition on  $z_2$

$$\begin{aligned} \mathcal{L}_\Phi z_2 &= \partial_{\bar{y}_a \bar{y}_a}^2 \psi^h \partial_h w_0 + 2 \partial_{\bar{y}_a} \psi^h \partial_{j_h}^2 w_0 \partial_{\bar{y}_a} \Phi_0^j - \frac{2}{3} \psi^h R_{ihtj} \zeta_t \partial_{ij}^2 w_0 \\ &+ \psi^h \left( \frac{2}{3} R_{illh} + R_{iaah} - \Gamma_a^b(E_i) \Gamma_b^a(E_h) \right) \partial_i w_0 + p \bar{\mu} \psi^h w_0^{p-1} \partial_h w_0 \\ &- 2 H_{ab} \Gamma_a^b(E_h) \partial_{\zeta_{n+1}} w_0 - 4 \zeta_{n+1} H_{aj} \partial_{\bar{y}_a} \psi^m \partial_{j_m}^2 w_0 + O(\varepsilon). \end{aligned} \tag{86}$$



In order to get solvability of this equation (in  $z_2$ ), we need to impose that the right-hand side is orthogonal to the kernel of  $\mathcal{L}_\Phi$  namely that, multiplying it by  $\partial_s w_0$  and integrating in  $\zeta$ ,  $s = 1, \dots, n$ , we must get zero. If we do this, reasoning as at the end of Section 3.2.1, we obtain the following condition on  $\psi$

$$C_0 \mathfrak{J}\psi = C_1 \bar{\mu}\psi, \quad \text{where } C_1 = p \int_{\mathbb{R}_+^{n+1}} w_0^{p-1} (\partial_1 w_0)^2 d\zeta,$$

and where  $C_0$  is given in (47). With the choices

$$\bar{\mu} = \frac{C_0}{C_1} \mu_l; \quad \psi = \psi_l,$$

where  $\mu_l$  is an eigenvalue of  $\mathfrak{J}$  with eigenfunction  $\psi_l$ , the right-hand side of (86) is perpendicular to the kernel of  $\mathcal{L}_\Phi$ , and we get solvability in  $z_2$ . Using the eigenvalue equation for  $\psi_l$ , (86) can be simplified as

$$\begin{aligned} \mathcal{L}_\Phi z_2 &= \mu_l \psi_l^h \partial_h w_0 \left( p \frac{C_0}{C_1} w_0^{p-1} - 1 \right) + 2 \partial_{\bar{y}_a} \psi_l^h (\partial_{\bar{y}_a} \Phi_0^j - 2 \zeta_{n+1} H_{aj}) \partial_{jh}^2 w_0 \\ &+ \frac{2}{3} \psi_l^h (R_{ijjh} \partial_i w_0 - R_{ihtj} \zeta_t \partial_{ij}^2 w_0 - 3 H_{ab} \Gamma_a^b(E_h) \partial_{\zeta_{n+1}} w_0). \end{aligned}$$

Next, we set

$$\begin{aligned} g_0^h(\bar{y}, \zeta) &= \mathcal{L}_\Phi^{-1} \left[ \partial_h w_0 \left( p \frac{C_0}{C_1} w_0^{p-1} - 1 \right) \right]; \\ g_1^h(\bar{y}, \zeta) &= 2 \mathcal{L}_\Phi^{-1} [(\partial_{\bar{y}_a} \Phi_0^j - 2 \zeta_{n+1} H_{aj}) \partial_{jh}^2 w_0]; \\ g_2^h(\bar{y}, \zeta) &= \frac{2}{3} \mathcal{L}_\Phi^{-1} [(R_{ijjh} \partial_i w_0 - R_{ihtj} \zeta_t \partial_{ij}^2 w_0 - 3 H_{ab} \Gamma_a^b(E_h) \partial_{\zeta_{n+1}} w_0)] \\ &+ \partial_h w_2(\bar{y}, \zeta' + \Phi(\bar{y}), \zeta_{n+1}), \end{aligned}$$

and

$$g_3^h(\bar{y}, \zeta) = \partial_h w_1(\bar{y}, \zeta' + \Phi(\bar{y}), \zeta_{n+1}).$$

We notice that, by the definitions of  $C_0, C_1$ , the computations in Section 3.2.2 and by oddness, the arguments of  $\mathcal{L}_\Phi^{-1}$  in the definitions of  $g_0^h, g_1^h$  and  $g_2^h$  are all perpendicular to the kernel of  $\mathcal{L}_\Phi$ , and therefore  $g_0, g_1$  and  $g_2$  are well defined.

Finally, with this notation, we define the approximate eigenfunction  $\Psi_l$  as  $v_\varepsilon$  times a suitable cutoff function of  $\zeta$ , namely

$$\begin{aligned} \Psi_l(\bar{y}, \zeta) &= \chi_\varepsilon(|\zeta|) \{ \psi_l^h(\bar{y}) [\partial_h w_0 + \varepsilon g_3^h(\bar{y}, \zeta) + \varepsilon^2 g_2^h(\bar{y}, \zeta)] + \varepsilon^2 \mu_l \psi_l^h(\bar{y}) g_0^h(\bar{y}, \zeta) \\ &+ \varepsilon^2 \partial_{\bar{y}_a} \psi_l^h(\bar{y}) g_1^h(\bar{y}, \zeta) \}, \end{aligned} \tag{87}$$

where  $\chi_\varepsilon$  is as in (32), and, as usual,  $\bar{y} = \varepsilon y$ .

A more accurate analysis, which we omit, shows that the above error terms are not only of order  $\varepsilon^3$ , but they decay exponentially to zero as  $|\zeta|$  tends to infinity. Moreover, as we already remarked, in the above estimates one can replace  $u_{2,\varepsilon}$  with  $u_{1,\varepsilon}$ . Precisely, one can prove the following result.

**Lemma 5.4.** *If  $\Psi_l$  is given in (87), then there exist a polynomial  $P(\zeta)$  and a sequence of positive constants  $(C_l)_l$ , depending on  $\Omega$ ,  $K$ ,  $p$  and  $I$  such that*

$$\left| -\Delta_{g_\varepsilon} \Psi_l + \Psi_l - pu_{1,\varepsilon}^{p-1} \Psi_l - \varepsilon^2 \frac{C_0}{C_1} \mu_l (-\Delta_{g_\varepsilon} \Psi_l + \Psi_l) \right| \leq C_l \varepsilon^3 P(\zeta) e^{-|\zeta|}.$$

### 5.3. A splitting of the functional space

In the previous subsection we expanded in  $\varepsilon$  some of the eigenvalues of  $T_{\Sigma_\varepsilon}$ , precisely those which are the counterparts of the  $\sigma_{l,\varepsilon}$ 's for  $T_{S_\varepsilon}$ . Actually,  $T_{S_\varepsilon}$  possesses another type of resonant eigenvalues, namely the  $\eta_{j,\varepsilon}$ 's for suitable values of  $j$ , which in principle could approach zero even faster. One of the differences between these two families of eigenvalues is that the eigenfunctions corresponding to the resonant  $\sigma_{l,\varepsilon}$ 's oscillate *slowly* along  $\partial\Omega_\varepsilon$ , and this allowed us to perform the above expansion. On the contrary, the eigenfunctions related to the  $\eta_{j,\varepsilon}$ 's possess only high Fourier modes, and therefore such an expansion is not possible anymore. Nevertheless, we can deal with the counterparts of these eigenvalues applying Kato's theorem, which on the other hand requires to characterize the corresponding eigenfunctions up to some extent.

The purpose of the present subsection is to identify appropriate subspaces of  $H_{\Sigma_\varepsilon}$  with respect to which  $T_{\Sigma_\varepsilon}$  is approximately in block form. Recalling the definitions in Proposition 4.5, in (73) and in (87) (and also our convention about the range of an integer index), for  $\delta \in (0, k)$ ,  $\bar{C} \in (0, 1)$ , we define the following subspaces

$$H_1 = \text{span}\{\phi_i(\varepsilon y)u_{i,\varepsilon}(\zeta), i = 0, \dots, \infty\}; \tag{88}$$

$$\hat{H}_2 = \text{span}\{\Psi_l, l = 0, \dots, \varepsilon^{-\delta}\};$$

$$\tilde{H}_2 = \text{span}\left\{\psi_j^m(\varepsilon y)\hat{v}_{j,\varepsilon}(|\zeta|)\frac{\zeta^m}{|\zeta|}, j = \varepsilon^{-\delta} + 1, \dots, \bar{C}\varepsilon^{-k}\right\}; \tag{89}$$

$$H_2 = \hat{H}_2 \oplus \tilde{H}_2; \quad H_3 = (H_1 \oplus H_2)^\perp, \tag{90}$$

where  $X^\perp$  denotes the orthogonal complement to the subspace  $X$  with respect to the scalar product in  $H_{\Sigma_\varepsilon}$ . We have the following result, which is the counterpart of Proposition 4.2 in [33]. The proof follows the same arguments, but for the reader's convenience we prefer to give details since the notation and the estimates are affected by the different dimensions and codimensions we are dealing with.

**Proposition 5.5.** *There exists a small value of the constant  $\bar{C} > 0$  in (89), depending on  $\Omega$ ,  $K$  and  $p$ , such that the following property holds. For  $\varepsilon$  sufficiently small and choosing  $\delta \in (k/2, k)$  in (89), every function  $u \in H_{\Sigma_\varepsilon}$  decomposes uniquely as*

$$u = u_1 + u_2 + u_3, \quad \text{with } u_1 \in H_1, u_2 \in H_2, u_3 \in H_3.$$

Moreover there exists a positive constant  $C$ , also depending on  $\Omega$ ,  $K$  and  $p$  such that

$$(T_{\Sigma_\varepsilon} u_3, u_3) \geq \frac{1}{C\bar{C}^{\frac{2}{k}}} \|u_3\|_{H_{\Sigma_\varepsilon}}^2.$$

The proof requires some preliminary lemmas. Before stating them, we recall our convention about the symbol  $\sum_c^d$ , for two positive real values  $c$  and  $d$ .

**Lemma 5.6.** *Let  $\tilde{u}_2 = \sum_{j=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_j \psi_j^m(\varepsilon y) \hat{v}_{j,\varepsilon}(|\zeta|) \zeta_m / |\zeta| \in \tilde{H}_2$ . Then*

$$\|\tilde{u}_2\|_{H_{\Sigma_\varepsilon}}^2 = (1 + O(\varepsilon^{1-\gamma})) \frac{1}{\varepsilon^k} \sum_{j=\varepsilon^{-\delta}}^{\bar{C}\varepsilon^{-k}} \beta_j^2. \tag{91}$$

**Proof.** By Lemma 5.1, it is sufficient to estimate  $\|\tilde{u}_2\|_{H_{S_\varepsilon}}^2$ . We notice that by (26) there holds

$$-\Delta_K^N \psi_j = \mathfrak{J} \psi_j + (\mathfrak{B} - \mathfrak{R}) \psi_j = \mu_l \psi_j + ((\mathfrak{B} - \mathfrak{R}) \psi)_j.$$

Integrating by parts, using (70) and the last formula one finds that  $\|\tilde{u}_2\|_{H_{S_\varepsilon}}^2$  becomes

$$\begin{aligned} & - \int_{S_\varepsilon} \sum_{j,l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \Delta_{\tilde{g}_\varepsilon} \left( \sum_{m=1}^n \beta_j \psi_j^m(\varepsilon y) \hat{v}_{j,\varepsilon}(|\zeta|) \frac{\zeta_m}{|\zeta|} \right) \cdot \left( \sum_{h=1}^n \beta_l \psi_l^h(\varepsilon y) \hat{v}_{l,\varepsilon}(|\zeta|) \frac{\zeta_h}{|\zeta|} \right) \\ & + \int_{S_\varepsilon} \sum_{j,l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \left( \sum_{m=1}^n \beta_j \psi_j^m(\varepsilon y) \hat{v}_{j,\varepsilon}(|\zeta|) \frac{\zeta_m}{|\zeta|} \right) \cdot \left( \sum_{h=1}^n \beta_l \psi_l^h(\varepsilon y) \hat{v}_{j,\varepsilon}(|\zeta|) \frac{\zeta_h}{|\zeta|} \right) \\ & = A_1 + A_2, \end{aligned} \tag{92}$$

where

$$\begin{aligned} A_1 &= \int_{S_\varepsilon} \sum_{j,l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \left[ (-\Delta_\zeta + (1 + \varepsilon^2 \mu_j)) \left( \sum_{m=1}^n \beta_j \psi_j^m(\varepsilon y) \hat{v}_{j,\varepsilon}(|\zeta|) \frac{\zeta_m}{|\zeta|} \right) \right] \\ & \quad \times \left( \sum_{h=1}^n \beta_l \psi_l^h(\varepsilon y) \hat{v}_{j,\varepsilon}(|\zeta|) \frac{\zeta_h}{|\zeta|} \right); \\ A_2 &= \varepsilon^2 \int_{S_\varepsilon} \sum_{j,l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \left( \sum_{m=1}^n \beta_j ((\mathfrak{B} - \mathfrak{R}) \psi_j)^m(\varepsilon y) \hat{v}_{j,\varepsilon}(|\zeta|) \frac{\zeta_m}{|\zeta|} \right) \\ & \quad \times \left( \sum_{h=1}^n \beta_l \psi_l^h(\varepsilon y) \hat{v}_{j,\varepsilon}(|\zeta|) \frac{\zeta_h}{|\zeta|} \right). \end{aligned}$$

Looking at  $A_1$ , the integral over any fiber  $N_y K_\varepsilon$  is non-zero if and only if  $m = h$  (and by symmetry, when computing the integral we can assume both the indices to be 1). Then, from (65) and from the orthogonality among different  $\psi_l$ 's (which now are scaled in  $\varepsilon$ ), recalling that  $\hat{v}_{j,\varepsilon}(|\zeta|)\zeta_m/|\zeta| = v_{j,\varepsilon,m}$ ,  $A_1$  becomes

$$\frac{1}{\varepsilon^k} \sum_{j=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_j^2 \|v_{j,\varepsilon,1}\|_{\varepsilon^2 \eta_{j,\varepsilon}}^2 = \frac{1}{\varepsilon^k} \sum_{j=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_j^2 \left[ \int_{\mathbb{R}_+^{n+1}} (|\nabla v_{j,\varepsilon,1}|^2 + (1 + \varepsilon^2 \mu_j) v_{j,\varepsilon,1}^2) \right].$$

Recalling the normalization (74) and the fact that  $\eta_j = \omega_j + O(1)$  (independently of  $j$ ), see Section 2.2, we obtain that

$$A_1 = \frac{1}{\varepsilon^k} \sum_{j=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} (1 + O(\varepsilon^2)) \beta_j^2. \tag{93}$$

We turn now to the estimate of  $A_2$ . By the orthogonality of the  $\psi_l$ 's, using again (65) and (74) one finds

$$\int_{S_\varepsilon} \tilde{u}_2^2 dV_{\tilde{g}_\varepsilon} = \frac{1}{\varepsilon^k} \sum_{j=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_j^2 \|v_{j,\varepsilon,1}\|_{L^2(\mathbb{R}_+^{n+1})}^2 \leq \frac{1}{\varepsilon^k} \sum_{j=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_j^2.$$

Working in a local system of coordinates  $(y, z)$  as in Section 4.2, it is also convenient to write  $\tilde{u}_2$  as

$$\tilde{u}_2(y, \zeta) = \sum_{m=1}^n f_m(y, |\zeta|) \zeta_m, \quad \text{where } f_m(y, |\zeta|) = \sum_{j=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_j \psi_j^m(\varepsilon y) \frac{\hat{v}_{j,\varepsilon}(|\zeta|)}{|\zeta|}.$$

If  $\mathcal{U}$  is a neighborhood of some point  $q$  in  $K$ , where the coordinates  $y$  are defined, letting  $\mathcal{U}_\varepsilon = \frac{1}{\varepsilon} \mathcal{U}$ , one has

$$\int_{N\mathcal{U}_\varepsilon} \tilde{u}_2^2 dV_{\tilde{g}_\varepsilon} = \sum_{m=1}^n \int_{\mathcal{U}_\varepsilon} \left( \int_{\mathbb{R}_+^{n+1}} f_m^2(y, |\zeta|) \zeta_1^2 d\zeta \right) dV_{\tilde{g}_\varepsilon}(y),$$

so it follows that

$$\sum_{m=1}^n \int_{\mathcal{U}_\varepsilon} \left( \int_{\mathbb{R}_+^{n+1}} f_m^2(y, |\zeta|) \zeta_1^2 d\zeta \right) dV_{\tilde{g}_\varepsilon}(y) \leq \int_{S_\varepsilon} \tilde{u}_2^2 dV_{\tilde{g}_\varepsilon} \leq \frac{1}{\varepsilon^k} \sum_{j=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_j^2. \tag{94}$$

Now, we can write

$$A_2 = \varepsilon^2 \int_{S_\varepsilon} \tilde{u}_2 \tilde{u}_2 dV_{\tilde{g}_\varepsilon}, \quad \text{where } \tilde{u}_2 = \sum_{j=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_j ((\mathfrak{B} - \mathfrak{R}) \psi_j)^m(\varepsilon y) \hat{v}_{j,\varepsilon}(|\zeta|) \frac{\zeta_m}{|\zeta|}.$$

As for  $\tilde{u}_2$ , we can write  $\tilde{u}_2 = \sum_{m=1}^n f_m(y, |\zeta|)\zeta_m$ , where  $f_m = \sum_{j=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} (\mathfrak{B} - \mathfrak{A})_{mj} f_j(y, |\zeta|)$ , and compute

$$\int_{N\mathcal{U}_\varepsilon} \tilde{u}_2^2 dV_{\tilde{g}_\varepsilon} = \sum_{m=1}^n \int_{\mathcal{U}_\varepsilon} \left( \int_{\mathbb{R}^{n+1}} f_m^2(y, |\zeta|)\zeta_1^2 d\zeta \right) dV_{\tilde{g}_\varepsilon}(y).$$

In conclusion, from the Hölder inequality, from (94), covering  $K_\varepsilon$  with finitely-many  $\mathcal{U}_\varepsilon$ 's we derive

$$|A_2| \leq \varepsilon^2 \left( \int_{S_\varepsilon} \tilde{u}_2^2 dV_{\tilde{g}_\varepsilon} \right)^{\frac{1}{2}} \left( \int_{S_\varepsilon} \tilde{u}_2^2 dV_{\tilde{g}_\varepsilon} \right)^{\frac{1}{2}} \leq C\varepsilon^2 \frac{1}{\varepsilon^k} \|\mathfrak{B} - \mathfrak{A}\|_{L^\infty} \sum_{j=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_j^2. \tag{95}$$

Then the conclusion follows from (93) and (95).  $\square$

In order to estimate the norm  $\|\hat{u}_2\|_{H_{\Sigma_\varepsilon}}$ , it is convenient to introduce an abstract result.

**Lemma 5.7.** For  $j \in \{0, \dots, \varepsilon^{-\delta}\}$ , and for a sequence  $(\beta_j)_j$ , let us consider a function  $u : S_\varepsilon \rightarrow \mathbb{R}$  of the form

$$u(y, \zeta) = \sum_{j=0}^{\varepsilon^{-\delta}} \sum_{m=1}^n \beta_j (L_{d, \bar{y}} \psi_j^m)(\bar{y}) g_m(\zeta),$$

where  $\bar{y} = \varepsilon y$ , where  $L_{d, \bar{y}}$  is a linear differential operator of order  $d$  with smooth coefficients in  $\bar{y}$ , and where the functions  $g_m(\zeta)$  are also smooth and have an exponential decay at infinity.

Then there exists a positive constant  $C$ , independent of  $\varepsilon$ ,  $\delta$  and  $(\beta_j)_j$  such that

$$\|u\|_{L^2(S_\varepsilon)}^2 \leq C \frac{1}{\varepsilon^k} \sum_{j=0}^{\varepsilon^{-\delta}} (1 + \varepsilon^{2d} |\mu_j|^d) \beta_j^2.$$

**Proof.** The proof is similar in spirit to that of Lemma 5.6, but here we take advantage of the fact that the profile  $g_m(\zeta)$  is independent of the index  $j$  (this lemma applies in particular to each of the summands in the definition of  $\Psi_l$ , see (87)).

Using local coordinates, (65) and the exponential decay of the  $g_m$ 's, after integration in  $\zeta$  we find

$$\|u\|_{L^2(S_\varepsilon)}^2 = \sum_{j,l=0}^{\varepsilon^{-\delta}} \sum_{m,h=1}^n \beta_j \beta_l c_{mh} \int_{\mathcal{U}_\varepsilon} (L_{d, \bar{y}} \psi_j^m)(\bar{y}) (L_{d, \bar{y}} \psi_l^h)(\bar{y}) dV_{\tilde{g}_\varepsilon},$$

for some bounded coefficients  $(c_{mh})$ . As for (95) then we find  $\|u\|_{L^2(S_\varepsilon)} \leq C \|\psi\|_{H^d(K_\varepsilon, N_{K_\varepsilon})}$  and the last quantity, with a change of variables and by (30), can be estimated with

$$\frac{C}{\varepsilon^k} \sum_{j=0}^{\varepsilon^{-\delta}} (1 + \varepsilon^{2d} |\mu_j|^d) \beta_j^2.$$

This concludes the proof.  $\square$

**Lemma 5.8.** *Let  $u_2 = \hat{u}_2 + \tilde{u}_2 = \sum_{j=0}^{\varepsilon^{-\delta}} \beta_j \Psi_j(\varepsilon y, \zeta) + \sum_{j=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_j \psi_j^m(\varepsilon y) \hat{v}_{j,\varepsilon}(|\zeta|) \frac{\zeta_m}{|\zeta|} \in H_2$ . Then, choosing  $\delta \in (k/2, k)$  in (89), one has*

$$\|u_2\|_{H_{S_\varepsilon}}^2 = \frac{1}{\varepsilon^k} (1 + O(\varepsilon^{1-\gamma} + \varepsilon^{2-\frac{2\delta}{k}})) \left[ \sum_{j=0}^{\varepsilon^{-\delta}} \beta_j^2 \|\partial_1 w_0\|_{H^1(\mathbb{R}_+^{n+1})} + \sum_{j=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_j^2 \right]. \tag{96}$$

**Proof.** We first claim that the following formula holds

$$\|\hat{u}_2\|_{H_{S_\varepsilon}}^2 = \frac{1}{\varepsilon^k} \sum_{j=0}^{\varepsilon^{-\delta}} \beta_j^2 (1 + O(\varepsilon^{2-2\gamma} + \varepsilon^{2-\frac{2\delta}{k}})) \|\partial_1 w_0\|_{H^1(\mathbb{R}_+^{n+1})}^2. \tag{97}$$

**Proof of (97).** We write

$$\hat{u}_2 = \hat{u}_{2,1} + \hat{u}_{2,2} := \sum_{j=0}^{\varepsilon^{-\delta}} \beta_j \psi_j^m(\varepsilon y) \partial_m w_0(\zeta) \chi_\varepsilon(|\zeta|) + \sum_{j=0}^{\varepsilon^{-\delta}} \beta_j \bar{\Psi}_j(\varepsilon y, \zeta),$$

where  $\bar{\Psi}_j$  is the term of order  $\varepsilon$  (and higher) in  $\Psi_j$ . Reasoning as in the proof of Lemma 5.6 we get

$$\begin{aligned} \|\hat{u}_{2,1}\|_{H_{S_\varepsilon}}^2 &= \frac{1}{\varepsilon^k} \sum_{j=0}^{\varepsilon^{-\delta}} \beta_j^2 (1 + \varepsilon^2 \mu_j + O(\varepsilon^2)) \|\partial_m w_0 \chi_\varepsilon\|_{H^1(\mathbb{R}_+^{n+1})}^2 \\ &= \frac{1}{\varepsilon^k} \sum_{j=0}^{\varepsilon^{-\delta}} \beta_j^2 (1 + O(\varepsilon^{2-\frac{2\delta}{k}})) \|\partial_1 w_0\|_{H^1(\mathbb{R}_+^{n+1})}^2, \end{aligned} \tag{98}$$

where the last equality follows from the Weyl’s asymptotic formula (29).

On the other hand, using Lemma 5.7, the Weyl’s formula and some computations, one also finds

$$\begin{aligned} \varepsilon^k \|\hat{u}_{2,2}\|_{H_{S_\varepsilon}}^2 &\leq C \varepsilon^2 \sum_{j=0}^{\varepsilon^{-\delta}} \beta_j^2 (1 + \varepsilon^2 |\mu_j|) + C \varepsilon^4 \sum_{j=0}^{\varepsilon^{-\delta}} \beta_j^2 \mu_j^2 (1 + \varepsilon^2 |\mu_j|) \\ &\quad + C \varepsilon^4 \sum_{j=0}^{\varepsilon^{-\delta}} \beta_j^2 (1 + |\mu_j| + \varepsilon^2 |\mu_j|^3) \leq C (\varepsilon^2 + \varepsilon^{4-\frac{4\delta}{k}} + \varepsilon^{6-\frac{6\delta}{k}}) \sum_{j=0}^{\varepsilon^{-\delta}} \beta_j^2. \end{aligned}$$

By our choice of  $\delta$ , the last formula reads

$$\|\hat{u}_{2,2}\|_{H_{S_\varepsilon}}^2 \leq \frac{C}{\varepsilon^k} \varepsilon^{4-\frac{4\delta}{k}} \sum_{j=0}^{\varepsilon^{-\delta}} \beta_j^2. \tag{99}$$

Finally, from (98) and (99) we also obtain

$$(\hat{u}_{2,1}, \hat{u}_{2,2})_{H_{S_\varepsilon}} \leq \frac{C}{\varepsilon^k} \sum_{j=0}^{\varepsilon^{-\delta}} \beta_j^2 (\varepsilon + O(\varepsilon^{2-\frac{2\delta}{k}})),$$

which concludes the proof of (97).  $\square$

**Proof of (96).** We write again  $\hat{u}_2 = \hat{u}_{2,1} + \hat{u}_{2,2}$ . Then, by the orthogonality relations among the  $\psi_j$ 's, reasoning as in the proof of Lemma 5.6, we get that  $(\tilde{u}_2, \hat{u}_{2,1})_{H_{S_\varepsilon}}$  becomes

$$\varepsilon^2 \sum_{j=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \sum_{l=0}^{\varepsilon^{-\delta}} \int_{S_\varepsilon} \left( \sum_{m=1}^n \beta_j ((\mathfrak{B} - \mathfrak{R})\psi_j)^m (\varepsilon y) \hat{v}_{j,\varepsilon}(|\zeta|) \frac{\zeta^m}{|\zeta|} \right) \cdot \left( \chi_\varepsilon(|\zeta|) \sum_{h=1}^n \beta_l \psi_l^h(\varepsilon y) \partial_h w_0 \right).$$

As above, with some computations we find

$$(\tilde{u}_2, \hat{u}_{2,1})_{H_{S_\varepsilon}} = O(\varepsilon^2) \|\tilde{u}_2\|_{H_{S_\varepsilon}} \|\hat{u}_{2,1}\|_{H_{S_\varepsilon}} = O(\varepsilon^2) \frac{1}{\varepsilon^k} \sum_{j=0}^{\bar{C}\varepsilon^{-k}} \beta_j^2.$$

From Lemma 5.6 and (99) we also find

$$(\tilde{u}_2, \hat{u}_{2,1})_{H_{S_\varepsilon}} \leq C \frac{1}{\varepsilon^k} \left( \sum_{j=0}^{\bar{C}\varepsilon^{-k}} (1 + O(\varepsilon^{1-\gamma})\beta_j^2) \right)^{\frac{1}{2}} \varepsilon^{2-\frac{2\delta}{k}} \left( \sum_{j=0}^{\bar{C}\varepsilon^{-k}} \beta_j^2 \right)^{\frac{1}{2}}.$$

The result follows from the last two formulas.  $\square$

**Remark 5.9.** From the proof of (96) it also follows that every function  $u_2 \in H_2$  can be written uniquely as  $u_2 = \hat{u}_2 + \tilde{u}_2$ , with  $\hat{u}_2 \in \hat{H}_2$  and  $\tilde{u}_2 \in \tilde{H}_2$ .

**Proof of Proposition 5.5.** In order to prove the uniqueness of the decomposition it is sufficient to show that, for  $\varepsilon$  small

$$(u_1, u_2)_{H_{S_\varepsilon}} = o_\varepsilon(1) \|u_1\|_{H_{S_\varepsilon}} \|u_2\|_{H_{S_\varepsilon}}, \quad u_1 \in H_1, \quad u_2 \in H_2, \tag{100}$$

where  $o_\varepsilon(1) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Indeed, by Lemma 5.1 we have

$$(u_1, u_2)_{H_{S_\varepsilon}} = (u_1, u_2)_{H_{S_\varepsilon}} + O(\varepsilon^{1-\gamma}) \|u_1\|_{H_{S_\varepsilon}} \|u_2\|_{H_{S_\varepsilon}},$$

and since the functions  $\partial_h w_0$ ,  $g_0^h$ ,  $g_3^h$  and  $v_{l,\varepsilon,i}$  are odd in  $\zeta'$  (and so also  $\tilde{u}_2$  and  $\hat{u}_{2,1}$ ), we get

$$(u_1, u_2)_{H_{S_\varepsilon}} = (u_1, \hat{u}_{2,2})_{H_{S_\varepsilon}},$$

where we have used the notation in the proof of Lemma 5.8. Hence from the last three formulas, (99) and form (96) we deduce

$$(u_1, u_2)_{H_{S_\varepsilon}} \leq C (\varepsilon^{1-\gamma} + \varepsilon^{2-\frac{2\delta}{k}}) \|u_1\|_{H_{S_\varepsilon}} \|u_2\|_{H_{S_\varepsilon}}, \tag{101}$$

which implies (100), since  $\delta \in (k/2, k)$ .

To prove the second statement, it is sufficient to show that

$$(u_3, v)_{H_{S_\varepsilon}} \leq \frac{1}{2} \|u_3\|_{H_{S_\varepsilon}} \|v\|_{H_{S_\varepsilon}} \quad \text{as } \varepsilon \rightarrow 0, \tag{102}$$

for all  $u_3 \in H_3$  and for all the functions  $v$  of the form

$$v = \sum_{l=0}^{\frac{1}{2}\bar{C}\varepsilon^{-k}} \tilde{\beta}_l \varphi_l^m(\varepsilon y) v_{l,\varepsilon,m}(\zeta).$$

In fact, if we write  $u_3 = u_{3,0} + u_{3,1} + u_{3,2}$  as in Remark 4.7 (with an obvious change of notation),

$$u_{3,0} = \sum_{j=0}^{\infty} \alpha_j u_{j,\varepsilon}(|\zeta|) \phi_j(\varepsilon y); \quad u_{3,1} = \sum_{l=0}^{\infty} \beta_l v_{l,\varepsilon,i}(\zeta) \varphi_l^i(\varepsilon y),$$

by (79) we find

$$\|u_3\|_{S_\varepsilon}^2 = \frac{1}{\varepsilon^k} \sum_{l=0}^{\infty} (\alpha_l^2 + \beta_l^2) + \|u_{3,2}\|_{H_{S_\varepsilon}}^2. \tag{103}$$

From (79), from Lemma 5.1 and from the fact that  $u_3$  is perpendicular in  $H_{S_\varepsilon}$  to  $u_{3,0} \in H_1$ , we deduce

$$\frac{1}{\varepsilon^k} \sum_{l=0}^{\infty} \alpha_l^2 = (u_{3,0}, u_{3,0})_{H_{S_\varepsilon}} = (u_{3,0}, u_3)_{H_{S_\varepsilon}} = O(\varepsilon^{1-\gamma}) \|u_3\|_{H_{S_\varepsilon}} \|u_{3,0}\|_{H_{S_\varepsilon}} \leq C \varepsilon^{1-\gamma} \|u_3\|_{H_{S_\varepsilon}}^2.$$

Moreover from (102), choosing  $v = \sum_{l=0}^{\frac{1}{2}\bar{C}\varepsilon^{-k}} \beta_l \varphi_l^m(\varepsilon y) v_{l,\varepsilon,m}(\zeta)$ , and using (103) we get

$$\frac{1}{\varepsilon^k} \sum_{l \leq \frac{1}{2}\bar{C}\varepsilon^{-k}} \beta_l^2 = (u_3, v)_{H_{S_\varepsilon}} \leq \frac{1}{2} \|u_3\|_{S_\varepsilon}^2.$$

The last two formulas and (103) then imply

$$\|u_3\|_{H_{S_\varepsilon}}^2 \leq C \left( \sum_{l > \frac{1}{2}\bar{C}\varepsilon^{-k}} \beta_l^2 + \|u_{3,2}\|_{H_{S_\varepsilon}}^2 \right), \tag{104}$$

for some fixed constant  $C$ .

On the other hand, by (80) we also have

$$(T_{S_\varepsilon} u_3, u_3)_{S_\varepsilon} \geq \frac{1}{\varepsilon^k} \sum_{l > \frac{1}{2}\bar{C}\varepsilon^{-k}+1} \sigma_{l,\varepsilon} \beta_l^2 + \frac{1}{C} \|u_{3,2}\|_{H_{S_\varepsilon}}^2.$$



Using the fact that  $\sigma_{i,\varepsilon} \sim \sigma_{\varepsilon^2 w_i, \varepsilon} \sim \varepsilon^2 i^{2/k}$  by Proposition 4.5, from (104) and the last formula it follows that

$$(T_{S_\varepsilon} u_3, u_3)_{S_\varepsilon} \geq \frac{1}{\varepsilon^k} \frac{1}{C \bar{C}_k^2} \sum_{i > \frac{1}{2} \bar{C}_k \varepsilon^{-k+1}} \beta_l^2 + \frac{1}{C} \|u_{3,2}\|_{H_{S_\varepsilon}}^2 \geq \frac{1}{C \bar{C}_k^2} \|u_3\|_{H_{S_\varepsilon}}^2.$$

This yields our conclusion, hence we are reduced to prove (102).

**Proof of (102).** By the form of  $v$  and by (79), we have

$$\|v\|_{H_{S_\varepsilon}}^2 = \frac{1}{\varepsilon^k} \sum_{l=0}^{\frac{1}{2} \bar{C}_k \varepsilon^{-k}} \tilde{\beta}_l^2. \tag{105}$$

Using the  $L^2$  basis  $(\psi_l)_l$  of eigenfunctions of  $\mathfrak{J}$ , we define the function  $\varphi$  and the coefficients  $\{\beta_l\}_{l=1, \dots, \infty}$  as

$$\varphi(\bar{y}) = \sum_{l=0}^{\frac{1}{2} \bar{C}_k \varepsilon^{-k}} \tilde{\beta}_l \varphi_l(\bar{y}) = \sum_{l=0}^{\infty} \beta_l \psi_l(\bar{y}) := \sum_{l=0}^{\infty} \beta_l \psi_l^h(\bar{y}) E_h(\bar{y}),$$

so we have

$$\|\varphi\|_{L^2(K; NK)}^2 = \sum_{l=0}^{\frac{1}{2} \bar{C}_k \varepsilon^{-k}} \tilde{\beta}_l^2 = \sum_{l=0}^{\infty} \beta_l^2. \tag{106}$$

Using these new coefficients  $\beta_j$ , we set (see (73))

$$\tilde{v}(y, \zeta) = \bar{C}_0 \sum_{j=0}^{\varepsilon^{-\delta}} \beta_j \Psi_j(\varepsilon y, \zeta) + \sum_{j=\varepsilon^{-\delta}+1}^{\bar{C}_k \varepsilon^{-k}} \beta_j \psi_j^h(\varepsilon y) \hat{v}_{j,\varepsilon}(|\zeta|) \frac{\zeta_h}{|\zeta|} \in H_2,$$

where  $\bar{C}_0$  is given in Remark 4.4. Hence we can write

$$v - \tilde{v} = A_1 + A_2 + A_3 + A_4 + A_5,$$

with

$$A_1 = \sum_{l=0}^{\frac{1}{2} \bar{C}_k \varepsilon^{-k}} \tilde{\beta}_l \varphi_l^m(\varepsilon y) [v_{l,\varepsilon,m}(\zeta) - v_{0,\varepsilon,m}(\zeta)]; \quad A_2 = \sum_{l=\bar{C}_k \varepsilon^{-k}+1}^{\infty} \beta_l \psi_l^h(\varepsilon y) v_{0,\varepsilon,h}(\zeta);$$

$$A_3 = -\bar{C}_0 \sum_{j=0}^{\varepsilon^{-\delta}} \beta_j \bar{\Psi}_j(\varepsilon y, \zeta); \quad A_4 = \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}_k \varepsilon^{-k}} \beta_l \psi_l^h(\varepsilon y) (v_{0,\varepsilon,h} - v_{l,\varepsilon,h});$$

$$A_5 = \sum_{l=0}^{\varepsilon^{-\delta}} \beta_l \psi_l^h (v_{0,\varepsilon,h} - \bar{C}_0 \chi_\varepsilon(|\zeta|) \partial_h w_0),$$

and where  $\bar{\Psi}_j$  is defined in the proof of Lemma 5.8. Since  $u_3$  is orthogonal to  $H_2$ , we get  $(u_2, \tilde{v})_{H_{\Sigma_\varepsilon}} = 0$ , and so

$$(u_3, v)_{H_{\Sigma_\varepsilon}} = (u_3, A_1)_{H_{\Sigma_\varepsilon}} + (u_3, A_2)_{H_{\Sigma_\varepsilon}} + (u_3, A_3)_{H_{\Sigma_\varepsilon}} + (u_3, A_4)_{H_{\Sigma_\varepsilon}} + (u_3, A_5)_{H_{\Sigma_\varepsilon}}. \tag{107}$$

We prove now that  $\|A_i\|_{H_{\Sigma_\varepsilon}}$  is small for every  $i = 1, \dots, 5$ . From (65), the proofs of Propositions 4.5, 4.6 and (105) there holds

$$\|A_1\|_{H_{\Sigma_\varepsilon}}^2 = \frac{1}{\varepsilon^k} \sum_{l=0}^{\frac{1}{2}\bar{C}\varepsilon^{-k}} \tilde{\beta}_l^2 \|v_{l,\varepsilon,1} - v_{0,\varepsilon,1}\|_{l,\varepsilon}^2 \leq C\bar{C}^2(1 + \bar{C}^2) \|v\|_{H_{\Sigma_\varepsilon}}^2 < \frac{1}{16} \|v\|_{H_{\Sigma_\varepsilon}}^2,$$

provided  $\bar{C}$  is sufficiently small.

To estimate  $A_2$  we can use Lemma 5.7 and some computations to find

$$\|A_2\|_{H_{\Sigma_\varepsilon}}^2 \leq C \frac{1}{\varepsilon^k} \sum_{l=\bar{C}\varepsilon^{-k}+1}^{\infty} \beta_l^2 (1 + \varepsilon^2 |\mu_l|). \tag{108}$$

We now set  $\tilde{\varphi} = \sum_{l=\bar{C}\varepsilon^{-k}+1}^{\infty} \beta_l \psi_l$ . Since  $\mathfrak{J} = -\Delta_K^N + O(1)$ , for any integer  $m$  one finds

$$\begin{aligned} (\mathfrak{J}^m \tilde{\varphi}, \tilde{\varphi})_{L^2(K;NK)} &\leq (\mathfrak{J}^m \varphi, \varphi)_{L^2(K)} \\ &\leq ((-\Delta_K^N)^m \varphi, \varphi)_{L^2(K;NK)} \\ &\quad + C_m [((-\Delta_K^N)^{m-1} \varphi, \varphi)_{L^2(K;NK)} + (\varphi, \varphi)_{L^2(K;NK)}]. \end{aligned}$$

Since  $\varphi = \sum_{l=0}^{\frac{1}{2}\bar{C}\varepsilon^{-k}} \tilde{\beta}_l \varphi_l$ , from (106) we deduce that

$$\begin{aligned} (\mathfrak{J}^m \tilde{\varphi}, \tilde{\varphi})_{L^2(K;NK)} &\leq \left(\frac{\bar{C}}{2}\right)^{\frac{2m}{k}} \varepsilon^{-2m} \|\varphi\|_{L^2(K;NK)}^2 + O(\varepsilon^{-2(m-1)}) \|\varphi\|_{L^2(K;NK)}^2 \\ &\leq \left[ \left(\frac{\bar{C}}{2}\right)^{\frac{2m}{k}} \varepsilon^{-2m} + O(\varepsilon^{-2(m-1)}) \right] \left( \sum_{l=0}^{\frac{1}{2}\bar{C}\varepsilon^{-k}} \tilde{\beta}_l^2 \right). \end{aligned} \tag{109}$$

On the other hand, since in the basis  $(\psi_l)_l$ , the function  $\tilde{\varphi}$  has non-zero components only when  $l \geq \bar{C}\varepsilon^{-k}$ , by the Weyl’s asymptotic formula we have also that

$$(\mathfrak{J}^m \tilde{\varphi}, \tilde{\varphi})_{L^2(K;NK)} \geq \begin{cases} \sum_{l=\bar{C}\varepsilon^{-k}+1}^{\infty} \mu_l^m \beta_l^2; \\ C\bar{C}^{\frac{2m}{k}} \varepsilon^{-2m} \sum_{l=\bar{C}\varepsilon^{-k}+1}^{\infty} \beta_l^2. \end{cases} \tag{110}$$

Using (109) and the first inequality in (110) with  $m = 1$  we get

$$\varepsilon^2 \sum_{l=\bar{C}\varepsilon^{-k}+1}^{\infty} \mu_l \beta_l^2 \leq (C\bar{C}^{\frac{2}{k}} + o_\varepsilon(1)) \sum_{l=0}^{\frac{1}{2}\bar{C}\varepsilon^{-k}} \tilde{\beta}_l^2.$$

Moreover, using (109) and the second inequality in (110) with  $m$  arbitrary one also finds

$$\sum_{l=\bar{C}\varepsilon^{-k}+1}^{\infty} \beta_l^2 \leq \left( \left( \frac{1}{2} \right)^{\frac{2m}{k}} + o_\varepsilon(1) \right) \sum_{l=0}^{\frac{1}{2}\bar{C}\varepsilon^{-k}} \tilde{\beta}_l^2.$$

Using (105), (108) and the last two inequalities (for the second one we take  $m$  large enough), for sufficiently small  $\bar{C}$  we find  $\|A_2\|_{H_{S_\varepsilon}} < \frac{1}{16} \|v\|_{H_{S_\varepsilon}}$ .

Now we estimate  $\|A_3\|_{H_{S_\varepsilon}}$ . Reasoning as for (99), from (105) and (106) we get

$$\|A_3\|_{H_{S_\varepsilon}}^2 \leq C \frac{1}{\varepsilon^k} \varepsilon^{4-4\frac{\delta}{k}} \sum_0^{\varepsilon^{-\delta}} \beta_j^2 \leq C \varepsilon^{4-4\frac{\delta}{k}} \|v\|_{H_{S_\varepsilon}}^2.$$

Next, similarly to the estimate of  $A_1$ , for small  $\bar{C}$  we find

$$\|A_4\|_{H_{S_\varepsilon}}^2 \leq \frac{1}{\varepsilon^k} C \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_l^2 \|\hat{v}_{0,\varepsilon,1} - v_{l,\varepsilon,1}\|_{l,\varepsilon}^2 \leq C\bar{C}^2(1 + \bar{C}^2) \|v\|_{H_{S_\varepsilon}}^2 < \frac{1}{16} \|v\|_{H_{S_\varepsilon}}^2.$$

Finally, from Proposition 4.5 and reasoning as for  $A_2$ , we obtain also

$$\|A_5\|_{H_{S_\varepsilon}}^2 \leq \frac{1}{\varepsilon^k} C e^{-C^{-1}\varepsilon^{-\gamma}} \sum_{l=0}^{\varepsilon^{-\delta}} \beta_l^2 (1 + \varepsilon^2 \omega_l) \bar{C} \varepsilon^{-k} l^{-\frac{2l}{k}} \leq C \varepsilon^{-k} e^{-C^{-1}\varepsilon^{-\gamma}} \|v\|_{H_{S_\varepsilon}}^2.$$

Taking (107) into account, this concludes the proof of (102), provided we choose  $\bar{C}$  and  $\varepsilon$  sufficiently small.  $\square$

### 6. Diagonalization of $T_{\Sigma_\varepsilon}$ and applications

In this section we study how the operator  $T_{\Sigma_\varepsilon}$  behaves with respect to the above splitting of  $H_{\Sigma_\varepsilon}$  in the three subspaces  $H_1, H_2$  and  $H_3$ . We prove that its form is almost diagonal and we apply this analysis to study its invertibility for suitable values of  $\varepsilon$ .

#### 6.1. Diagonalization

Integrating by parts, we can evaluate the operator  $T_{\Sigma_\varepsilon}$  multiplying a test function by the following quantity

$$\mathfrak{S}_\varepsilon(u) = \sqrt{\det g} (-\Delta_g u + u - pu_{l,\varepsilon}^{p-1} u) \tag{111}$$

and integrating in the variables  $y$  and  $\zeta$  (using (65)). In Lemma 5.4 we studied  $\mathfrak{S}_\varepsilon$  acting on the functions  $\Psi_l$ , for any  $l$  fixed. In that lemma, our estimates depend on the value of the index  $l$ , and in general one can expect that they become worse and worse as  $l$  increases. The goal of this subsection is to derive estimates in terms of both  $\varepsilon$  and  $l$  and, evaluating  $\mathfrak{S}_\varepsilon(u)$  on the functions  $\hat{u}_2 \in \hat{H}_2$ , we will keep track also of the terms of order  $\varepsilon^3$  and higher.

In the following, we will sometimes omit the factor  $\chi_\varepsilon$  appearing in (87) since this will only produce error terms exponentially small in  $\varepsilon$ , which are negligible for our purposes.

**Lemma 6.1.** *There exist linear differential operators  $L_1, L_2, L_3$  (acting on the variables  $\bar{y}$ ) of order 1, 2 and 3 respectively, whose coefficients (independent of  $l$ ) are smooth and satisfy the bounds*

$$c_\alpha(L_i) \leq C(1 + |\zeta|^C)e^{-\frac{|\zeta|}{C}}, \tag{112}$$

and such that in local coordinates we have the following expression for  $\mathfrak{S}_\varepsilon(\Psi_l)$

$$\begin{aligned} \mathfrak{S}_\varepsilon(\Psi_l) = & \varepsilon^2 \frac{C_0}{C_1} \mu_l w_0^{p-1} \partial_h w_0 \psi_l^h \\ & - 2\varepsilon^3 (\zeta_i \Gamma_a^b(E_i) - \zeta_{n+1} H_{ab} + \zeta_{n+1} H_\alpha^\alpha \delta_{ab}) (\partial_{\bar{y}_a \bar{y}_b}^2 \psi_l^h) \partial_h w_0 - \varepsilon^3 (\partial_{\bar{y}_a \bar{y}_a}^2 \psi_l^h) \partial_h w_1 \\ & + \varepsilon^3 \zeta_{n+1} H_\alpha^\alpha \mu_l \psi_l^h g_0^h (1 - p w_0^{p-1}) - \varepsilon^3 p(p-1) w_0^{p-2} w_1 \mu_l \psi_l^h g_0^h \\ & - \varepsilon^4 \mu_l (\partial_{\bar{y}_a \bar{y}_a}^2 \psi_l^h) g_0^h + \varepsilon^3 L_1 \psi_l + \varepsilon^4 L_3 \psi_l + \varepsilon^4 \mu_l L_1 \psi_l + \varepsilon^5 \mu_l L_2 \psi_l, \end{aligned} \tag{113}$$

where  $C_0, C_1$  are as in Section 5.2.

**Proof.** As for the construction of the approximate solutions  $u_{l,\varepsilon}$ , we can expand formally  $\mathfrak{S}_\varepsilon(\Psi_l)$  in powers of  $\varepsilon$  and check carefully all the error terms, paying particular attention to the ones involving derivatives in the variables  $\bar{y}_a$ , which produce larger and larger terms (as  $l$  increases) in the Fourier modes. When we differentiate with respect to the variables  $\zeta$ , the quantities appearing will be considered as coefficients (depending smoothly on  $\zeta$ , with exponential decay) of the functions  $\psi_l$  or their derivatives in  $\bar{y}$ .

We recall that the functions  $w_0$  and  $(g_i)_i$  in (87) are shifted in the  $\zeta'$  variable by the (smooth) normal section  $\Phi(\bar{y})$ . Hence, when differentiating with respect to  $\bar{y}$ , the derivatives of  $\Phi$  might appear through the chain rule, see also Section 3.2. This fact will be assumed understood, and it will not be mentioned anymore since it does not create any serious difficulty, or any difference in the estimates.

By our construction of  $\Psi_l$ , all the terms multiplying powers of  $\varepsilon$  less or equal than 2 reduce to

$$\varepsilon^2 \frac{C_0}{C_1} \mu_l (-\Delta_\zeta (\psi_l^h \partial_h w_0) + \psi_l^h \partial_h w_0) = \varepsilon^2 p \frac{C_0}{C_1} \mu_l w_0^{p-1} \partial_h w_0 \psi_l^h,$$

so we are left to consider the powers (of  $\varepsilon$ ) of order 3 and higher. In the remainder of the proof, we use the symbol  $\mathcal{A}_2(\varepsilon)$  to denote terms of order 1,  $\varepsilon$  or  $\varepsilon^2$ : since they all generate a single term, we do not need to compute them separately.

We begin by considering the terms where derivatives in  $\bar{y}$  appear. Since  $\mathfrak{S}_\varepsilon$  is linear in  $u$ , we can deal with each summand in  $\Psi_l$  separately. Looking at  $-\sqrt{\det g} \Delta_g(\psi_l^h(\bar{y})\partial_h w_0)$ , second derivatives in  $\bar{y}$  appear only in the expression  $-\sqrt{\det g} g^{ab} u_{ab}$ , so from Lemma 3.3 and Remark 3.4(b) we find that

$$-\sqrt{\det g} \Delta_g(\psi_l^h(\bar{y})\partial_h w_0) = \mathcal{A}_2(\varepsilon) - 2\varepsilon^3(\zeta_i \Gamma_a^b(E_i) - \zeta_{n+1} H_{ab} + \zeta_{n+1} H_\alpha^\alpha)(\partial_{\bar{y}_a \bar{y}_b}^2 \psi_l^h) \partial_h w_0 + \varepsilon^3 L_1 \psi_l + \varepsilon^4 L_2 \psi_l,$$

where  $L_1, L_2$  are as in the statement of the lemma.

Similarly one finds

$$\begin{aligned} -\sqrt{\det g} \Delta_g(\varepsilon \psi_l^h(\bar{y}) g_3^h(\bar{y}, \zeta)) &= \mathcal{A}_2(\varepsilon) - \varepsilon^3 \partial_{\bar{y}_a \bar{y}_a}^2 \psi_l^h \partial_h w_1 + \varepsilon^3 L_1 \psi_l + \varepsilon^4 L_2 \psi_l; \\ -\sqrt{\det g} \Delta_g(\varepsilon^2 \psi_l^h(\bar{y}) g_2^h(\bar{y}, \zeta)) &= \mathcal{A}_2(\varepsilon) + \varepsilon^4 L_2 \psi_l + \varepsilon^3 L_1 \psi_l; \\ -\sqrt{\det g} \Delta_g(\varepsilon^2 \mu_l \psi_l^h(\bar{y}) g_0^h(\bar{y}, \zeta)) &= \mathcal{A}_2(\varepsilon) - \varepsilon^4 (\partial_{\bar{y}_a \bar{y}_a}^2 \psi_l^h) g_0 + \varepsilon^4 \mu_l L_1 \psi_l + \varepsilon^5 \mu_l L_2 \psi_l; \\ -\sqrt{\det g} \Delta_g(\varepsilon^2 (\partial_{\bar{y}_a} \psi_l^h(\bar{y})) g_1^h(\bar{y}, \zeta)) &= \mathcal{A}_2(\varepsilon) + \varepsilon^4 L_3 \psi_l. \end{aligned}$$

At this point we are left with the terms (of order  $\varepsilon^3$  and higher) which do not involve derivatives of  $\psi_l$  in  $\bar{y}$ : these will appear as multipliers of the summands in the expression of  $\Psi_l$ . The ones involving  $\partial_h w_0, g_1, g_2$  and  $g_3$  are included in the expression  $\varepsilon^3 L_1 \psi_l$ , so it remains to consider  $\varepsilon^2 \mu_l \psi_l^h g_0^h$ . Recalling that  $\sqrt{\det g} = 1 + \varepsilon \zeta_n H_\alpha^\alpha + O(\varepsilon^2)$  (see the proof of Lemma 3.3), and expanding  $-p u_{l,\varepsilon}^{p-1}$  as

$$-p \left[ w_0^{p-1} + \varepsilon(p-1) w_0^{p-2} w_1 + \varepsilon^2(p-1) w_0^{p-2} w_2 + \frac{1}{2} \varepsilon^2(p-1)(p-2) w_0^{p-3} w_1^2 \right] + O(\varepsilon^3),$$

we obtain

$$\begin{aligned} \sqrt{\det g} (1 - p u_{l,\varepsilon}^{p-1}) \varepsilon^2 \mu_l \psi_l^h g_0^h &= \mathcal{A}_2(\varepsilon) + \varepsilon^3 \zeta_{n+1} H_\alpha^\alpha \mu_l \psi_l^h g_0^h (1 - p w_0^{p-1}) \\ &\quad - \varepsilon^3 p(p-1) w_0^{p-2} w_1 \mu_l \psi_l^h g_0^h + \varepsilon^4 \mu_l L_0 \psi_l, \end{aligned}$$

where  $L_0$  is a multiplication operator with coefficients also satisfy (112). This concludes the proof of the lemma.  $\square$

Next, using the above characterization, if  $\hat{u}_2$  is a suitable linear combination of the  $\Psi_l$ 's, we can estimate the scalar products of  $T_{\Sigma_\varepsilon} \hat{u}_2$  (in  $H_{\Sigma_\varepsilon}$ ) with some other elements belonging to the subspaces  $H_1, \hat{H}_2, \tilde{H}_2$  and  $H_3$ , see (88)–(90).

**Lemma 6.2.** *For some arbitrary real coefficients  $(\alpha_l)_l$  and  $(\beta_l)_l$ , we consider functions  $u_1 \in H_1, \hat{u}_2 \in \hat{H}_2$  and  $\tilde{u}_2 \in \tilde{H}_2$  of the form*

$$u_1 = \sum_{j=0}^{\infty} \alpha_j \phi_j(\varepsilon y) u_{j,\varepsilon}(|\zeta|); \quad \hat{u}_2 = \sum_{l=0}^{\varepsilon^{-\delta}} \beta_l \Psi_l; \quad \tilde{u}_2 = \sum_{\varepsilon^{-\delta}+1}^{\bar{C}_\varepsilon^{-k}} \beta_l \psi_l^m(\varepsilon y) \hat{v}_{l,\varepsilon,m}(\zeta).$$

We also let  $u_3 \in H_3$ . Then, for  $\delta \in (\frac{k}{2} + \gamma, \frac{2}{3}k - \gamma)$  and  $\gamma$  sufficiently small, we have the following relations

$$(T_{\Sigma_\varepsilon} \hat{u}_2, u_1)_{H_{\Sigma_\varepsilon}} = o(\varepsilon^2) \left( \frac{1}{\varepsilon^k} \sum_{l=0}^{\varepsilon^{-\delta}} |\mu_l| \beta_l^2 \right)^{\frac{1}{2}} \|u_1\|_{H_{\Sigma_\varepsilon}}; \tag{114}$$

$$(T_{\Sigma_\varepsilon} \hat{u}_2, \hat{u}_2)_{H_{\Sigma_\varepsilon}} = C_0(1 + o_\varepsilon(1)) \frac{1}{\varepsilon^k} \sum_{l=0}^{\varepsilon^{-\delta}} \varepsilon^2 \mu_l \beta_l^2; \tag{115}$$

$$\begin{aligned} (T_{\Sigma_\varepsilon} \hat{u}_2, \tilde{u}_2)_{H_{\Sigma_\varepsilon}} &= O(\varepsilon^3) \frac{1}{\varepsilon^k} \left( \sum_{l=0}^{\varepsilon^{-\delta}} (\mu_l^2 + \varepsilon^2 \mu_l^4) \beta_l^2 \right)^{\frac{1}{2}} \left( \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_l^2 \right)^{\frac{1}{2}} \\ &= o(\varepsilon^{\frac{4}{3}}) \|\hat{u}_2\|_{H_{\Sigma_\varepsilon}} \|\tilde{u}_2\|_{H_{\Sigma_\varepsilon}}; \end{aligned} \tag{116}$$

$$(T_{\Sigma_\varepsilon} \hat{u}_2, u_3)_{H_{\Sigma_\varepsilon}} = O(1) \|u_3\|_{H_{\Sigma_\varepsilon}} \left( \frac{1}{\varepsilon^k} \sum_{l=0}^{\varepsilon^{-\delta}} (\varepsilon^6 \mu_l^2 + \varepsilon^8 \mu_l^4) \beta_l^2 \right)^{\frac{1}{2}}. \tag{117}$$

**Proof.** We recall that, by Lemma 5.1, (79), (91) and (97) there holds

$$\begin{aligned} \|u_1\|_{H_{\Sigma_\varepsilon}}^2 &= \frac{1 + o_\varepsilon(1)}{\varepsilon^k} \sum_{j=0}^{\infty} \alpha_j^2; & \|\hat{u}_2\|_{H_{\Sigma_\varepsilon}}^2 &= \frac{1 + o_\varepsilon(1)}{\varepsilon^k} \|\partial_1 w_0\|_{H^1(\mathbb{R}_+^{n+1})}^2 \sum_{l=0}^{\varepsilon^{-\delta}} \beta_l^2; \\ \|\tilde{u}_2\|_{H_{\Sigma_\varepsilon}}^2 &= \frac{1 + o_\varepsilon(1)}{\varepsilon^k} \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_l^2. \end{aligned} \tag{118}$$

We show first (114). Since  $u_1$  is even in  $\zeta'$ , when we use the expression of  $\mathfrak{S}_\varepsilon(\Psi_l)$  in (113) we have to consider only  $-2\varepsilon^3 \zeta_i \Gamma_a^b(E_i) \partial_{\bar{y}_a \bar{y}_b}^2 \psi_l^h \partial_j w_0 = \varepsilon^3 L_2 \psi_l$  and the errors  $\varepsilon^\delta L_1 \psi_l$ , since the products of all the others terms with  $u_1$  will vanish by oddness. Therefore we leave this term as it is, and we estimate the error terms only. So we get

$$\begin{aligned} (T_{\Sigma_\varepsilon} \hat{u}_2, u_1)_{H_{\Sigma_\varepsilon}} &= \frac{1}{\varepsilon^k} \sum_{j,l} \alpha_j \beta_l \int_K \int_{\mathbb{R}_+^{n+1}} u_{j,\varepsilon}(|\zeta|) \phi_j(\bar{y}) \\ &\quad \times (\varepsilon^3 L_2 \psi_l + \varepsilon^4 L_3 \psi_l + \varepsilon^4 \mu_l L_1 \psi_l + \varepsilon^5 \mu_l L_2 \psi_l) d\bar{y} d\zeta. \end{aligned}$$

Reasoning as in Lemma 5.7 (avoiding the scaling in  $\varepsilon$ , which has been already taken care of) one can show that, for any integer  $m$

$$\int_K \int_{\mathbb{R}_+^{n+1}} \left( \sum_{l=0}^{\varepsilon^{-\delta}} \beta_l L_m \psi_l \right)^2 \leq C \sum_{l=0}^{\varepsilon^{-\delta}} (1 + |\mu_l|^m \beta_l^2). \tag{119}$$

From the Hölder inequality and the last three formulas we deduce that

$$(T_{\Sigma_\varepsilon} \hat{u}_2, u_1)_{H_{\Sigma_\varepsilon}} \leq C \|u_1\|_{H_{\Sigma_\varepsilon}} \left[ \frac{1}{\varepsilon^k} \sum_{l=0}^{\varepsilon^{-\delta}} (\varepsilon^6(1 + |\mu_l|^2) + \varepsilon^8|\mu_l|^3 + \varepsilon^{10}|\mu_l|^4) \beta_l^2 \right]^{\frac{1}{2}}.$$

Now, from the Weyl’s asymptotic formula and from the fact that  $\delta \in (\frac{k}{2} + \gamma, \frac{2}{3}k - \gamma)$ , one finds that for  $l \leq \varepsilon^{-\delta}$  there holds  $\varepsilon^2|\mu_l|^2 = o_\varepsilon(1)|\mu_l|$ , that  $\varepsilon^4|\mu_l|^3 = o_\varepsilon(1)$  and that  $\varepsilon^6|\mu_l|^4 = o_\varepsilon(1)$ , so (114) follows.

We turn now to (115). It is convenient first to evaluate some  $L^2$  norms. Writing  $\mathfrak{S}_\varepsilon(\Psi_l) = \varepsilon^2 p \frac{C_0}{C_1} \mu_l w_0^{p-1} \partial_h w_0 \psi_l^h + \tilde{\mathfrak{S}}_\varepsilon(\Psi_l)$ , and  $\Psi_l = \chi_\varepsilon(|\zeta|) \psi_l^h \partial_h w_0 + \bar{\Psi}_l$ , from (119) we find ( $l$  runs between 0 and  $\varepsilon^{-\delta}$ )

$$\left\| \sum \beta_l \Psi_l \right\|_{L^2}^2, \left\| \sum \beta_l \psi_l^h \partial_h w_0 \right\|_{L^2}^2 \leq \frac{C}{\varepsilon^k} \sum_l (1 + \varepsilon^2 + \varepsilon^4|\mu_l|^2) \beta_l^2 \leq \frac{C}{\varepsilon^k} \sum_l \beta_l^2; \tag{120}$$

$$\left\| \sum \beta_l \bar{\Psi}_l \right\|_{L^2}^2 \leq \frac{C}{\varepsilon^k} \sum_l (\varepsilon^2 + \varepsilon^4|\mu_l|^2) \beta_l^2 \leq \frac{C}{\varepsilon^k} \varepsilon^2 \sum_l (1 + \varepsilon^2 \mu_l^2) \beta_l^2; \tag{121}$$

$$\left\| \sum \beta_l \mathfrak{S}_\varepsilon(\Psi_l) \right\|_{L^2}^2 \leq \frac{C}{\varepsilon^k} \sum_l (\varepsilon^4|\mu_l|^2 + \varepsilon^6|\mu_l|^2 + \varepsilon^8|\mu_l|^4 + \varepsilon^{10}|\mu_l|^4) \beta_l^2 \leq \frac{C}{\varepsilon^k} \varepsilon^4 \sum_l \mu_l^2 \beta_l^2; \tag{122}$$

$$\left\| \sum \beta_l \tilde{\mathfrak{S}}_\varepsilon(\Psi_l) \right\|_{L^2}^2 \leq \frac{C}{\varepsilon^k} \sum_l (\varepsilon^6|\mu_l|^2 + \varepsilon^8|\mu_l|^4 + \varepsilon^{10}|\mu_l|^4) \beta_l^2 \leq \frac{C}{\varepsilon^k} \varepsilon^6 \sum_l (|\mu_l|^2 + \varepsilon^2|\mu_l|^4) \beta_l^2. \tag{123}$$

Using the orthogonality of the  $\psi_l$ ’s, (65) and recalling the definition of  $C_1$  in Section 5.2, we find

$$(T_{\Sigma_\varepsilon}(\Psi_l), \Psi_j)_{H_{\Sigma_\varepsilon}} = \varepsilon^2 C_0 \mu_l \delta_{lj} + (\tilde{\mathfrak{S}}_\varepsilon(\Psi_l), \psi_j^h \partial_h w_0)_{L^2} + (\mathfrak{S}_\varepsilon(\Psi_l), \bar{\Psi}_j)_{L^2}. \tag{124}$$

Multiplying by the coefficients  $\beta$ ’s, using the Hölder inequality and (120)–(123) we get

$$(T_{\Sigma_\varepsilon} \hat{u}_2, \hat{u}_2)_{H_{\Sigma_\varepsilon}} = C_0 \sum_l \varepsilon^2 \mu_l \beta_l^2 + \frac{1}{\varepsilon^k} O(\varepsilon^3) \left[ \left( \sum_l (\mu_l^2 + \varepsilon^2 \mu_l^4) \beta_l^2 \right)^{\frac{1}{2}} \left( \sum_l \beta_l^2 \right)^{\frac{1}{2}} + \left( \sum_l \mu_l^2 \beta_l^2 \right)^{\frac{1}{2}} \left( \sum_l (1 + \varepsilon^2 \mu_l^2) \beta_l^2 \right)^{\frac{1}{2}} \right].$$

Recalling the Weyl’s asymptotic formula and the fact that  $\delta \in (\frac{k}{2} + \gamma, \frac{2}{3}k - \gamma)$ , we obtain  $\varepsilon^2 \mu_l^2 = o(\mu_l)$ ,  $\varepsilon^4 \mu_l^4 = o(\mu_l)$  for  $l \leq \varepsilon^{-\delta}$ , so the last formula implies (115).

To prove (116) we notice that, by the orthogonality of the  $\psi_l$ 's, the term of order  $\varepsilon^2$  in  $\mathfrak{S}_\varepsilon(\Psi_l)$ , once multiplied by  $\tilde{u}_2$  and integrated, vanishes identically. Therefore, from the Hölder inequality, (118) and (123) we find

$$(T_{\Sigma_\varepsilon} \hat{u}_2, \tilde{u}_2)_{H_{\Sigma_\varepsilon}} = O(\varepsilon^3) \frac{1}{\varepsilon^k} \left( \sum_{l=0}^{\varepsilon^{-\delta}} (\mu_l^2 + \varepsilon^2 \mu_l^4) \beta_l^2 \right)^{\frac{1}{2}} \left( \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_l^2 \right)^{\frac{1}{2}},$$

which is precisely (116).

It remains to prove (117). Using (42), the formulas in the proof of Lemma 3.3 and the fact that (linearizing (3) at  $w_0$ )  $-\Delta_\zeta(\partial_h w_0) + \partial_h w_0 = p w_0^{p-1} \partial_h w_0$ , one finds

$$\begin{aligned} \sqrt{\det g_\varepsilon}(-\Delta_{g_\varepsilon} \Psi_l + \Psi_l) &= p w_0^{p-1} \psi_l^h \partial_h w_0 + \varepsilon L_0 \psi_l + \varepsilon^2 (L_2 \psi_l + \mu_l L_0 \psi_l) + \varepsilon^3 L_2 \psi_l \\ &\quad + \varepsilon^4 (\mu_l L_2 \psi_l + L_3 \psi_l). \end{aligned} \tag{125}$$

Hence from (113) it follows that

$$\begin{aligned} \mathfrak{S}_\varepsilon(\Psi_l) &= \varepsilon^2 \frac{C_0}{C_1} \frac{\mu_l}{p} \sqrt{\det g_\varepsilon}(-\Delta_{g_\varepsilon} \Psi_l + \Psi_l) + \varepsilon^3 \mu_l L_0 \psi_l + \varepsilon^4 \mu_l (L_2 \psi_l + \mu_l L_0 \psi_l) \\ &\quad + \varepsilon^5 \mu_l L_2 \psi_l + \varepsilon^6 \mu_l (\mu_l L_2 \psi_l + L_3 \psi_l) + \tilde{\mathfrak{S}}_\varepsilon(\Psi_l). \end{aligned}$$

Since  $u_3$  is orthogonal to  $\hat{H}_2$  in  $H_{\Sigma_\varepsilon}$ , integrating by parts we have  $\int_{\Sigma_\varepsilon} u_3 (-\Delta_{g_\varepsilon} \Psi_l + \Psi_l) \times \sqrt{\det g_\varepsilon} dy d\zeta = 0$  for  $l = 0, \dots, \varepsilon^{-\delta}$ . Hence from (119) and (123) we get

$$(T_{\Sigma_\varepsilon} \hat{u}_2, u_3)_{H_{\Sigma_\varepsilon}} = O(1) \|u_3\|_{H_{\Sigma_\varepsilon}} \left( \frac{1}{\varepsilon^k} \sum_{l=0}^{\varepsilon^{-\delta}} (\varepsilon^6 \mu_l^2 + \varepsilon^8 \mu_l^4 + \varepsilon^{12} \mu_l^6) \beta_l^2 \right)^{\frac{1}{2}}.$$

As shown before,  $\varepsilon^2 \mu_l^2 = o_\varepsilon(1)$  for  $l \leq \varepsilon^{-\delta}$ , so we have  $\varepsilon^{12} \mu_l^6 = o(\varepsilon^8 \mu_l^4)$ , and the conclusion holds.  $\square$

We have now the counterpart of Lemma 6.2 with  $\tilde{u}_2$  replacing  $\hat{u}_2$ .

**Lemma 6.3.** *For some arbitrary real coefficients  $(\alpha_l)_l$  and  $(\beta_l)_l$ , we consider functions  $u_1 \in H_1$ ,  $\hat{u}_2 \in \hat{H}_2$  and  $\tilde{u}_2 \in \tilde{H}_2$  of the form*

$$u_1 = \sum_{j=0}^{\infty} \alpha_j \phi_j(\varepsilon y) u_{j,\varepsilon}(|\zeta|); \quad \hat{u}_2 = \sum_{l=0}^{\varepsilon^{-\delta}} \beta_l \Psi_l; \quad \tilde{u}_2 = \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_l \psi_l^m(\varepsilon y) \hat{v}_{l,\varepsilon,m}(\zeta).$$

Suppose also that  $u_3 \in H_3$ . Then, for  $\delta \in (\frac{k}{2} + \gamma, \frac{2}{3}k - \gamma)$  and  $\gamma$  sufficiently small, we have the following relations



$$(T_{\Sigma_\varepsilon} \tilde{u}_2, u_1)_{H_{\Sigma_\varepsilon}} = O(\varepsilon^{1-\gamma}) \|u_1\|_{H_{\Sigma_\varepsilon}} \left( \frac{1}{\varepsilon^k} \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_l^2 \right)^{\frac{1}{2}}; \tag{126}$$

$$(T_{\Sigma_\varepsilon} \tilde{u}_2, \tilde{u}_2)_{H_{\Sigma_\varepsilon}} \geq \frac{C^{-1}}{\varepsilon^k} \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \varepsilon^2 \mu_l \beta_l^2; \tag{127}$$

$$(T_{\Sigma_\varepsilon} \tilde{u}_2, u_3)_{H_{\Sigma_\varepsilon}} = O(\varepsilon^{1-\gamma}) \|u_3\|_{H_{\Sigma_\varepsilon}} \left( \frac{1}{\varepsilon^k} \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_l^2 \right)^{\frac{1}{2}}. \tag{128}$$

**Proof.** We show first (126). Since  $u_1$  and  $\tilde{u}_2$ , for any fixed  $y$  are linear combinations of spherical harmonics (in  $\zeta/|\zeta|$ ) of different type, from the arguments of Section 4.2 it follows that

$$(u_1, \tilde{u}_2)_{H_{S_\varepsilon}} = 0; \quad \int_{S_\varepsilon} w_0^{p-1}(|\zeta|) u_1 \tilde{u}_2 dV_{\tilde{g}_\varepsilon} = 0,$$

so we clearly have that  $(T_{S_\varepsilon} u_1, \tilde{u}_2)_{H_{S_\varepsilon}} = 0$ . Then (126) follows immediately from Lemma 5.1.

To prove (127), we reason as for the proof of Lemma 5.6 to find

$$(T_{S_\varepsilon} \tilde{u}_2, w)_{H_{S_\varepsilon}} = \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3, \tag{129}$$

where  $w \in H_{S_\varepsilon}$  is arbitrary, and where

$$\begin{aligned} \tilde{A}_1(w) &= \int_{S_\varepsilon} \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \left[ (-\Delta_\zeta + (1 + \varepsilon^2 \omega_l) - p w_0^{p-1}) \left( \sum_{m=1}^n \beta_l \psi_l^m(\varepsilon y) \hat{v}_{l,\varepsilon}(|\zeta|) \frac{\zeta_m}{|\zeta|} \right) \right] w; \\ \tilde{A}_2(w) &= \varepsilon^2 \int_{S_\varepsilon} \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \left( \sum_{m=1}^n \beta_l ((\mathfrak{B} - \mathfrak{R}) \psi_l)^m(\varepsilon y) \hat{v}_{l,\varepsilon}(|\zeta|) \frac{\zeta_m}{|\zeta|} \right) w; \\ \tilde{A}_3(w) &= \varepsilon^2 \int_{S_\varepsilon} \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \left( \sum_{m=1}^n \beta_l (\mu_l - \omega_l) \psi_l^m(\varepsilon y) \hat{v}_{l,\varepsilon}(|\zeta|) \frac{\zeta_m}{|\zeta|} \right) w. \end{aligned}$$

As for (95), since  $|\mu_l - \omega_l|$  is uniformly bounded one finds

$$|\tilde{A}_2(w)| + |\tilde{A}_3(w)| \leq C \varepsilon^2 \|\tilde{u}_2\|_{H_{S_\varepsilon}} \|w\|_{H_{S_\varepsilon}} \tag{130}$$

for a fixed positive constant  $C$ . Taking  $w = \tilde{u}_2$ , by the orthogonality of the  $\psi_l$ 's, by the fact that  $T_{\varepsilon^2 \omega_l} v_{l,\varepsilon,m} = \sigma_{\varepsilon^2 \omega_l, \varepsilon} v_{l,\varepsilon,m}$  (see Proposition 4.5) and by (74), with an integration by parts we have

$$\tilde{A}_1(\tilde{u}_2) = \frac{1}{\varepsilon^k} \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \sigma_{\varepsilon^2 \omega_l, \varepsilon} \beta_l^2 \|v_{l,\varepsilon,1}\|_{\varepsilon^2 \omega_l, \varepsilon} = \frac{1}{\varepsilon^k} \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \sigma_{\varepsilon^2 \omega_l, \varepsilon} \beta_l^2.$$

From (28), Propositions 4.2 and 4.5, which provide estimates on  $\sigma_{\varepsilon^2\omega_l, \varepsilon}$ , we obtain

$$\tilde{A}_1(\tilde{u}_2) \geq \frac{C^{-1}}{\varepsilon^k} \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \varepsilon^2 \mu_l \beta_l^2 \tag{131}$$

for some fixed  $C > 0$ . Then (127) follows from (130), (131), Lemmas 5.6 and 5.1 (since  $\varepsilon^2 \mu_l \gg \varepsilon^{1-\gamma}$  for  $l > \varepsilon^{-\delta}$  and for  $\gamma$  sufficiently small).

We turn now to (128). By (130), taking  $w = u_3$ , it is sufficient to estimate  $\tilde{A}_1(u_3) + \tilde{A}_3(u_3)$ . From  $T_{\varepsilon^2\omega_l} v_{l, \varepsilon, m} = \sigma_{\varepsilon^2\omega_l, \varepsilon} v_{l, \varepsilon, m}$  in  $H_{\varepsilon^2\omega_l, \varepsilon}$ , with an integration by parts we find

$$\begin{aligned} \tilde{A}_1(u_3) + \tilde{A}_3(u_3) &= \int_{S_\varepsilon} \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \sigma_{\varepsilon^2\omega_l} \left[ (-\Delta_\zeta + (1 + \varepsilon^2 \mu_l) - p w_0^{p-1}) \right. \\ &\quad \left. \times \left( \sum_{m=1}^n \beta_l \psi_l^m(\varepsilon y) \hat{v}_{l, \varepsilon}(|\zeta|) \frac{\zeta_m}{|\zeta|} \right) \right] u_3. \end{aligned}$$

From (67) and from the fact that  $-\Delta_K^N \psi_l = \mu_l \psi_l + (\mathfrak{R} - \mathfrak{B}) \psi_l$ , one finds

$$\varepsilon^2 \mu_l \psi_l^m \hat{v}_{l, \varepsilon}(|\zeta|) \frac{\zeta_m}{|\zeta|} = -\varepsilon^2 \Delta_K^N \psi_l^m \hat{v}_{l, \varepsilon}(|\zeta|) \frac{\zeta_m}{|\zeta|} + \varepsilon^2 ((\mathfrak{R} - \mathfrak{B}) \psi_l)^m \hat{v}_{l, \varepsilon}(|\zeta|) \frac{\zeta_m}{|\zeta|}.$$

Therefore, integrating by parts we obtain

$$\tilde{A}_1(u_3) + \tilde{A}_3(u_3) = (\tilde{U}_2, u_3)_{H_{S_\varepsilon}} + \tilde{A}_4(u_3), \tag{132}$$

where

$$\tilde{A}_4(u_3) = \varepsilon^2 \int_{S_\varepsilon} \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \left( \sum_{m=1}^n \sigma_{\varepsilon^2\omega_l, \varepsilon} \beta_l ((\mathfrak{B} - \mathfrak{R}) \psi_l)^m(\varepsilon y) \hat{v}_{l, \varepsilon}(|\zeta|) \frac{\zeta_m}{|\zeta|} \right) u_3,$$

and where  $\tilde{U}_2 = \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \sigma_{\varepsilon^2\omega_l, \varepsilon} \beta_l \psi_l^m(\varepsilon y) \hat{v}_{l, \varepsilon, m}(\zeta) \in H_2$ . Now, as for  $\tilde{u}_2$  it is possible to prove that there exists a fixed  $C > 0$  such that

$$\|\tilde{U}_2\|_{H_{S_\varepsilon}}^2 \leq \frac{C}{\varepsilon^k} \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \sigma_{\varepsilon^2\omega_l, \varepsilon}^2 \beta_l^2 \leq \frac{C}{\varepsilon^k} \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_l^2,$$

where we used the fact that  $\sigma_{\varepsilon^2\omega_l, \varepsilon}$  is uniformly bounded for  $l \leq \bar{C}\varepsilon^{-k}$ . Since  $u_3$  is orthogonal in  $H_{S_\varepsilon}$  to  $H_2$ , from Lemma 5.1, these observations and the last two formulas it follows that

$$(\tilde{U}_2, u_3)_{H_{S_\varepsilon}} = O(\varepsilon^{1-\gamma}) \|\tilde{U}_2\|_{H_{S_\varepsilon}} \|u_3\|_{H_{S_\varepsilon}} \leq C \varepsilon^{1-\gamma} \left( \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_l^2 \right)^{\frac{1}{2}} \|u_3\|_{H_{S_\varepsilon}}.$$

The arguments of the proof of Lemma 5.6 yield  $\tilde{A}_4(u_3) \leq C\varepsilon^4(\sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_l^2)^{1/2} \|u_3\|_{H_{S_\varepsilon}}$ . Hence from (129), (132) and Lemma 5.1 we find that

$$(T_{\Sigma_\varepsilon} \tilde{u}_2, u_3)_{H_{S_\varepsilon}} = (\tilde{U}_2, u_3)_{H_{S_\varepsilon}} + O(\varepsilon^{1-\gamma}) \left( \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_l^2 \right)^{\frac{1}{2}} \|u_3\|_{H_{S_\varepsilon}},$$

which concludes the proof.  $\square$

### 6.2. Applications

In this subsection we apply the results in Lemmas 5.1, 6.2 and 6.3 to estimate the Morse index of  $T_{\Sigma_\varepsilon}$  as  $\varepsilon$  tends to zero, and to characterize the eigenfunctions of  $T_{\Sigma_\varepsilon}$  corresponding to *resonant* eigenvalues.

From Proposition 4.2 we know that there exists a unique positive number  $\bar{\alpha}$  such that  $\eta_{\bar{\alpha}} = 0$ . If  $C_k$  is the constant given in (27), we also let

$$\Theta = \left( \frac{\bar{\alpha}}{C_k} \right)^{\frac{k}{2}} \text{Vol}(K). \tag{133}$$

Then we have the following result.

**Proposition 6.4.** *Let  $\Theta$  be the constant given in (133), and let  $T_{\Sigma_\varepsilon}$  be the operator given in (81). Then, as  $\varepsilon$  tends to zero, the Morse index of  $T_{\Sigma_\varepsilon}$  is asymptotic to  $\Theta\varepsilon^{-k}$ .*

**Proof.** For any  $m \in \mathbb{N}$ , the  $m$ th eigenvalue  $\lambda_m$  of  $T_{\Sigma_\varepsilon}$ , and the  $m$ th eigenvalue  $\tilde{\lambda}_m$  of  $T_{S_\varepsilon}$  can be evaluated via the classical Rayleigh quotients

$$\lambda_m = \inf_{\dim M_m=m} \sup_{u \in M_m} \frac{(T_{\Sigma_\varepsilon} u, u)_{H_{\Sigma_\varepsilon}}}{(u, u)_{H_{\Sigma_\varepsilon}}}; \quad \tilde{\lambda}_m = \inf_{\dim M_m=m} \sup_{u \in M_m} \frac{(T_{S_\varepsilon} u, u)_{H_{S_\varepsilon}}}{(u, u)_{H_{S_\varepsilon}}}, \tag{134}$$

where  $M_m$  is a vector subspace of  $H_{\Sigma_\varepsilon}$ . Choosing  $M_m = \tilde{M}_m$  to be the span of the first  $m$  eigenfunctions of  $T_{S_\varepsilon}$ , from the above formula for  $\lambda_m$  and from Lemma 5.1 we get

$$\lambda_m \leq \sup_{u \in \tilde{M}_m} \frac{(T_{\Sigma_\varepsilon} u, u)_{H_{\Sigma_\varepsilon}}}{(u, u)_{H_{\Sigma_\varepsilon}}} = \sup_{u \in \tilde{M}_m} \frac{(T_{S_\varepsilon} u, u)_{H_{S_\varepsilon}} + O(\varepsilon^{1-\gamma})(u, u)_{H_{S_\varepsilon}}}{(1 + O(\varepsilon^{1-\gamma}))(u, u)_{H_{S_\varepsilon}}} \leq \tilde{\lambda}_m + O(\varepsilon^{1-\gamma}).$$

Reasoning in the same way we also find  $\tilde{\lambda}_m \leq \lambda_m + O(\varepsilon^{1-\gamma})$ , and hence it follows that

$$|\lambda_m - \tilde{\lambda}_m| \leq C\varepsilon^{1-\gamma} \quad \text{for all } m \in \mathbb{N} \text{ and for } \varepsilon \text{ small,} \tag{135}$$

where  $C > 0$  is a fixed constant.

Now we let  $N_1(\varepsilon)$  denote the number of eigenvalues  $\tilde{\lambda}_m$  less or equal than  $-\varepsilon^{(1-\gamma)/2}$ , and by  $N_2(\varepsilon)$  the number of eigenvalues  $\tilde{\lambda}_m$  less or equal than  $\varepsilon^{(1-\gamma)/2}$ . From Proposition 4.6 it

follows that  $N_1(\varepsilon)$  is the number of the  $\eta_{l,\varepsilon}$ 's which are smaller than  $-\varepsilon^{(1-\gamma)^2}$ . Reasoning as in Corollary 4.8 one finds that, as  $\varepsilon$  tends to zero

$$N_1(\varepsilon) \simeq \left(\frac{\bar{\alpha}}{C_k}\right)^{\frac{k}{2}} \text{Vol}(K)\varepsilon^{-k}.$$

On the other hand, still by Proposition 4.6 we have that  $N_2(\varepsilon) = N_{2,1}(\varepsilon) + N_{2,2}(\varepsilon)$ , where  $N_{2,1}(\varepsilon)$  is the number of  $\eta_{l,\varepsilon}$ 's which are smaller than  $\varepsilon^{(1-\gamma)/2}$ , and  $N_{2,2}(\varepsilon)$  the number of  $\sigma_{l,\varepsilon}$ 's which are smaller than  $\varepsilon^{(1-\gamma)/2}$ . From (27), (28) and Proposition 4.5 we obtain, for  $\varepsilon$  small

$$N_{2,1}(\varepsilon) \simeq \left(\frac{\bar{\alpha}}{C_k}\right)^{\frac{k}{2}} \text{Vol}(K)\varepsilon^{-k}; \quad N_{2,2}(\varepsilon) \simeq \left(\frac{1}{C_{N-1,k}}\right)^{\frac{k}{2}} \text{Vol}(K)\varepsilon^{\frac{k(1-\gamma)}{4}-k} = o(\varepsilon^{-k}).$$

From the last formula we deduce that also

$$N_2(\varepsilon) \simeq \left(\frac{\bar{\alpha}}{C_k}\right)^{\frac{k}{2}} \text{Vol}(K)\varepsilon^{-k}.$$

Since by (135) the Morse index of  $T_{\Sigma_\varepsilon}$  is between  $N_1(\varepsilon)$  and  $N_2(\varepsilon)$ , the conclusion follows.  $\square$

We can now characterize the eigenfunctions of  $T_{\Sigma_\varepsilon}$  corresponding to eigenvalues close to zero.

**Proposition 6.5.** *For  $\varepsilon$  sufficiently small, let  $\lambda$  be an eigenvalue of  $T_{\Sigma_\varepsilon}$  such that  $|\lambda| \leq \varepsilon^\varsigma$ , for some  $\varsigma > 2$ , and let  $u \in H_{\Sigma_\varepsilon}$  be an eigenfunction of  $T_{\Sigma_\varepsilon}$  corresponding to  $\lambda$  with  $\|u\|_{H_{\Sigma_\varepsilon}} = 1$ . In the above notation, let  $u = u_1 + u_2 + u_3$ , with  $u_i \in H_i$ ,  $i = 1, 2, 3$ . Then, if  $u_1 = \sum_{j=0}^\infty \alpha_j \phi_j(\varepsilon y) u_{j,\varepsilon}(|\zeta|)$ , one has*

$$\left\| u - \sum_{\{j: |\eta_{j,\varepsilon}| \leq \varepsilon^{(1-\gamma)/2}\}} \alpha_j \phi_j u_{j,\varepsilon} \right\|_{H_{\Sigma_\varepsilon}} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \tag{136}$$

**Proof.** We show that  $u_2, u_3$  tend to zero as  $\varepsilon$  tends to zero. This clearly implies  $\|u - u_1\|_{H_{\Sigma_\varepsilon}} \rightarrow 0$ . Once this verified, (136) can be proved as in [35, Proposition 4.1].

To prove that  $u_3$  tends to zero as  $\varepsilon \rightarrow 0$ , we take the scalar product of the eigenvalue equation  $T_{\Sigma_\varepsilon} u = \lambda u$  with  $u_3$ . Using the above arguments (in particular Lemma 5.1) we easily find

$$\frac{1}{C\bar{C}^{\frac{k}{2}}} \|u_3\|_{H_{\Sigma_\varepsilon}}^2 + O(\varepsilon^{1-\gamma}) \|u\|_{H_{\Sigma_\varepsilon}} \|u_3\|_{H_{\Sigma_\varepsilon}} \leq (T_{\Sigma_\varepsilon} u, u_3)_{H_{\Sigma_\varepsilon}} = \lambda (u, u_3)_{H_{\Sigma_\varepsilon}} = \lambda \|u_3\|_{H_{\Sigma_\varepsilon}}^2.$$

This implies

$$\|u_3\|_{H_{\Sigma_\varepsilon}}^2 = O(\varepsilon^{1-\gamma}) \|u\|_{H_{\Sigma_\varepsilon}} \|u_3\|_{H_{\Sigma_\varepsilon}},$$

and hence

$$\|u_3\|_{H_{\Sigma_\varepsilon}} \leq C\varepsilon^{1-\gamma} \|u\|_{H_{\Sigma_\varepsilon}} \leq C\varepsilon^{1-\gamma}.$$

Next we take the scalar product of the eigenvalue equation with  $u_2$ . From Lemmas 6.2 and 6.3 we find

$$(T_{\Sigma_\varepsilon} u_2, u_2)_{H_{\Sigma_\varepsilon}} \geq \frac{C_0(1 + o_\varepsilon(1))}{\varepsilon^k} \sum_{l=0}^{\varepsilon^{-\delta}} \varepsilon^2 \mu_l \beta_l^2 + \frac{O(1)}{\varepsilon^k} \left( \varepsilon^5 \sum_{l=0}^{\varepsilon^{-\delta}} (\mu_l^2 + \varepsilon^2 \mu_l^4) \beta_l^2 \right)^{\frac{1}{2}} \left( \varepsilon \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \beta_l^2 \right)^{\frac{1}{2}} + \frac{C^{-1}}{\varepsilon^k} \sum_{l=\varepsilon^{-\delta}+1}^{\bar{C}\varepsilon^{-k}} \varepsilon^2 \mu_l \beta_l^2.$$

Since  $\varepsilon^5 \mu_l^2 + \varepsilon^7 \mu_l^4 = o_\varepsilon(1) |\mu_l|$  for  $l \leq \varepsilon^{-\delta}$  and  $\varepsilon = o(\varepsilon^2 \mu_l)$  for  $l > \varepsilon^{-\delta}$  (recall that  $\delta \in (k/2 + \gamma, k - \gamma)$ ), it follows that

$$(T_{\Sigma_\varepsilon} u_2, u_2)_{H_{\Sigma_\varepsilon}} \geq C^{-1} \frac{1}{\varepsilon^k} \sum_{l=0}^{\bar{C}\varepsilon^{-k}} \varepsilon^2 \mu_l \beta_l^2 \tag{137}$$

for a fixed positive constant  $C$ . Finally, still from Lemmas 6.2, 6.3, from the fact that  $\varepsilon^4 |\mu_l| + \varepsilon^6 |\mu_l|^3 = o_\varepsilon(1)$  for  $l \leq \varepsilon^{-\delta}$  and  $\varepsilon^{2-2\gamma} = o(\varepsilon^2 \mu_l) \gg 1$  for  $l > \varepsilon^{-\delta}$  (taking  $\gamma$  sufficiently small) we have also that

$$(T_{\Sigma_\varepsilon} u_2, u_1 + u_3)_{H_{\Sigma_\varepsilon}} = o_\varepsilon(1) (\|u_1\|_{H_{\Sigma_\varepsilon}} + \|u_3\|_{H_{\Sigma_\varepsilon}}) \left( \frac{1}{\varepsilon^k} \sum_{l=0}^{\bar{C}\varepsilon^{-k}} \varepsilon^2 |\mu_l| \beta_l^2 \right)^{\frac{1}{2}}. \tag{138}$$

From (137) and (138) and the fact that  $T_{\Sigma_\varepsilon}$  is self-adjoint we deduce that

$$\begin{aligned} & \frac{C^{-1}}{\varepsilon^k} \sum_{l=0}^{\bar{C}\varepsilon^{-k}} \varepsilon^2 \mu_l \beta_l^2 + o_\varepsilon(1) \left( \sum_{l=0}^{\bar{C}\varepsilon^{-k}} \varepsilon^2 |\mu_l| \beta_l^2 \right)^{\frac{1}{2}} (\|u_1\|_{H_{\Sigma_\varepsilon}} + \|u_3\|_{H_{\Sigma_\varepsilon}}) \\ & \leq (T_{\Sigma_\varepsilon} u, u_2)_{H_{\Sigma_\varepsilon}} = \lambda(u, u_2)_{H_{\Sigma_\varepsilon}} \leq C \varepsilon^\zeta \|u\|_{H_{\Sigma_\varepsilon}} \|u_2\|_{H_{\Sigma_\varepsilon}}. \end{aligned}$$

Also, from Lemma 5.4, testing the eigenvalue equation on  $\sum_{l \leq l_0} \beta_l \Psi_l$ , where  $l_0$  is the biggest integer such that  $\mu_{l_0} < 0$ , one finds

$$\frac{1}{\varepsilon^k} \varepsilon^2 \sum_{l \leq l_0} \beta_l^2 |\mu_l| = O(\varepsilon^3) \|u\|_{H_{\Sigma_\varepsilon}}.$$

The last two formulas imply that  $\frac{1}{\varepsilon^k} \sum_{l=0}^{\bar{C}\varepsilon^{-k}} \beta_l^2 = o_\varepsilon(1)$ , namely that  $\|u_2\|_{H_{\Sigma_\varepsilon}}$  tends to zero as  $\varepsilon$  tends to zero. This concludes the proof.  $\square$

### 6.3. Proof of Theorem 1.1

Once Propositions 6.4 and 6.5 have been established, the proof goes as in [34, Section 8] (see also [33, Section 5]) and therefore we will limit ourselves to sketch the main ideas.

First of all, using Kato’s theorem, see [25, p. 445], one can prove that the eigenvalues of  $T_{\Sigma_\varepsilon}$  are differentiable with respect to  $\varepsilon$ , and if  $\lambda$  is such an eigenvalue, then there holds

$$\frac{\partial \lambda}{\partial \varepsilon} = \{\text{eigenvalues of } Q_\lambda\}, \tag{139}$$

where  $Q_\lambda : H_\lambda \times H_\lambda \rightarrow \mathbb{R}$  is the quadratic form given by

$$Q_\lambda(u, v) = (1 - \lambda) \frac{2}{\varepsilon} \int_{\Sigma_\varepsilon} \nabla u \cdot \nabla v - p(p - 1) \int_{\Sigma_\varepsilon} uvu_{I,\varepsilon}^{p-2} \left( \frac{\partial \bar{u}_{I,\varepsilon}}{\partial \varepsilon} \right) (\varepsilon \cdot). \tag{140}$$

Here  $H_\lambda \subseteq H_{\Sigma_\varepsilon}$  stands for the eigenspace of  $T_{\Sigma_\varepsilon}$  corresponding to  $\lambda$  and the function  $\bar{u}_{I,\varepsilon} : \Omega \rightarrow \mathbb{R}$  is defined by the scaling  $\bar{u}_{I,\varepsilon}(x) = u_{I,\varepsilon}(\varepsilon x)$ , where  $u_{I,\varepsilon}$  is as in Section 3. Notice that, since  $\lambda$  might have multiplicity bigger than 1, when we vary  $\varepsilon$  this eigenvalue can split into a multiplet, which is allowed by formula (139).

Taking  $\lambda$  as in Proposition 6.5, we can apply (139), and evaluate the quadratic form in (140) on the couples of eigenfunctions in  $H_\lambda$ , which are characterized by (136). Reasoning as in [33], Proposition 5.1 one can prove the following result.

**Proposition 6.6.** *Let  $\lambda$  be as in Proposition 6.5. Then for  $\varepsilon$  small one has*

$$\frac{\partial \lambda}{\partial \varepsilon} = \frac{1}{\varepsilon} (\bar{F} + o_\varepsilon(1)),$$

where  $\bar{F}$  is a positive constant depending on  $N, k$  and  $p$ .

Now we are in position to prove the following proposition, which states the invertibility of  $T_{\Sigma_\varepsilon}$  for suitable values of  $\varepsilon$ .

**Proposition 6.7.** *For a suitable sequence  $\varepsilon_j \rightarrow 0$ , the operator  $J''_\varepsilon(u_{I,\varepsilon}) : H^1(\Omega_\varepsilon) \rightarrow H^1(\Omega_\varepsilon)$  is invertible and the inverse operator satisfies*

$$\|J''_{\varepsilon_j}(u_{I,\varepsilon_j})^{-1}\|_{H^1(\Omega_{\varepsilon_j})} \leq \frac{C}{\min\{\varepsilon_j^k, \varepsilon_j^5\}},$$

for all  $j \in \mathbb{N}$ .

**Proof.** From Proposition 6.4 we have that, letting  $N_\varepsilon$  denote the Morse index of  $T_{\Sigma_\varepsilon}$ , there holds  $N_\varepsilon \simeq (\bar{\alpha}/C_k)^{k/2} \text{Vol}(K)\varepsilon^{-k}$ . For  $l \in \mathbb{N}$ , let  $\varepsilon_l = 2^{-l}$ . Then we have

$$N_{\varepsilon_{l+1}} - N_{\varepsilon_l} \simeq \left(\frac{\bar{\alpha}}{C_k}\right)^{\frac{k}{2}} \text{Vol}(K)(2^{k(l+1)} - 2^{kl}) \simeq \left(\frac{\bar{\alpha}}{C_k}\right)^{\frac{k}{2}} \text{Vol}(K)(2^k - 1)\varepsilon_l^{-k}. \tag{141}$$

By Proposition 6.6, the eigenvalues  $\lambda$  of  $T_{\Sigma_\varepsilon}$  with  $|\lambda| \leq \varepsilon^5$  are strictly monotone functions of  $\varepsilon$  so by the last equation the number of eigenvalues which cross 0, when  $\varepsilon$  decreases from  $\varepsilon_l$  to  $\varepsilon_{l+1}$ , is of order  $\varepsilon_l^{-k}$ . Now we define

$$A_l = \{\varepsilon \in (\varepsilon_{l+1}, \varepsilon_l) : \ker T_{\Sigma_\varepsilon} \neq \emptyset\}; \quad B_l = (\varepsilon_{l+1}, \varepsilon_l) \setminus A_l.$$

By Proposition 6.6 and (141) we deduce that  $\text{card}(A_l) < C\varepsilon_l^{-k}$ , and hence there exists an interval  $(a_l, b_l)$  such that

$$(a_l, b_l) \subseteq B_l; \quad |b_l - a_l| \geq C^{-1} \frac{\text{meas}(B_l)}{\text{card}(A_l)} \geq C^{-1} \varepsilon_l^{k+1}. \tag{142}$$

From Proposition 6.6, then it follows that every eigenvalue of  $T_{\Sigma_{(a_l+b_l)/2}}$  in absolute value is bigger than  $C^{-1} \min\{\varepsilon^k, \varepsilon^5\}$  for some  $C > 0$ . By Lemma 5.2 then the same is true for the eigenvalues of  $J''_\varepsilon(u_{l,\varepsilon})$  so the conclusion follows taking  $\varepsilon_j = (a_j + b_j)/2$ .  $\square$

**Remark 6.8.** The arguments in the proof of Proposition 6.5 can be easily adapted to the case in which  $|\lambda| \leq C^{-1}\varepsilon^2$  with  $C$  is sufficiently large. Therefore the result of Proposition 6.7 can be improved to  $\|J''_{\varepsilon_j}(u_{l,\varepsilon_j})^{-1}\|_{H^1(\Omega_{\varepsilon_j})} \leq C/\min\{\varepsilon_j^k, \varepsilon_j^2\}$ , for all  $j \in \mathbb{N}$ .

Below,  $\|\cdot\|$  denotes the standard norm of  $H^1(\Omega_\varepsilon)$ . For the values of  $\varepsilon$  such that  $J''_\varepsilon(u_{l,\varepsilon})$  is invertible, it is sufficient to apply the contraction mapping theorem. Writing  $\varepsilon = \varepsilon_j$ , we find a solution  $\tilde{u}_\varepsilon$  of  $(\tilde{P}_\varepsilon)$  in the form  $\tilde{u}_\varepsilon = u_{l,\varepsilon} + w$ , with  $w \in H^1(\Omega_\varepsilon)$  small in norm. Since  $J''_\varepsilon(u_{l,\varepsilon})$  is invertible we have that  $J'_\varepsilon(u) = 0$  if and only if  $w = -(J''_\varepsilon(u_{l,\varepsilon}))^{-1}[J'_\varepsilon(u_{l,\varepsilon}) + G(w)]$ , where

$$G(w) = J'_\varepsilon(u_{l,\varepsilon} + w) - J'_\varepsilon(u_{l,\varepsilon}) - J''_\varepsilon(u_{l,\varepsilon})[w].$$

Note that

$$G(w)[v] = - \int_{\Omega_\varepsilon} [(u_{l,\varepsilon} + w)^p - u_{l,\varepsilon}^p - pu_{l,\varepsilon}^{p-1}w]v; \quad v \in H^1(\Omega_\varepsilon).$$

Reasoning as in the last section of [35], we find the following estimates, which are based on elementary inequalities

$$\|G(w)\| \leq \begin{cases} C\|w\|^p & \text{for } p \leq 2, \\ C\|w\|^2 & \text{for } p > 2; \end{cases} \quad \|w\| \leq 1; \tag{143}$$

$$\|G(w_1) - G(w_2)\| \leq \begin{cases} C(\|w_1\|^{p-1} + \|w_2\|^{p-1})\|w_1 - w_2\|, & p \leq 2, \\ C(\|w_1\| + \|w_2\|)\|w_1 - w_2\|, & p > 2; \end{cases} \tag{144}$$

$$\|w_1\|, \|w_2\| \leq 1.$$

Defining  $F_\varepsilon : H^1(\Omega_\varepsilon) \rightarrow H^1(\Omega_\varepsilon)$  as

$$F_\varepsilon(w) = -(J''_\varepsilon(u_{l,\varepsilon}))^{-1}[J'_\varepsilon(u_{l,\varepsilon}) + G(w)], \quad w \in H^1(\Omega_\varepsilon),$$

we will show that  $F_\varepsilon$  is a contraction in some closed ball of  $H^1(\Omega_\varepsilon)$ . From (40), Proposition 6.7 (with Remark 6.8) and (143)–(144) we get

$$\|F_\varepsilon(w)\| \leq \begin{cases} C\varepsilon^{-(k+1)}(\varepsilon^{I+1-\frac{k}{2}} + \|w\|^p) & \text{for } p \leq 2, \\ C\varepsilon^{-(k+1)}(\varepsilon^{I+1-\frac{k}{2}} + \|w\|^2) & \text{for } p > 2; \end{cases} \quad \|w\| \leq 1; \tag{145}$$

$$\|F_\varepsilon(w_1) - F_\varepsilon(w_2)\| \leq \begin{cases} C\varepsilon^{-(k+1)}(\|w_1\|^{p-1} + \|w_2\|^{p-1})\|w_1 - w_2\|, & p \leq 2, \\ C\varepsilon^{-(k+1)}(\|w_1\| + \|w_2\|)\|w_1 - w_2\|, & p > 2; \end{cases}$$

$$\|w_1\|, \|w_2\| \leq 1. \tag{146}$$

Now we choose integers  $d$  and  $k$  such that

$$d > \begin{cases} \frac{k+1}{p-1} & \text{for } p \leq 2, \\ k+1 & \text{for } p > 2; \end{cases} \quad I > d - 1 + \frac{3}{2}k, \tag{147}$$

and we set

$$\mathcal{B} = \{w \in H^1(\Omega_\varepsilon): \|w\| \leq \varepsilon^d\}.$$

From (145)–(146) we deduce that  $F_\varepsilon$  is a contraction in  $\mathcal{B}$  for  $\varepsilon$  small, so the existence of a critical point  $\tilde{u}_\varepsilon$  of  $J_\varepsilon$  near  $u_{I,\varepsilon}$  follows. All the properties listed in Theorem 1.1, including the positivity of the solutions, follow from the construction of  $u_{I,\varepsilon}$  and standard arguments. As in [35], when  $p$  is supercritical one can use truncations and  $L^\infty$  estimates to apply the above argument working in the function space  $H^1(\Omega_\varepsilon) \cap L^\infty(\Omega_\varepsilon)$ .

**Remark 6.9.** With the arguments given in Section 5 we could obtain sharp estimates on the Morse index of  $T_{\Sigma_\varepsilon}$  and on the eigenfunctions corresponding to resonant eigenvalues. In particular about the latter we showed that the components in  $H_2, H_3$  are small, and that in  $H_1$  the Fourier modes are localized near some precise frequencies. This allowed us to prove Proposition 6.7 using Kato’s theorem.

Even if we did not work the computations out, it seems it should be possible to give a more rough characterization of these eigenfunctions (in particular on the  $H_2$  component) and to prove a (non-sharp) estimate on the derivatives of the eigenvalues, still obtaining invertibility. This might slightly simplify the proof of existence, although most of the delicate estimates will be shifted from the analysis of  $T_{\Sigma_\varepsilon}$  to that of the quadratic form  $Q_\lambda$  defined in (140).

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**References**

- [1] A. Ambrosetti, A. Malchiodi, *Perturbation Methods and Semilinear Elliptic Problems on  $\mathbb{R}^n$* , Progr. Math., vol. 240, Birkhäuser, 2005.
- [2] A. Ambrosetti, A. Malchiodi, W.M. Ni, Singularly perturbed elliptic equations with symmetry: Existence of solutions concentrating on spheres, part I, *Comm. Math. Phys.* 235 (2003) 427–466.
- [3] A. Ambrosetti, A. Malchiodi, W.M. Ni, Singularly perturbed elliptic equations with symmetry: Existence of solutions concentrating on spheres, part II, *Indiana Univ. Math. J.* 53 (2) (2004) 297–329.
- [4] T. Aubin, *Some Nonlinear Problems in Differential Geometry*, Springer Monogr. Math., Springer, Berlin, 1998.
- [5] M. Badiale, T. D’Aprile, Concentration around a sphere for a singularly perturbed Schrödinger equation, *Nonlinear Anal.* 49 (7) (2002) 947–985.
- [6] T. Bartsch, S. Peng, Semiclassical symmetric Schrödinger equations: Existence of solutions concentrating simultaneously on several spheres, preprint.



- [7] V. Benci, T. D'Aprile, The semiclassical limit of the nonlinear Schrödinger equation in a radial potential, *J. Differential Equations* 184 (1) (2002) 109–138.
- [8] H. Berestycki, P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, *Arch. Ration. Mech. Anal.* 82 (4) (1983) 313–345.
- [9] R.G. Casten, C.J. Holland, Instability results for reaction diffusion equations with Neumann boundary conditions, *J. Differential Equations* 27 (2) (1978) 266–273.
- [10] I. Chavel, *Riemannian Geometry, a Modern Introduction*, Cambridge Tracts in Math., vol. 108, Cambridge Univ. Press, Cambridge, 1993.
- [11] E.N. Dancer, S. Yan, Multipeak solutions for a singularly perturbed Neumann problem, *Pacific J. Math.* 189 (2) (1999) 241–262.
- [12] E.N. Dancer, S. Yan, A new type of concentration solutions for a singularly perturbed elliptic problem, preprint.
- [13] T. D'Aprile, On a class of solutions with non-vanishing angular momentum for nonlinear Schrödinger equations, *Differential Integral Equations* 16 (3) (2003) 349–384.
- [14] M. Del Pino, P. Felmer, J. Wei, On the role of the mean curvature in some singularly perturbed Neumann problems, *SIAM J. Math. Anal.* 31 (1999) 63–79.
- [15] M. Del Pino, M. Kowalczyk, J. Wei, Concentration at curves for nonlinear Schrödinger equations, *Comm. Pure Appl. Math.*, in press.
- [16] M. Do Carmo, *Riemannian Geometry*, Math. Theory Appl., Birkhäuser Boston, Boston, MA, 1992 (translated from the second Portuguese edition by Francis Flaherty).
- [17] A. Floer, A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential, *J. Funct. Anal.* 69 (1986) 397–408.
- [18] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* 68 (1979) 209–243.
- [19] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in  $R^n$ , in: *Mathematical Analysis and Applications*, part A, in: *Adv. Math. Suppl. Stud.*, vol. 7a, Academic Press, New York–London, 1981, pp. 369–402.
- [20] A. Gierer, H. Meinhardt, A theory of biological pattern formation, *Kybernetik (Berlin)* 12 (1972) 30–39.
- [21] M. Grossi, A. Pistoia, J. Wei, Existence of multipeak solutions for a semilinear Neumann problem via nonsmooth critical point theory, *Calc. Var. Partial Differential Equations* 11 (2) (2000) 143–175.
- [22] C. Gui, Multipeak solutions for a semilinear Neumann problem, *Duke Math. J.* 84 (1996) 739–769.
- [23] C. Gui, J. Wei, On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems, *Canad. J. Math.* 52 (3) (2000) 522–538.
- [24] C. Gui, J. Wei, M. Winter, Multiple boundary peak solutions for some singularly perturbed Neumann problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 17 (1) (2000) 47–82.
- [25] T. Kato, *Perturbation Theory for Linear Operators*, second ed., Grundlehren Math. Wiss., Band 132, Springer, Berlin–New York, 1976.
- [26] M.K. Kwong, Uniqueness of positive solutions of  $-\Delta u + u + u^p = 0$  in  $\mathbb{R}^n$ , *Arch. Ration. Mech. Anal.* 105 (1989) 243–266.
- [27] Peter Li, Shing-Tung Yau, On the Schrödinger equation and the eigenvalue problem, *Comm. Math. Phys.* 88 (1983) 309–318.
- [28] Y.Y. Li, On a singularly perturbed equation with Neumann boundary conditions, *Comm. Partial Differential Equations* 23 (1998) 487–545.
- [29] Y.Y. Li, L. Nirenberg, The Dirichlet problem for singularly perturbed elliptic equations, *Comm. Pure Appl. Math.* 51 (1998) 1445–1490.
- [30] C.-S. Lin, W.-M. Ni, I. Takagi, Large amplitude stationary solutions to a chemotaxis systems, *J. Differential Equations* 72 (1988) 1–27.
- [31] F. Mahmoudi, A. Malchiodi, Concentration at manifolds of arbitrary dimension for a singularly perturbed Neumann problem, *Rend. Mat. Acc. Lincei*, in press.
- [32] F. Mahmoudi, R. Mazzeo, F. Pacard, Constant mean curvature hypersurfaces condensing along a submanifold, *G.A.F.A.*, in press.
- [33] A. Malchiodi, Concentration at curves for a singularly perturbed Neumann problem in three-dimensional domains, *G.A.F.A.* 15–16 (2005) 1162–1222.
- [34] A. Malchiodi, M. Montenegro, Boundary concentration phenomena for a singularly perturbed elliptic problem, *Comm. Pure Appl. Math.* 15 (2002) 1507–1568.
- [35] A. Malchiodi, M. Montenegro, Multidimensional boundary-layers for a singularly perturbed Neumann problem, *Duke Math. J.* 124 (1) (2004) 105–143.

- [36] A. Malchiodi, W.M. Ni, J. Wei, Multiple clustered layer solutions for semilinear Neumann problems on a ball, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 22 (2005) 143–163.
- [37] H. Matano, Asymptotic behavior and stability of solutions of semilinear diffusion equations, *Publ. Res. Inst. Math. Sci.* 15 (2) (1979) 401–454.
- [38] R. Mazzeo, F. Pacard, Foliations by constant mean curvature tubes, *Comm. Anal. Geom.* 13 (4) (2005) 633–670.
- [39] S. Minakshisundaram, A. Pleijel, Some properties of the eigenfunctions of the Laplace operator on Riemannian manifolds, *Canad. J. Math.* 1 (1949) 242–256.
- [40] R. Molle, D. Passaseo, Concentration phenomena for solutions of superlinear elliptic problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 23 (1) (2006) 63–84.
- [41] W.M. Ni, Diffusion, cross-diffusion, and their spike-layer steady states, *Not. Amer. Math. Soc.* 45 (1) (1998) 9–18.
- [42] W.M. Ni, I. Takagi, On the shape of least-energy solution to a semilinear Neumann problem, *Comm. Pure Appl. Math.* 41 (1991) 819–851.
- [43] W.M. Ni, I. Takagi, Locating the peaks of least-energy solutions to a semilinear Neumann problem, *Duke Math. J.* 70 (1993) 247–281.
- [44] W.M. Ni, I. Takagi, E. Yanagida, Stability of least energy patterns of the shadow system for an activator–inhibitor model, in: *Recent Topics in Mathematics Moving toward Science and Engineering*, *Japan J. Indust. Appl. Math.* 18 (2) (2001) 259–272.
- [45] Y.-G. Oh, On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potentials, *Comm. Math. Phys.* 131 (1990) 223–253.
- [46] S.I. Pohožaev, On the eigenfunctions of the equation  $\Delta u + \lambda f(u) = 0$ , *Dokl. Akad. Nauk SSSR* 165 (1965) 36–39 (in Russian).
- [47] J. Shi, Semilinear Neumann boundary value problems on a rectangle, *Trans. Amer. Math. Soc.* 354 (8) (2002) 3117–3154.
- [48] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, second ed., Publish or Perish, Wilmington, Del., 1979.
- [49] Walter A. Strauss, Existence of solitary waves in higher dimensions, *Comm. Math. Phys.* 55 (2) (1977) 149–162.
- [50] A.M. Turing, The chemical basis of morphogenesis, *Philos. Trans. R. Soc. London Ser. B Biol. Sci.* 237 (1952) 37–72.
- [51] Z.-Q. Wang, On the existence of multiple, single-peaked solutions for a semilinear Neumann problem, *Arch. Ration. Mech. Anal.* 120 (1992) 375–399.
- [52] J. Wei, On the construction of single-peaked solutions to a singularly perturbed semilinear Dirichlet problem, *J. Differential Equations* 129 (1996) 315–333.
- [53] J. Wei, On the boundary spike layer solutions of a singularly perturbed semilinear Neumann problem, *J. Differential Equations* 134 (1997) 104–133.