On the automorphism group of the generalized conformal structure of a symmetric $R$-space*

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Abstract: In this paper we define a canonical locally flat generalized conformal structure on a symmetric $R$-space of the rank greater than 1. We prove that the group of automorphisms of this structure coincides with the noncompact group of automorphisms of the symmetric space.

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Introduction

In this paper we are concerned with (irreducible) symmetric $R$-spaces introduced by Nagano [14] and Takeuchi [17]. Symmetric $R$-spaces break up into two classes—complex symmetric $R$-spaces and real symmetric $R$-spaces. The latter ones are the real forms of the former ones which are no other than irreducible compact Hermitian symmetric spaces. Complex symmetric $R$-spaces have many remarkable properties—existence of invariant complex structures and invariant Kählerian structures. The (compact) automorphism group of the Kählerian structure can be extended up to the "big transformation group"—the noncompact group of holomorphic automorphisms and anti-holomorphic automorphisms. Also there is a duality between complex symmetric $R$-spaces and irreducible noncompact Hermitian symmetric spaces. A noncompact one can be realized as a homogeneous domain in the dual compact one.

All last properties (existence of the big transformation group containing the isometry group and existence of the dual homogeneous domain) are inherited by real symmetric $R$-spaces (Nagano [14], Takeuchi [17]). For example, the unit sphere $S^n$ in $\mathbb{R}^{n+1}$ is a real symmetric $R$-space. The orthogonal group $O(n+1)$ acts on it as isometries. The big transformation group is the Lorentz group $O(n+1, 1)$ acting on $S^n$ as conformal transformations. The noncompact dual of $S^n$ is the Lobatchevsky $n$-space, which is realized in $S^n$ as an open $O(n, 1)$-orbit. On the other hand, the correspondence between symmetric $R$-spaces and compact simple Jordan

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triple systems (JTS's) was established by Loos [12]. From the point of view of the split root systems for the associated affine symmetric pairs \((\mathfrak{g}, \mathfrak{g}_0)\) (cf. Sect. 1), symmetric \(R\)-spaces break up into two classes: those of type \(C\) and those of type \(BC\). The former ones correspond to simple Jordan algebras. Especially, a real symmetric \(R\)-space \(M\) of type \(C\) can be realized as the Šilov boundary of a simple irreducible pseudo-Hermitian symmetric domain in the complex symmetric \(R\)-space which is the complexification of \(M\). That pseudo-Hermitian symmetric domain admits a realization as a tube domain over (not necessarily convex) open cones either in the classical sense when it is Hermitian or in the sense of Kaneyuki [8] and D’Atri–Gindikin [3] when they are pseudo-Hermitian but not Hermitian. Real symmetric \(R\)-spaces of type \(BC\) do not correspond to simple Jordan algebras but simple JTS’s. Those symmetric \(R\)-spaces cannot be realized as the Šilov boundaries of simple irreducible pseudo-Hermitian symmetric domains. For example, let us consider the Cayley projective plane \(P^2(\mathbb{O}) = F_4/\text{Spin}(9)\), which is a real symmetric \(R\)-space of type \(BC\). \(P^2(\mathbb{O})\) admits \(E_{6(-26)}^\text{I}\) as the big transformation group, and it is realized as the real form of the complex symmetric \(R\)-space \(E_6/\text{spin}(10)-T^1\). Since there are no simple irreducible pseudo-Hermitian symmetric spaces of the group \(E_{6(-26)}\), \(P^2(\mathbb{O})\) is not the Šilov boundary of a simple irreducible pseudo-Hermitian symmetric domain.

Let \(\mathbb{F} = \mathbb{R}\) or \(\mathbb{C}\). By a generalized cone in \(\mathbb{F}^n\) we mean a conic (i.e, \(\mathbb{F}^*\)-stable) algebraic set \(V\) in \(\mathbb{F}^n\). Let \(G(V)\) be the group of \(\mathbb{F}\)-linear automorphisms of \(V\). If the group \(G(V)\) has an open orbit in \(\mathbb{F}^n\), then the cone \(V\) is called prehomogeneous. Let \(M\) be a \(\mathbb{F}\)-manifold. Let \(\mathcal{K}\) be a smooth assignment of prehomogeneous generalized cone \(V_p\) in the tangent space \(T_pM\) to each point \(p \in M\) such that all \(V_p\) are linearly equivalent. Then \(\mathcal{K}\) is called a generalized conformal structure (GCS) on \(M\). If a cone in \(\mathcal{K}\) is quadratic cone, then \(\mathcal{K}\) is the classical conformal structure. Let \(V\) be a generalized prehomogeneous cone in \(\mathbb{F}^n\). We assign to each point in \(\mathbb{F}^n\) the parallel transport of \(V\). The GCS on \(\mathbb{F}^n\) thus obtained is called the flat GCS on \(\mathbb{F}^n\) with typical cone \(V\), which is denoted by \(\mathcal{K}_{\text{fr}}\). Let \((M, \mathcal{K})\) be a manifold with GCS \(\mathcal{K}\). \(\mathcal{K}\) is called locally flat, if \((M, \mathcal{K})\) is locally equivalent to \((\mathbb{F}^n, \mathcal{K}_{\text{fr}})\).

In this paper we define a canonical locally flat GCS on a symmetric \(R\)-space \(M\), arising from the prehomogeneous cone \(\partial V_r\). We prove that the group of automorphisms of this GCS coincides with the big transformation group, provided that the rank of \(M\) is greater than 1 (Theorem 3.3). There are a few historical remarks. The origin of this problem is the classical result of Plücker which asserts that the manifold \(\text{Gr}(2, 4)\) of lines in the 3-dimensional projective space can be characterized as a compact conformally flat 4-manifold. Goncharov [4] solved the problem of the first author and defined locally flat GCS’s on complex symmetric \(R\)-spaces of rank greater than 1. Here we generalize this result to all symmetric \(R\)-spaces using another language. Let us remark that, in the case where a symmetric \(R\)-space is the Šilov boundary of a symmetric convex tube domain, the group of automorphisms of our GCS coincides with the group of automorphisms and anti-automorphisms of the causal structure defined by the second author [7].

Let us remark that there is another characterization of the big transformation group of a symmetric \(R\)-space in terms of the arithmetic distance (see Takeuchi [19]). By combining our result with that of Koecher [11] or Loos [13], it turns out that the big transformation group of a symmetric \(R\)-space is realized as the group of birational transformations on the tangent space at the origin leaving the flat GCS corresponding to \(\partial V_r\) invariant (Theorem 3.4). Theorem 3.4 can be viewed as a generalization of Liouville theorem for the classical conformal structure. In the case of a symmetric \(R\)-space of type \(C\), the essentially same result has been obtained by
1. Preliminaries

Let

\[ g = g_{-1} + g_0 + g_1 \]  

be a real simple graded Lie algebra (in short GLA). Let \((g, Z, \tau)\) be the associated graded triple. That is, \(Z \in g\) is the characteristic element of the GLA \(g\), which means that \(g_k (k = 0, \pm 1)\) is the \(k\)-eigenspace of the operator \(\text{ad} Z\), and \(\tau\) is a Cartan involution which satisfies \(\tau(Z) = -Z\). Let

\[ g = \mathfrak{k} + \mathfrak{h} \]  

be the Cartan decomposition of \(g\) by \(\tau\), where \(\tau|_{\mathfrak{k}} = 1\) and \(\tau|_{\mathfrak{h}} = -1\). Let \(\mathfrak{a}\) be a maximal abelian subspace of \(\mathfrak{h}\) containing \(Z\). Let \(\Delta\) be the root system for the pair \((g, \mathfrak{a})\), and let \(\Delta_\pm = \{\alpha \in \Delta : (\alpha, Z) = \pm 1\}\), where \((\cdot, \cdot)\) denotes the Killing form of \(g\). Then we have a partition of \(\Delta\):

\[ \Delta = \Delta_- \cup \Delta_0 \cup \Delta_+. \]  

Choose a linear order in \(\Delta\) in such a way that

\[ \Delta_1 \subset \Delta^+ \subset \Delta_0 \cup \Delta_1. \]  

where \(\Delta^+\) is the positive roots in \(\Delta\). One can choose a maximal system of strongly orthogonal roots \(\{\beta_1, \ldots, \beta_r\}\) in \(\Delta_+\) such that \(\beta_1\) is the highest root in \(\Delta\). Note that \(r\) is uniquely determined by the GLA \(g\), and that \(\beta_1, \ldots, \beta_r\) have the same length with respect to the inner product \((\cdot, \cdot)\).

Let

\[ \sigma = \text{Ad} \exp \pi i Z. \]  

Then \(\sigma\) is an involutive automorphism of \(g\), which is \(1\) on \(g_0\) and \(-1\) on \(m := g_{-1} + g_1\). Since \(\sigma\) and \(\tau\) commute, \(g\) is decomposed into four subspaces:

\[ g = \mathfrak{k}_0 + \mathfrak{m}_\tau + \mathfrak{n}_0 + \mathfrak{m}_\eta, \]  

where \(\mathfrak{k}_0 = g_0 \cap \mathfrak{k}, \mathfrak{n}_0 = g_0 \cap \mathfrak{n}, \mathfrak{m}_\tau = \mathfrak{m} \cap \mathfrak{k}\) and \(\mathfrak{m}_\eta = \mathfrak{m} \cap \mathfrak{n}\). Let \(g^\alpha\) be the root space for a root \(\alpha \in \Delta\). Choose a root vector \(E_i \in g^{\beta_i} \subset g_1 (1 \leq i \leq r)\) in such a way that

\[ [E_i, E_{-i}] = \frac{2}{(\beta_i, \beta_i)} \beta_i =: \tilde{\beta}_i, \]  

where \(E_{-i} = -\tau E_i \in g^{-\beta_i} \subset g_{-1}\). Let

\[ X_i = E_i + E_{-i} \in \mathfrak{m}_\eta, \]  

\[ c = \sum_{i=1}^r \mathbb{R} X_i. \]
We gather some well-known facts in a lemma.

**Lemma 1.1.** (1) \( c \) is a maximal abelian subspace in \( \mathfrak{m}_0 \), i.e., \( c \) is a split Cartan subalgebra of the symmetric triple \( (\mathfrak{g}, \mathfrak{g}_0, \sigma) \) ([5]).

(2) The root system \( \Delta(\mathfrak{g}, c) \) of the pair \( (\mathfrak{g}, c) \) (= the split root system of \( (\mathfrak{g}, \mathfrak{g}_0, \sigma) \)) is of type \( C \) or \( BC \) (Oshima-Sekiguchi [15]).

(3) \( \Delta(\mathfrak{g}, c) \) is of type \( C \), if and only if \( \mathfrak{g}_{-1} \) has the structure of a simple Jordan algebra ([9]).

(4) Let \( \mathfrak{g}^* := \mathfrak{e}_0 + \mathfrak{m}_0 \subset \mathfrak{g} \). Then \( \mathfrak{g}^* \) is a subalgebra of \( \mathfrak{g} \). The triple \( (\mathfrak{g}^*, \mathfrak{e}_0, \tau) \) is the noncompact dual of the symmetric triple \( (\mathfrak{e}, \mathfrak{e}_0, \sigma) \) (Nagano [14], Takeuchi [17]).

We consider the subgroups of the automorphism group \( \text{Aut} \mathfrak{g} \) of the Lie algebra \( \mathfrak{g} \) (cf. Takeuchi [19]):
- \( G_0 \), the group of grade-preserving automorphisms of the GLA \( \mathfrak{g} \);
- \( G := G_0 \text{ Inn } \mathfrak{g} \), which is an open subgroup of \( \text{Aut} \mathfrak{g} \);
- \( G' \), the Zariski connected component of \( \text{Aut} \mathfrak{g} \). Note that \( G' \subset G \);
- \( G'_0 := G_0 \cap G' \);
- \( G_0^0 \), the identity component of \( G_0 \) (or \( G'_0 \));
- \( K := \{ g \in G : g\tau = \tau g \} \), which is a maximal compact subgroup of \( G \). Note that \( \text{Lie} \ K = \mathfrak{e} \);
- \( K' := G' \cap K \);
- \( K_0 := G_0 \cap K \);
- \( K'_0 := G'_0 \cap K \);
- \( K^0_0 \), the identity component of \( K_0 \);
- \( G^* \), the analytic subgroup of \( G \) generated by \( \mathfrak{g}^* \);
- \( U := G_0 \exp \mathfrak{g}_1 \).

Let us consider the coset space

\[
M = G/U, \tag{1.9}
\]

which is a connected compact manifold, on which \( G \) acts effectively. \( M \) is called a symmetric \( R \)-space associated with the GLA \( \mathfrak{g} \). \( M \) has a symmetric coset space expression

\[
M = K/K_0. \tag{1.10}
\]

The group \( G \) is called the big transformation group of \( M \). Note that the number \( r \) is equal to the rank of the symmetric coset space (1.10).

**Lemma 1.2.** Let us consider the operator \( I = \text{ad}_m Z \) on \( \mathfrak{m}_0 \), and define a linear map \( \varphi : \mathfrak{m}_0 \to \mathfrak{g}_{-1} \) by

\[
\varphi(X) = \frac{1}{2}(X - IX), \quad X \in \mathfrak{m}_0. \tag{1.11}
\]

Then \( \varphi \) is a \( K^0_0 \)-isomorphism of \( \mathfrak{m}_0 \) onto \( \mathfrak{g}_{-1} \).

**Proof.** The inclusion \( \varphi(\mathfrak{m}_0) \subset \mathfrak{g}_{-1} \) follows from the equality \( I^2 = 1 \). The equality \( [\mathfrak{e}_0, Z] = 0 \) implies that \( \varphi \) commutes with the action of \( K^0_0 \). One can easily verify that the operator \( I \) interchanges \( \mathfrak{m}_0 \) with \( \mathfrak{m}_e \), from which it follows that \( \varphi \) is bijective. \( \square \)
Now let
\[ \mathfrak{a}_{-1} = \varphi(c) \subset \mathfrak{g}_{-1}. \] (1.12)
Then \( \mathfrak{a}_{-1} \) is spanned by \( E_{-1}, \ldots, E_{-r} \). Consider the Riemann symmetric space
\[ M^* = G^*/K_0^0, \] (1.13)
which is the noncompact dual of \( M \) (Lemma 1.1). Since \( c \) is a maximal abelian subspace of \( \mathfrak{m}_n \), one can consider the root system \( \Delta(g^*, c) \) for the pair \( (g^*, c) \). Let \( W(M^*) \) be the Weyl group for \( \Delta(g^*, c) \). Then we have a natural isomorphism
\[ W(M^*) \cong N_{K_0^0}(c)/C_{K_0^0}(c). \] (1.14)
when \( N_{K_0^0}(c) \) (resp. \( C_{K_0^0}(c) \)) is the normalizer (resp. centralizer) of \( c \) in \( K_0^0 \). \( W(M^*) \) acts on \( c \) as the signed permutations:
\[ X_i \mapsto \pm X_{\rho(i)}, \quad \rho \in \Sigma_r, \] (1.15)
where \( \Sigma_r \) denotes the permutation group of \( \{1, \ldots, r\} \). This action of \( W(M^*) \) is transferred onto \( \mathfrak{a}_{-1} \) by \( \varphi \):
\[ E_{-i} \mapsto \pm E_{\rho(i)}, \quad \rho \in \Sigma_r. \] (1.16)
Put
\[ O_{p,q} = \sum_{k=1}^{p} E_{-k} - \sum_{j=p+1}^{p+q} E_{-j} \in \mathfrak{a}_{-1} \subset \mathfrak{g}_{-1}, \] (1.17)
where \( p, q \geq 0, p + q \leq r \). Consider the union of \( G_0^0 \)-orbits in \( \mathfrak{g}_{-1} \):
\[ V_{\ell} = \bigcup_{p+q=\ell} G_0^0 \cdot O_{p,q} \subset \mathfrak{g}_{-1}. \quad 0 \leq \ell \leq r. \] (1.18)

**Lemma 1.3.** An element of \( \mathfrak{g}_{-1} \) can be transformed by the action of \( G_0^0 \) into at least one normal form \( O_{p,q} \), more precisely,
\[ \mathfrak{g}_{-1} = \bigcup_{\ell=0}^{r} V_{\ell}. \] (1.19)

**Proof.** (cf. Takeuchi [19]) By the conjugateness of Cartan subalgebras of \( M^* \), we have \( \mathfrak{m}_n = K_0^0 \mathfrak{c} \). Applying \( \varphi \) to the both sides, we get \( \mathfrak{g}_{-1} = K_0^0 \mathfrak{a}_{-1} \) (cf. Lemma 1.2). Let \( \Omega_\ell \) \( (0 \leq \ell \leq r) \) be the totality of elements \( X = \sum_{i=1}^{\ell} x_i E_{-i} \in \mathfrak{a}_{-1} \) such that the number of non-zero \( x_i \)'s is \( \ell \). Then we have \( \mathfrak{a}_{-1} = \bigcup_{\ell=0}^{r} \Omega_\ell \), and consequently
\[ \mathfrak{g}_{-1} = \bigcup_{\ell=0}^{r} K_0^0 \cdot \Omega_\ell. \] (1.20)
Let
\[ \mathfrak{a}_0 = \sum_{i=0}^{\infty} \mathbb{R} \beta_i \subset \mathfrak{a}. \] (1.21)
Note that \( \mathfrak{a} \subset \mathfrak{n}_0 \), since \( \mathfrak{z} \subset \mathfrak{n}_0 \). Therefore \( \mathfrak{a}_0 \subset \mathfrak{g}_0 \). By the strong orthogonality of \( \beta_i \)'s, the subgroup \( \exp \mathfrak{a}_0 \subset G_0^0 \) acts on \( \mathfrak{a}_{-1} \) as diagonal transformations with positive coefficients.
The Weyl group \( W(M^*) \) contains permutations among \( E_{-1}, \ldots, E_{-r} \). Therefore we have the expression
\[
\Omega_\ell = \bigcup_{p+q=\ell} (\exp a_0) W(M^*) O_{p,q}.
\]
Hence (1.19) follows from (1.20), (1.14) and (1.22). \( \square \)

2. The prehomogeneous cone \( \partial V_r \)

Let \( \varpi : a \rightarrow a_0 \) be the orthogonal projection of \( a \) onto \( a_0 \) with respect to the Killing form \( (\cdot, \cdot) \) of \( g \). Then it is known [18, 61] that
\[
\varpi (\Delta_1) = \left\{ \frac{1}{2} (\beta_i + \beta_j) : 1 \leq i \leq j \leq r \right\}
\]
or \( \left\{ \frac{1}{2} (\beta_i + \beta_j) (1 \leq i \leq j \leq r), \frac{1}{2} \beta_i (1 \leq i \leq r) \right\} \),
according as the split root system \( \Delta(g, c) \) is of type \( C \) or \( BC \).

Put
\[
a_{ij} = \sum_{\gamma \in \Delta_1 : \varpi (\gamma) = \frac{1}{2} (\beta_i + \beta_j)} g^{-\gamma}, \quad 1 \leq i \leq j \leq r,
\]
\[
c_i = \sum_{\gamma \in \Delta_1 : \varpi (\gamma) = \frac{1}{2} \beta_i} g^{-\gamma}, \quad 1 \leq i \leq r.
\]
Then we have the expression
\[
g_{-1} = \sum_{1 \leq i \leq j \leq r} a_{ij} + \sum_{1 \leq i \leq r} c_i,
\]
where the second term of the right-hand side appears only if \( \Delta(g, c) \) is of type \( BC \).

Lemma 2.1. ([5]) The dimensions of \( a_{ij} (i < j) \), \( a_{ii} \) and \( c_i \) are constants which do not depend on \( i \) and \( j \).

With the notation as at the beginning of Section 1, we define a triple product \( B_r \) on \( g_{-1} \) by putting
\[
B_r (X, Y, Z) = \frac{1}{2} [[\tau Y, X], Z], \quad X, Y, Z \in g_{-1}.
\]
Then it is known (Satake [16], Loos [12]) that the pair \( \mathfrak{B} := (g_{-1}, B_r) \) is a compact simple Jordan triple system (or shortly JTS) and that \( g \) is isomorphic to the Koecher–Kantor algebra for \( \mathfrak{B} \). (These two statements are contained in more general framework, i.e., for a simple GLA of the second kind and the corresponding generalized JTS; see [1, 10]).

For simplicity we write
\[
(XYZ) = B_r (X, Y, Z).
\]
As usual, we define the linear operator \( L_{XY} (X, Y \in g_{-1}) \) by
\[
L_{XY} (Z) = (XYZ)
\]
and the quadratic representation \( P : g_{-1} \to \text{End} \, g_{-1} \) by
\[
P(X)Y = (XYX).
\] (2.7)

The structure group \( \text{Str} \, \mathfrak{B} \) of the JTS \( \mathfrak{B} \) is, by definition, the totality of the elements \( g \in GL(g_{-1}) \) which satisfy
\[
g(XYZ) = ((gX)(g^{*-1}Y)(gZ)), \quad X, Y, Z \in g_{-1}.
\] (2.8)

where \( g^* \) denotes the adjoint operator of \( g \) with respect to the trace form of \( \mathfrak{B} \). It follows that
\[
\text{Str} \, \mathfrak{B} = \{ g \in GL(g_{-1}) : P(gX) = gP(X)g^*, X \in g_{-1} \}.
\] (2.9)

It is known (Satake [16]) that the group \( G_0 \) is isomorphic to \( \text{Str} \, \mathfrak{B} \). The isomorphism is given just by taking the restriction of the action of \( G_0 \) to \( g_{-1} \). We identify these two groups. We write \( e_i \) for \( E_{-i} \) (1 \( \leq i \leq r \)).

**Lemma 2.2.** \( e_i \) (1 \( \leq i \leq r \)) is an idempotent of the JTS \( \mathfrak{B} \).

**Proof.** We have to check that \( (e_i e_j e_i) = e_i \). But this follows easily from (2.4) and (1.7). \( \Box \)

**Lemma 2.3.** For decomposition (2.3), we have
\[
L_{e_i e_j} = \begin{cases} 
\frac{1}{2} (\delta_{ik} + \delta_{i\ell}) & \text{on } a_{kl} \ (1 \leq k \leq \ell \leq r), \\
\frac{1}{2} \delta_{ij} & \text{on } c_j \ (1 \leq j \leq r), \\
0 & \text{otherwise.}
\end{cases}
\]
(2.10)

Suppose \( i < j \). Then we have
\[
L_{e_i e_j} L_{e_j e_i} - L_{e_j e_j} L_{e_i e_i} = \begin{cases} 
\frac{1}{4} & \text{on } a_{ij}, \\
0 & \text{otherwise.}
\end{cases}
\]
(2.12)

**Proof.** Taking (2.2) into account, we choose an element \( X \in g^{-\gamma} \subset a_{kl} \), where \( \sigma(\gamma) = \frac{1}{2}(\beta_k + \beta_\ell) \). Then, by (1.7) we have
\[
L_{e_i e_j} X = (e_i e_j X) = -\frac{1}{2} (\tilde{\beta}_i, X) = \frac{1}{2} (\tilde{\beta}_i, \frac{1}{2} (\beta_k + \beta_\ell)) X = \frac{1}{2} (\delta_{ik} + \delta_{i\ell}) X,
\]
which implies the first case of (2.10). (2.11) follows from the strong orthogonality of \( \beta_i \) with \( \beta_j \). (2.12) is an easy consequence from (2.10). \( \Box \)

We need the following lemma for further argument.

**Lemma 2.4.** (Satake [16]) Let \( e \) be an idempotent in a JTS over a field of characteristic zero. Then the quadratic representation \( P(e) \) is a semisimple operator and its square is written as
\[
P(e)^2 = 2L_{ee}^2 - L_{ee}.
\]
(2.13)
Lemma 2.5. Let $s \leq r$. Then $\sum_{i=1}^{s} \epsilon_{i}e_{i}$ is an idempotent in $\mathfrak{B}$, where $\epsilon_{i} = \pm 1$ ($1 \leq i \leq s$).

Proof. By using Lemma 2.3, we have
\[
\left(\sum_{i=1}^{s} \epsilon_{i}e_{i}\right)\left(\sum_{k=1}^{s} \epsilon_{k}e_{k}\right) = \sum_{i,j,k} \epsilon_{i}\epsilon_{k}\epsilon_{\ell} (e_{i}e_{k}e_{\ell}) = \sum_{i,j,k} \epsilon_{i}\epsilon_{k}\epsilon_{\ell} L_{e_{i}e_{k}e_{\ell}} = \sum_{i,\ell} \epsilon_{i}\epsilon_{\ell} L_{e_{i}e_{\ell}}
\]
\[
= \sum_{i,\ell} \epsilon_{\ell} e_{\ell} e_{i} = \sum_{i=1}^{s} \epsilon_{i} e_{i}.
\]

Lemma 2.6. (1) $P(O_{s,0}) = \sum_{i=1}^{s} 1_{a_{i}} \oplus \sum_{1 \leq i < j \leq s} 1_{a_{ij}}$,
(2) rank $P(O_{s,0}) = s \dim a_{11} + \frac{1}{2} s(s-1) \dim a_{12}$,
(3) rank $P(O_{s,0}) \neq \text{rank } P(O_{t,0})$ if and only if $s \neq t$.

Proof. By (2.13) and (2.11) we have
\[
P(e_{1} + \cdots + e_{s})^{2} = 2(L_{e_{1}+\cdots+e_{s},e_{1}+\cdots+e_{s}}) - L_{e_{1}+\cdots+e_{s},e_{1}+\cdots+e_{s}}
\]
\[
= 2\left(\sum_{i,j=1}^{s} L_{e_{i}e_{j}}\right)^{2} - \sum_{i,j=1}^{s} L_{e_{i}e_{j}}
\]
\[
= 2\left(\sum_{i=1}^{s} L_{e_{i}e_{i}}\right)^{2} - \sum_{i=1}^{s} L_{e_{i}e_{i}}
\]
\[
= \sum_{i=1}^{s} (2L_{e_{i}e_{i}} - L_{e_{i}e_{i}}) + 4 \sum_{1 \leq i < j \leq s} L_{e_{i},e_{j}}.
\]
Therefore the assertion (1) follows from Lemma 2.3. The assertion (2) is a direct consequence of (1) and Lemma 2.1. Note that the operator $P(O_{s,0})$ is semisimple. (3) follows from (2). □

Lemma 2.7. $P\left(\sum_{i=1}^{s} \epsilon_{i}e_{i}\right)^{2} = P(O_{s,0})^{2}$. In particular, we have
\[
\text{rank } P\left(\sum_{i=1}^{s} \epsilon_{i}e_{i}\right) = \text{rank } P(O_{s,0}).
\] (2.15)

Proof. From Lemmas 2.5, 2.4, 2.3 and (2.14) it follows that
\[
P\left(\sum_{i=1}^{s} \epsilon_{i}e_{i}\right)^{2} = 2L_{(\sum_{i=1}^{s} \epsilon_{i}e_{i})(\sum_{j=1}^{s} \epsilon_{j}e_{j})} - L_{(\sum_{i=1}^{s} \epsilon_{i}e_{i})(\sum_{j=1}^{s} \epsilon_{j}e_{j})}
\]
\[
= 2\left(\sum_{i,j} \epsilon_{i}\epsilon_{j} L_{e_{i}e_{j}}\right)^{2} - \sum_{i,j} \epsilon_{i}\epsilon_{j} L_{e_{i}e_{j}}
\]
\[
= 2\left(\sum_{i=1}^{s} \epsilon_{i}^{2} L_{e_{i}e_{i}}\right)^{2} - \sum_{i=1}^{s} \epsilon_{i}^{2} L_{e_{i}e_{i}} = P\left(\sum_{i=1}^{s} \epsilon_{i}e_{i}\right)^{2}.
\]
Therefore (2.15) is a direct consequence from Lemmas 2.5 and 2.4. □
Corollary 2.8.  rank $P(O_{p,q}) = rank P(O_{p+q,0}).$

Theorem 2.9.  Let $g = g_1 + g_0 + g_1$ be a real simple GIA and let $G_0$ be the subgroup of Aut $g$ consisting of all grade-preserving automorphisms of $g$. Let $V_0$ ($0 \leq \ell \leq r$) be the subset of $g_{-1}$ given by (1.18), $r$ being the split rank of the symmetric pair $(g, g_0)$. Then

$$V_{\ell} = \{ X \in g_{-1} : rank P(X) = i_{\ell} \}, \quad 0 \leq \ell \leq r,$$

where $i_{\ell} = rank P(O_{\ell,0})$. $V_{\ell}$ is $G_0$-stable. The closure $\overline{V}_{\ell}$ is given by

$$\overline{V}_{\ell} = \{ X \in g_{-1} : rank P(X) \leq i_{\ell} \}, \quad 0 \leq \ell \leq r.$$

Proof.  Let us denote by $V'_\ell$ the set on the right-hand side of (2.16). First we will show the inclusion $V_\ell \subset V'_\ell$. Let $X \in V_\ell$. Then one can write $X = g_0 \ldots g_r$ where $g \in G_0$ and $p + q = \ell$. By (2.9) and Corollary 2.8, we have

$$rank P(X) = rank P(g_0 \ldots g_r) = rank P(g_0) = rank P(O_{p,q,0}) = i_{\ell},$$

which implies $X \in V'_\ell$. Hence the equality $V_\ell = V'_\ell$ follows from (1.19) and the mutual disjointness of $V'_0, \ldots, V'_r$. $V'_\ell$ is $G_0$-stable by (2.9). Next let us denote by $P_\ell$ the set on the right-hand side of (2.17). Then, from (2.16) and (1.19) it follows that

$$P_\ell = V_0 \cup V_1 \cup \cdots \cup V_r, \quad 0 \leq \ell \leq r.$$

We want to prove the inclusion $\overline{V}_{\ell} \supset P_\ell$. Let $X \in V_\ell$, where $\ell_1 < \ell$. Then there exists $g \in G_0$ such that $gX = O_{p_1,q_1}$ with $p_1 + q_1 = \ell_1$. Choose two integers $p, q$ such that $p + q = \ell$, $p \geq p_1$ and $q \geq q_1$. Let

$$Y_n = O_{p_1,q_1} + \sum_{k=q_1+p_1+1}^{q_1+p} \frac{1}{n} e_k - \sum_{t=p+q_1+1}^{p+q} \frac{1}{n} e_t, \quad n = 1, 2, \ldots.$$

Then there exists an element in $G_0$ sending $Y_n$ to $O_{p,q}$. Therefore $rank P(Y_n) = rank P(O_{p,q}) = rank P(O_{p,q,0}) = i_{\ell}$. By (2.16) we have $Y_n \in V_\ell$. On the other hand, $Y_n \to O_{p,q,0} = gX$ as $n \to \infty$. Hence $g^{-1}Y_n \to X$ as $n \to \infty$. This implies $X \in \overline{V}_{\ell}$. The inclusion $\overline{V}_{\ell} \subset P_\ell$ follows easily from (2.16). \qed 

In the proof of Theorem 2.9, we have proved the following

Corollary 2.10.  (Takeuchi [19])

$$g_{-1} = V_0 \cup V_1 \cup \cdots \cup V_r.$$

$$\overline{V}_{\ell} = V_0 \cup V_1 \cup \cdots \cup V_{\ell-1} \cup V_\ell, \quad 0 \leq \ell \leq r.$$

Corollary 2.11.  $V_{\ell}$ is open in $g_{-1}$ and the boundary $\partial V_{\ell}$ of $V_{\ell}$ is given by

$$\partial V_{\ell} = \{ X \in g_{-1} : rank P(X) \leq i_{\ell-1} \}.$$

$\partial V_{\ell}$ is a prehomogeneous generalized cone. Suppose that $\Delta(g, c)$ is of type C (Lemma 1.1(3)). Then we have

$$V_{\ell} = \{ X \in g_{-1} : det P(X) \neq 0 \},$$

$$\partial V_{\ell} = \{ X \in g_{-1} : det P(X) = 0 \}.$$
Proof. By (1.19) and (2.16) we have that rank \( P(X) \leq i_r \) for any \( X \in \mathfrak{g}_{-1} \). Therefore the relation \( V_r = \{ X \in \mathfrak{g}_{-1} : \text{rank} \, P(X) = i_r \} \) implies that \( V_r \) is open in \( \mathfrak{g}_{-1} \). Since \( \partial V_r = \tilde{V}_r - V_r = \mathfrak{g}_{-1} - V_r \), we get (2.23). (2.23) implies that \( \partial V_r \) is \( G_0 \)-stable conic algebraic set in \( \mathfrak{g}_{-1} \). Since \( V_r \) is open, it follows that each \( G_0 \)-orbit in \( V_r \) is open (cf. (1.18)). This implies that \( \partial V_r \) is a prehomogeneous generalized cone. Suppose next that \( \Delta(\mathfrak{g}, \mathfrak{c}) \) is of type \( C \). As is noted just before Lemma 2.1, this condition implies that (2.3) takes the form

\[
\mathfrak{g}_{-1} = \sum_{1 \leq i < j < r} a_{ij}.
\]  

(2.26)

Therefore, from Lemma 2.6(2) we have that \( i_r = \text{rank} \, P(O_{r,0}) = \dim \mathfrak{g}_{-1} \). (2.24) now follows from (2.16). \( \square \)

Proposition 2.12. Let \( GL(\mathfrak{g}_{-1}, \partial V_r) \) be the subgroup of \( GL(\mathfrak{g}_{-1}) \) consisting of all elements leaving \( \partial V_r \) stable. Then we have

\[
GL(\mathfrak{g}_{-1}, \partial V_r) = G_0, \quad r \geq 2.
\]

(2.27)

Proof. Let us consider the subgroup of \( GL(\mathfrak{g}_{-1}) \):

\[
GL(V_0, \ldots, V_r) = \{ g \in GL(\mathfrak{g}_{-1}) : g V_\ell = V_\ell, 0 \leq \ell \leq r \}.
\]

(2.28)

Takeuchi [19] determined this group as follows:

\[
GL(V_0, \ldots, V_r) = G_0, \quad \text{if } r \geq 2.
\]

(2.29)

So, what we have to do is to prove the coincidence \( GL(\mathfrak{g}_{-1}, \partial V_r) = GL(V_0, \ldots, V_r) \). To prove the inclusion \( \subset \), let \( g \in GL(\mathfrak{g}_{-1}, \partial V_r) \). \( g \) leaves the complement of \( \partial V_r \) stable, that is, \( g(V_r) = V_r \). By Corollary 2.10 we have \( \partial V_r = \tilde{V}_r - 1 \), which is an affine algebraic variety in \( \mathfrak{g}_{-1} \). By (2.17), \( g \) acts on \( \tilde{V}_r - 1 \) as a morphism of the algebraic variety. Therefore \( g \) sends its regular points to its regular points and its singular points to its singular points, that is, \( g(V_{r-1}) = V_{r-1} \) and \( g(V_{r-2}) = V_{r-2} \). Repeat this procedure, we have \( g(V_\ell) = V_\ell \) for each \( \ell \). The converse inclusion is obvious. \( \square \)

Tanaka [22] has obtained Proposition 2.12 for the case where the split root system of \( (\mathfrak{g}, \mathfrak{g}_0) \) is of type \( C \) with \( \mathfrak{g} \) classical.

3. Generalized conformal structures on symmetric \( R \)-spaces

Let \( M \) be an \( n \)-dimensional smooth manifold and \( TM \) denote the tangent bundle of \( M \). Let \( V \) be a prehomogeneous generalized cone in \( \mathbb{R}^n \). There \( \mathcal{K} \) is said to be a generalized conformal structure \((GCS)\) on \( M \) with typical cone \( V \), if \( \mathcal{K} \) is a smooth assignment of a cone \( V_p \) in the tangent space \( T_p M \) which is linearly equivalent to \( V \), to each point \( p \in M \). Here, by a smooth assignment we mean that for each point \( p \in M \) there exist a neighborhood \( U \) of \( p \) in \( M \) and a smooth map \( \varphi_U \) of \( U \times \mathbb{R}^n \to TM|_U \) such that \( \varphi_U(q, \cdot) : v \mapsto \varphi_U(q, v) \) is a linear isomorphism of \( \mathbb{R}^n \) onto \( T_q M \) which sends \( V \) to \( V_q \). In the above case, we occasionally write \( \mathcal{K} = \{ V_p \}_{p \in M} \). Let \( (M, \mathcal{K}) \) denote a manifold \( M \) with GCS \( \mathcal{K} \). Now consider two manifolds \( (M, \mathcal{K}) \) and \( (M', \mathcal{K}') \), where \( \mathcal{K} = \{ V_p \}_{p \in M} \) and \( \mathcal{K}' = \{ V'_q \}_{q \in M'} \). A smooth map \( \varphi \) of \( M \) to \( M' \)
is said to be \textit{conformal} (with respect to \(\mathcal{K}\) and \(\mathcal{K}'\)), if \(\varphi_p V_p = V'_{\psi(p)}\) for each point \(p \in M\). A diffeomorphism \(\psi : M \to M\) is called a \(\mathcal{K}\)-\textit{conformal automorphism}, if \(\psi\) is conformal with respect to \(\mathcal{K}\). We denote by \(\text{Aut}(M, \mathcal{K})\) the group of \(\mathcal{K}\)-conformal automorphisms. If a Lie group \(G\) acts on \(M\) as \(\mathcal{K}\)-conformal automorphisms, then \(\mathcal{K}\) is said to be \(G\)-\textit{invariant}.

**Lemma 3.1.** Let \(M = G/U\) be a symmetric \(R\)-space given by (1.9). Then \(M\) has a \(G\)-invariant \(GCS\) \(\mathcal{K}\) with \(\partial V_r\) as a typical cone.

**Proof.** Since \(\text{Lie } U = \mathfrak{g}_0 + \mathfrak{g}_1\), the tangent space \(T_0 M\) at the origin \(o \in M\) may be identified with \(\mathfrak{g}_{-1}\). Let \(\rho\) be the linear isotropy representation of \(U\) at \(o\). Then \(\text{Ker } \rho = \exp \mathfrak{g}_1\) and \(\rho\) is faithful on \(G_0 \subset U\). It is easy to see that \(\rho(U) = \rho(G_0) = G_0|_{\mathfrak{g}_{-1}}\), which implies that the cone \(\partial V_r\) is invariant under the linear isotropy group. We translate \((\partial V_r)_o := \partial V_r\) to each point \(p \in M\) by the action of \(G\), that is, we put

\[(\partial V_r)_p := g(p)(\partial V_r)_o, \quad (3.1)\]

where \(g(o) = p\). The right-hand side of (3.1) is well-defined by the invariance of \(\partial V_r\) under the linear isotropy group. Thus the assignment \(\mathcal{K} : p \mapsto (\partial V_r)_p, \ p \in M\) is the desired \(G\)-invariant \(GCS\) on \(M\).

**Lemma 3.2.** Let \(\mathcal{K}'\) be the flat \(GCS\) on \(\mathfrak{g}_{-1}\) with \(\partial V_r\) as a typical cone, and let us define a map \(\xi\) of \(\mathfrak{g}_{-1}\) \(M\) by putting \(\xi(X) = \exp X \cdot o\), where \(o\) is the origin of \(M\). Then \(\xi\) is a conformal (open dense) imbedding of \((\mathfrak{g}_{-1}, \mathcal{K}')\) into \((M, \mathcal{K})\), and thus \((M, \mathcal{K})\) is a conformal compactification of \((\mathfrak{g}_{-1}, \mathcal{K}')\). Also the \(GCS\) \(\mathcal{K}\) is locally flat.

**Proof.** Let \(\tau_X\) be the parallel translation in \(\mathfrak{g}_{-1}\) along the vector \(X\). Let \(\mathcal{K}' = \{K_X\}_{X \in \mathfrak{g}_{-1}}\). Then \(K_X = \partial V_r + X = \tau_X(\partial V_r)\). It is easy to see that \(\xi \cdot \tau_X = \exp X \cdot \xi\). Hence, under the identification of \(\mathfrak{g}_{-1}\) with \(T_0 M\), we have

\[
\xi \cdot \tau_X(K_X) = \xi \cdot \tau_X(\partial V_r) = (\exp X)_{o_0}(\partial V_r) = (\partial V_r)_{\xi(X)},
\]

which implies that \(\xi\) is conformal. That \(\xi\) is an open dense imbedding of \(\mathfrak{g}_{-1}\) into \(M\) is well-known. The local flatness of \(\mathcal{K}\) follows from the facts that \(\xi\) is conformal and \(G\) is transitive on \(M\).

**Theorem 3.3.** Let \(M = G/U\) be a symmetric \(R\)-space given in (1.9), and \(\mathcal{K}\) be the \(G\)-invariant \(GCS\) on \(M\) with \(\partial V_r\) as a typical cone. Suppose that rank \(M > 1\), that is, \(M\) is not the \(n\)-sphere \(S^n\), nor the projective \(n\)-space \(P^n(F)\), where \(F = \mathbb{R}, \mathbb{C}, \mathbb{H}\) or \(\mathbb{O}\) (the Cayley algebra) with \(n = 2\). Then the group of \(\mathcal{K}\)-conformal automorphisms of \(M\) coincides with \(G\):

\[
\text{Aut}(M, \mathcal{K}) = G. \quad (3.3)
\]

**Proof.** The proof is similar to the case of causal structures of Šilov boundaries [7]. Let \(F(M)\) be the frame bundle of \(M\) and choose a reference frame \(z_0 \in F(M)\) at the origin \(o \in M\). Consider the \(G\)-orbit \(Q\) through \(z_0\) in \(F(M)\). Then \(Q\) is a \(G_0\)-structure of \(M\). Now we need the following result of Tanaka [20,21]: If \(M\) is not \(P^n(\mathbb{R})\), nor \(P^n(\mathbb{C})\), then

\[
\text{Aut } Q = G. \quad (3.4)
\]
where \( \text{Aut } Q \) is the group of the diffeomorphisms \( g \) of \( M \) whose liftings \( \tilde{g} \) to \( F(M) \) leave \( Q \) stable. By (3.4) we have only to prove \( \text{Aut}(M, \mathcal{K}) \subset \text{Aut } Q \), since \( G \) is contained in \( \text{Aut}(M, \mathcal{K}) \). Let \( g \in \text{Aut}(M, \mathcal{K}) \). Since \( G \) is transitive on \( M \), there exist \( a \in G \) such that \( ag(o) = o \). Put \( g' := ag \in \text{Aut}(M, \mathcal{K}) \). Then \( g' \) is \( \mathcal{K} \)-conformal and leaves the origin fixed. Therefore

\[
g'\cdot\partial V_o = (\partial V_o)_{g' \cdot o} = (\partial V_o)_o,
\]

which implies that \( g'_{\partial V} \in GL(g^{-1}, \partial V) \). By the assumption \( r = \text{rank } M > 1 \), one can apply Proposition 2.12 to get \( g'_{\partial V} \in G_0 = \rho(U) \). Therefore there exist \( h \in U \) such that \( h_{\partial V} = g'_{\partial V} \). That means that \( \tilde{h} = \tilde{g}' \) at \( z_0 \). We have \( \tilde{a}\tilde{g}(z_0) = \tilde{g}'(z_0) = \tilde{h}(z_0) \), and hence \( \tilde{g}(z_0) = \tilde{a}^{-1}\tilde{h}(z_0) = a^{-1}h(z_0) \in G_0 = \tilde{Q} \), which implies that \( \text{Aut}(M, \mathcal{K})z_0 \subset Q \). Now take an arbitrary point \( z \in Q \) and write \( z = b(z_0) \), where \( b \in G \). Then \( \tilde{g}(z) = \tilde{g}b(z_0) \in \text{Aut}(M, \mathcal{K})z_0 \subset \tilde{Q} \), which shows that \( g \in \text{Aut } Q = G \). \( \Box \)

We wish to give another geometric meaning of the group \( G \). Let \( \text{Rat}(g^{-1}) \) denote the group of birational transformations of \( g^{-1} \). For a birational transformation \( \varphi \in \text{Rat}(g^{-1}) \), we denote by \( \text{Dom}(\varphi) \) the domain of definition of \( \varphi \).

**Theorem 3.4.** Let \( M \) be a symmetric \( R \)-space given in (1.9), and let \( \xi \) be the conformal imbedding of \( g^{-1} \) into \( M \) given in Lemma 3.2. Let \( \text{Rat}(g^{-1}, \mathcal{K}') \) be the subgroup of birational transformations \( \varphi \in \text{Rat}(g^{-1}) \) which are \( \mathcal{K}' \)-conformal on \( \text{Dom}(\varphi) \). Suppose that \( r = \text{rank } M > 1 \).

Then

\[
\xi^{-1}G\xi = \text{Rat}(g^{-1}, \mathcal{K}').
\]

**Proof:** The inclusion \( \xi^{-1}G\xi \subset \text{Rat}(g^{-1}) \) was proved by Koecher [11] and Loos [13]. Actually the group \( \xi^{-1}G\xi \) is no other than the group of essential automorphisms in the sense of Koecher. Now let \( g \in G \). Since \( g \) is \( \mathcal{K} \)-conformal and since \( \mathcal{K}' \) and \( \mathcal{K} \) are \( \xi \)-related (cf.(3.2)), it follows that \( \xi^{-1}g\xi \) is \( \mathcal{K}' \)-conformal on \( \text{Dom}(\xi^{-1}g\xi) \). Conversely, let \( \varphi \in \text{Rat}(g^{-1}, \mathcal{K}') \), and let \( x_0 \in \text{Dom}(\varphi) \). Choose a connected neighborhood \( U \) of \( x_0 \) in \( \text{Dom}(\varphi) \). Then the diffeomorphism \( \xi\varphi\xi^{-1} : \xi(U) \rightarrow \xi(\varphi(U)) \) is \( \mathcal{K}' \)-conformal, since \( \mathcal{K}' \) and \( \mathcal{K} \) are \( \xi \)-related. Since \( r > 1 \), a result of Tanaka [20] shows that the local transformation \( \xi\varphi\xi^{-1} \) extends to a unique element \( g \in G \), that is, \( \xi(\varphi|_U)\xi^{-1} = g|_{\xi(U)} \), or equivalently, \( \varphi|_U = (\xi^{-1}g\xi)|_U \). Therefore we have \( \varphi = \xi^{-1}g\xi \) as birational transformations. \( \Box \)

Theorem 3.4 gives not only a geometric meaning of the group of essential automorphisms of a simple JTS, but a generalization of the Liouville theorem for classical conformal structure.

**Proposition 3.5.** Let \( \text{Aut}(g^{-1}, \mathcal{K}') \) be the group of \( \mathcal{K}' \)-conformal diffeomorphisms of \( g^{-1} \). Suppose that \( r = \text{rank } M > 1 \). Then we have

\[
\text{Aut}(g^{-1}, \mathcal{K}') = G_0 \ltimes g^{-1}.
\]

**Proof:** By Theorem 3.4, \( \text{Aut}(g^{-1}, \mathcal{K}') \) consists of elements \( \varphi \in \xi^{-1}G\xi \) with \( \text{Dom}(\varphi) = g^{-1} \). As is well known (see for instance [5]), \( \xi^{-1}G_0\xi \) (resp. \( \exp g^{-1} \)) acts on \( g^{-1} \) as linear transformations (resp. parallel translations), while \( \xi^{-1}(\exp g_1)\xi \) acts on \( g^{-1} \) as non-affine birational transformations. Therefore we have that \( \text{Aut}(g^{-1}, \mathcal{K}') = G_0 \exp g^{-1} = G_0 \ltimes g^{-1} \). \( \Box \)
Automorphism group of generalized conformal structure

References