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On Inequivalent Balanced Incomplete Block Designs, I

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ABSTRACT

The existence of at least two inequivalent balanced incomplete block designs is established for certain designs with $\lambda = 1$ and block size $m + 1$ where m is a prime power. An asymptotic result for the number of inequivalent solutions of such designs is also proved.

1. INTRODUCTION

A balanced incomplete block design (BIBD) is an arrangement of v symbols in b subsets, called blocks, of k distinct symbols each ($k < v$) satisfying the condition that any two distinct symbols occur together in exactly λ blocks. It then follows that each symbol occurs in exactly r blocks and that

$$vr = bk,$$

$$\lambda(v - 1) = r(k - 1).$$

In view of these relations we will call a balanced incomplete block design with parameters v, b, r, k, λ as a (v, k, λ) configuration.

Two (v, k, λ) configurations are said to be *equivalent* if one can be obtained from the other by a permutation of v symbols; otherwise they are said to be *inequivalent*. The members of a non-empty family of (v, k, λ) configurations are said to be inequivalent if no two members are equivalent. Two (v, k, λ) configurations are said to be *distinct* if in each configuration there is a block which is not in the other configuration. The members of a non-empty family of (v, k, λ) configurations are said to be distinct if every two members of it are distinct.

A balanced incomplete block design with $k = 3$ and $\lambda = 1$ is known as a Steiner triple system. Assmus and Mattson [1] have proved that, for $v = 2^q - 1, q \geqslant 4$, there are at least two inequivalent Steiner triple systems and that the number of inequivalent Steiner triple systems goes to infinity with q.

The object of this paper is to prove in detail the existence of at least two inequivalent $(v, 4, 1)$ configurations for every

$$v = \frac{3^{q+1} - 1}{3 - 1}, \qquad q \geqslant 3,$$

and that the number of inequivalent quadruple systems tends to infinity with q, and then to prove that in general this result is true for a $(v, p^n + 1, 1)$ configuration where p^n is a prime power, $p^n > 2$, and

$$v = \frac{(p^n)^{q+1} - 1}{p^n - 1}, \qquad q \geqslant 3.$$

The case $p^n = 2$ has been considered by Assmus and Mattson [1].

2. Orthogonal Arrays on Three Symbols and Quadruple Systems

An arrangement of v symbols in an array with h rows and λv^2 columns is called an orthogonal array of strength 2 and index λ if in any 2 rows all possible 2-tuples on v symbols occur λ times each. We denote this arrangement by $[\lambda v^2, h, v, 2]$. It is well known that the existence of $h - 2$ mutually orthogonal Latin squares on v symbols is equivalent to the existence of a $[v^2, h, v, 2]$ [see 4]. It is also well known that, if $v = p^n$, a prime power, then there is a complete set of $v - 1$ mutually orthogonal Latin squares of order v and hence an orthogonal array $[v^2, v + 1, v, 2]$ exists [5].

Consider the following orthogonal array $[9, 4, 3, 2]$ in its standard form constructed from a complete set of 2 mutually orthogonal Latin squares of order 3 in their standard form where the first row in each Latin square is (0 1 2).

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \end{bmatrix}.$$

The permutations (0 1) and (0 2) transform A_0 into the arrays A_1 and A_2, respectively, where

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 2 & 2 & 2 \\ 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 & 2 & 1 & 2 & 1 & 0 \\ 1 & 0 & 2 & 2 & 1 & 0 & 0 & 2 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 2 & 0 & 2 & 1 \\ 2 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 2 \end{bmatrix}.$$

A quadruple system on v symbols is a $(v, 4, 1)$ configuration. Let $S(v)$ be a set of cardinality v and let $Q(v)$ be a quadruple system on $S(v)$. Let $S(3v + 1)$ be the set of $3v + 1$ symbols (w_0, w_1, w_2) and $(*)$ where w runs through $S(v)$. Let (w, x, y, z) be a block of $Q(v)$. Then the nine blocks that we get after adjoining as suffixes the elements of the columns of A_0 to w, x, y, z, respectively, are said to be obtained by developing the block (w, x, y, z) by using the orthogonal array A_0.

If f is a function on the blocks of $Q(v)$ with values in the set $(0, 1, 2)$ and if $Q_f(3v + 1)$ denotes the set of blocks obtained by developing each block of $f^{-1}(0)$ by using the orthogonal array A_0, each block of $f^{-1}(1)$ by using the orthogonal array A_1, and each block of $f^{-1}(2)$ by using the orthogonal array A_2 together with the blocks of the form $(w_0, w_1, w_2, *)$ for each w in $S(v)$, then we have the following:

THEOREM 2.1. *$Q_f(3v + 1)$ is a quadruple system on $S(3v + 1)$.*

PROOF: We have only to show that any pair of distinct elements from $S(3v + 1)$ occurs exactly in one block of $Q_f(3v + 1)$. Let $a, b \in S(3v + 1)$, $a \neq b$.

CASE (i). If $a = x_i, b = x_j, i \neq j, 0 \leqslant i, j \leqslant 2$, and x is in $S(v)$, then $(x_0, x_1, x_2, *)$ is the unique block of $Q_f(3v + 1)$ that contains x_i and x_j.

CASE (ii). Let $a = x_i, b = y_j, 0 \leqslant i, j \leqslant 2, x, y \in S(v)$, and $x \neq y$. Now there exists a unique block, say $(x, y, w, z) \in Q(v)$, that contains x and y. Let $f((x, y, w, z)) = m, 0 \leqslant m \leqslant 2$. Since any two rows of each of the orthogonal arrays A_0, A_1, and A_2 contain the ordered pair (i, j) as a column exactly once, it follows that there is a unique block in $Q_f(3v + 1)$ which contains x_i and y_j.

CASE (iii). Let $a = x_i, 0 \leqslant i \leqslant 2$, and $b = *$. Then $(x_0, x_1, x_2, *)$ is the unique block of $Q_f(3v + 1)$ that contains x_i and $*$.

This theorem is a slight generalization of a theorem due to Bose and Shrikhande [3].

It is well known that a finite projective geometry offers a series of balanced incomplete block designs [2]. In particular, if PG(t, m), $m = p^n$, a prime power, denotes the projective geometry of dimension t based on the Galois field GF(p^n), then by treating the points as symbols and lines as blocks we have a balanced incomplete block design with

$$v = \frac{m^{t+1} - 1}{m - 1}, \qquad b = \frac{(m^{t+1} - 1)(m^t - 1)}{(m^2 - 1)(m - 1)},$$

$$r = \frac{m^t - 1}{m - 1}, \qquad k = m + 1, \qquad \lambda = 1.$$

For $t \geqslant 2$, PG(t, m) is known to be Desarguesian [6].

THEOREM 2.2. *For every $v = (3^{q+1} - 1)/(3 - 1)$, $q \geqslant 3$, there are at least two inequivalent quadruple systems on $S(v)$.*

PROOF: Let $v = (3^{t+1} - 1)/(3 - 1)$, $t \geqslant 2$. Let $Q(v)$ denote the quadruple system on $S(v)$ obtained by taking the points of PG(t, 3) as symbols and the lines in it as blocks. Let $Q(3v + 1)$ be the quadruple system on $S(3v + 1)$ obtained by taking the points of PG($t + 1$, 3) as symbols and the lines in it as blocks. Both $Q(v)$ and $Q(3v + 1)$ are Desarguesian.

Let the triangles with vertices a, b, c and with vertices d, e, f be in perspective with the point p as the center of perspectivity in PG(t, 3). Then $[p, a, d]$, $[p, b, e]$ and $[p, c, f]$ are lines in PG(t, 3). Let the lines $[a, b]$, $[d, e]$ meet in x; $[b, c]$, $[e, f]$ meet in y and $[a, c]$, $[d, f]$ in z. Then $[x, y, z]$ is a line in PG(t, 3).

Let F be a function on the blocks of $Q(v)$ with values in the set (0, 1, 2) which takes the value 0 on each of the lines $[a, b, x]$, $[b, c, y]$, $[a, c, z]$, $[d, e, x]$, $[e, f, y]$, $[d, f, z]$, $[p, a, d]$, $[p, b, e]$, $[p, c, f]$ and the value 1 on the line $[x, y, z]$ and any value elsewhere.

We speak of the blocks in the quadruple system $Q_F(3v + 1)$ as lines. Consider the lines $[a_0, b_0, x_0]$, $[d_0, e_0, x_0]$, $[p_0, a_0, d_0]$, and $[p_0, b_0, e_0]$ which are the developments of the corresponding lines $[a, b, x]$, $[d, e, x]$, $[p, a, d]$, and $[p, b, e]$ from the first column of the orthogonal array A_0 and the lines $[b_0, c_1, y_1]$, $[a_0, c_1, z_1]$, $[e_0, f_1, y_1]$, $[d_0, f_1, z_1]$, $[p_0, c_1, f_1]$ which are the developments of the corresponding lines $[b, c, y]$, $[a, c, z]$, $[e, f, y]$, $[d, f, z]$, and $[p, c, f]$ from the second column of the orthogonal array A_0. Clearly the lines $[a_0, b_0, x_0]$, $[b_0, c_1, y_1]$, $[a_0, c_1, z_1]$ form a triangle through the vertices a_0, b_0, c_1. Also the lines $[d_0, e_0, x_0]$, $[e_0, f_1, y_1]$, $[d_0, f_1, z_1]$ form a triangle through the vertices d_0, e_0, f_1. These two triangles are in perspective from the point p_0 and the corresponding lines meet in x_0, y_1, and z_1, respectively. However the line through y_1 and z_1 does not pass through x_0 in $Q_F(3v + 1)$. (Note that, in A_1, two 1's occur only in the first column and so the line through y_1 and z_1 passes through x_1.) Therefore $Q_F(3v + 1)$ is not Desarguesian. An equivalence clearly preserves the Desarguesian property, hence $Q(3v+1)$ and $Q_F(3v+1)$ are inequivalent quadruple systems on $S(3v+1)$.

3. On the Number of Inequivalent Quadruple Systems

Lemma 3.1. *Let $Q(v)$ be a quadruple system on $S(v)$ and let f and g be two functions on the blocks of $Q(v)$ with values in the set* $(0, 1, 2)$. *If $f \neq g$, then $Q_f(3v+1)$ and $Q_g(3v+1)$ are distinct.*

Proof: We say that two orthogonal arrays $[v^2, v+1, v, 2]$ on the same set of symbols are distinct if each contains a column regarded as a $(v+1)$-tuple which is not in the other. In fact, if two orthogonal arrays $[v^2, v+1, v, 2]$ on the same set of symbols are distinct then there exist i_1 and i_2 such that the columns which contain i_1 and i_2 in the first two positions in these two arrays are distinct. The members of a non-empty family of orthogonal arrays $[v^2, v+1, v, 2]$ on the same set of symbols are said to be distinct if every two of them are distinct. It is easy to see that A_0, A_1, and A_2 are distinct.

Let $(w, x, y, z) \in Q(v)$ and let $f((w, x, y, z)) = i$, $g((w, x, y, z)) = j$, $i \neq j$, $0 \leqslant i, j \leqslant 2$. Let i_1 and i_2 be such that the columns which contain i_1 and i_2 in the first two places in A_i and A_j are distinct. Let these columns be (i_1, i_2, i_3, i_4) and $(i_1, i_2, \bar{i}_3, \bar{i}_4)$. Then $(i_3, i_4) \neq (\bar{i}_3, \bar{i}_4)$. Now

$$(w_{i_1}, x_{i_2}, y_{i_3}, z_{i_4}) \in Q_f(3v+1) \qquad \text{and} \qquad (w_{i_1}, x_{i_2}, y_{\bar{i}_3}, z_{\bar{i}_4}) \in Q_g(3v+1)$$

and these are the unique blocks in $Q_f(3v+1)$ and $Q_g(3v+1)$, respectively, which contain w_{i_1} and x_{i_2}. These blocks are distinct and so $Q_f(3v+1)$ and $Q_g(3v+1)$ are distinct.

Lemma 3.2. *Let $Q(v)$ and $\bar{Q}(v)$ be two distinct quadruple systems on $S(v)$. If f and g are functions defined on the blocks of $Q(v)$ and $\bar{Q}(v)$, respectively, with values in the set* $(0, 1, 2)$ *then $Q_f(3v+1)$ and $\bar{Q}_g(3v+1)$ are distinct.*

Proof: As $Q(v)$ and $\bar{Q}(v)$ are distinct and since there is a unique block containing w and x both in $Q(v)$ and $\bar{Q}(v)$, let $(w, x, y, z) \in Q(v)$ and $(w, x, \bar{y}, \bar{z}) \in \bar{Q}(v)$ where the set (y, z) and the set $(\bar{y}, \bar{z})$ are not equal. We may assume that $z \neq \bar{z}$.

Let $f((w, x, y, z)) = i$, $g((w, x, \bar{y}, \bar{z})) = j$, $0 \leqslant i, j \leqslant 2$. Let $(0, 0, i_3, i_4)$ and $(0, 0, j_3, j_4)$ be the unique columns in A_i and A_j, respectively, containing $(0, 0)$ in the first two places. Then

$$(w_0, x_0, y_{i_1}, z_{i_2}) \in Q_f(3v+1) \qquad \text{and} \qquad (w_0, x_0, \bar{y}_{j_3}, \bar{z}_{j_4}) \in \bar{Q}_g(3v+1).$$

These are the unique blocks in $Q_f(3v+1)$ and $\bar{Q}_g(3v+1)$ containing w_0 and x_0. As $z_{i_4} \neq \bar{z}_{j_4}$, it follows that $Q_f(3v+1)$ and $\bar{Q}_g(3v+1)$ are distinct.

Let $v(t) = (3^{t+1}-1)/(3-1)$ and let $D(t)$ and $I(t)$ denote the number of distinct quadruple systems and the number of inequivalent quadruple systems, respectively, on $v(t)$ symbols. Clearly,

$$I(t) \geqslant \frac{D(t)}{(v(t))!}.$$

Now from Lemma 3.1 and Lemma 3.2 it follows that

$$D(t) \geqslant 3^{b(t-1)} D(t-1),$$

where $b(t-1)$ denotes the number of blocks in a quadruple system on $v(t-1)$ symbols. As $D(1) = 1$, we have

$$I(t) \geqslant \frac{3^{\phi(t)}}{(v(t))!},$$

where $\phi(t) = \sum_{h=1}^{t-1} b(h)$. We have

$$b(h) = \frac{(3^{h+1}-1)(3^h-1)}{(3^2-1)(3-1)}.$$

This gives

$$\begin{aligned}
\sum_{h=1}^{t-1} b(h) &= \sum_{h=1}^{t-1} \frac{(3^{h+1}-1)(3^h-1)}{(3^2-1)(3-1)} \\
&= \frac{1}{16} \sum_{h=1}^{t-1} (3^{2h+1} - 3^{h+1} - 3^h + 1) \\
&= \frac{3^3}{16} \frac{(3^2)^{t-1}-1}{3^2-1} - \frac{3^2}{16} \frac{3^{t-1}-1}{3-1} - \frac{3}{16} \frac{3^{t-1}-1}{3-1} + \frac{t-1}{16} \\
&= \frac{27}{128}(9^{t-1}-1) - \frac{3}{8}(3^{t-1}-1) + \frac{t-1}{16}.
\end{aligned}$$

We have $(v(t))! = \Gamma(v(t)+1)$, where $\Gamma(x)$ is the Euler's gamma function. Using Stirling's approximation for the gamma function, which is

$$\Gamma(x) = x^{x-1/2} e^{-x} \sqrt{2\pi}\,(1 + 0(1)).$$

We obtain the following result.

THEOREM 3.1. *$D(t) \geqslant 3^{\phi(t)}$, where*

$$\phi(t) = \frac{27}{128}(9^{t-1} - 1) - \frac{3}{8}(3^{t-1} - 1) + \frac{t-1}{16}$$

and hence $\lim_{t\to\infty} I(t) = \infty$.

4. INEQUIVALENT DESIGNS WITH

$$v = \frac{m^{q+1} - 1}{m - 1}, \quad m = p^n, \quad q \geqslant 3 \quad \text{and} \quad k = m + 1.$$

We know that

1. PG(t, m), $m = p^n$, gives a balanced incomplete block design with

$$v = \frac{m^{t+1} - 1}{m - 1}, \qquad b = \frac{(m^{t+1} - 1)(m^t - 1)}{(m^2 - 1)(m - 1)},$$

$$r = \frac{m^t - 1}{m - 1}, \qquad k = m + 1, \qquad \lambda = 1,$$

where we treat points as symbols and lines as blocks. If $Q(v)$ denotes this design then it is Desarguesian for $t \geqslant 2$.

2. An orthogonal array $[m^2, m + 1, m, 2]$ exists where $m = p^n$.

3. Let $m = p^n$. Let A_0 be the orthogonal array $[m^2, m + 1, m, 2]$ in its standard form. We have

$$A_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & j_1 & j_2 & \cdots & j_{(m-1)} \\ 0 & 1 & \cdots & (m-1) & & & & \\ \vdots & \vdots & \vdots & \vdots & B_1 & B_2 & \cdots & B_{(m-1)} \\ 0 & 1 & \cdots & (m-1) & & & & \end{bmatrix},$$

where j_i is a vector with m components each equal to i and all the B_i's are Latin squares of order m in their standard form. This means that in each of the columns of A_0 corresponding to those of B_i the element i appears exactly twice whereas each other element occurs only once, $0 < i \leqslant m - 1$. The element zero occurs only once in the 2nd, 3rd,..., m-th column of A_0. Therefore the only column that contains zero twice is the first column which has all its elements zero.

Similar considerations hold for every orthogonal array A_i, where A_i

is obtained by the permutation $(0\ i)$ on the symbols of A_0, $0 < i \leqslant m - 1$. The orthogonal arrays A_i, $0 \leqslant i \leqslant m - 1$, are distinct. Therefore if $i \neq j$ then there exist i_1 and i_2 such that the unique columns in A_i and A_j which contain i_1 and i_2 in the first two places are distinct.

4. From a $(v, m + 1, 1)$ configuration, $m = p^n$, by using a function f defined on its blocks with values in the set $(0, 1, \ldots, m - 1)$ and by developing each block in $f^{-1}(i)$ by using the orthogonal array A_i, $0 \leqslant i \leqslant m - 1$, we have a set of blocks each containing $m + 1$ symbols from the set $(w_0, w_1, w_2, \ldots, w_{m-1}) \sqcup (*)$ of $vp^n + 1$ symbols where w runs through $S(v)$. We denote this set of blocks by $(vp^n + 1, m + 1, 1)_f$.

From a careful examination of the proofs of the results in the previous sections we obtain the following results:

THEOREM 4.1. *$(vp^n + 1, m + 1, 1)_f$ is a balanced incomplete block design on $vp^n + 1$ symbols with $m + 1$ symbols in each block and any two symbols occurring together in exactly one block.*

THEOREM 4.2. *Let $v = (m^{q+1} - 1)/(m - 1)$, $m = p^n$, $m > 2$, $q \geqslant 3$. Then there exist at least two inequivalent $(v, m + 1, 1)$ configurations.*

LEMMA 4.1. *Let f and g be functions on the blocks of a $(v, m + 1, 1)$ configuration, $m = p^n$, with values in the set $(0, 1, \ldots, p^n - 1)$. If $f \neq g$ then the configurations $(vp^n + 1, m + 1, 1)_f$ and $(vp^n + 1, m + 1, 1)_g$ are distinct.*

LEMMA 4.2. *Let $(v, m + 1, 1)$ and $\overline{(v, m + 1, 1)}$ be two distinct configurations, $m = p^n$, and let f and g be two functions defined on the blocks of $(v, m + 1, 1)$ and $\overline{(v, m + 1, 1)}$, respectively, and with values in the set $(0, 1, \ldots, p^n - 1)$. Then the configurations $(vp^n + 1, m + 1, 1)_f$ and $\overline{(vp^n + 1, m + 1, 1)_g}$ are distinct.*

Let $v(t) = (m^{t+1} - 1)/(m - 1)$, $m = p^n$, and let $D(t)$ and $I(t)$ denote the number of distinct $(v(t), m + 1, 1)$ configurations and the number of inequivalent $(v(t), m + 1, 1)$ configurations, respectively. Then we have the following result:

THEOREM 4.3. *$D(t) \geqslant m^{\phi(t)}$ where*

$$\phi(t) = c_1((m^2)^{t-1} - 1) - c_2(m^{t-1} - 1) + \frac{t - 1}{c_3}$$

and c_1, c_2, c_3 are positive integers. Hence $\lim_{t \to \infty} I(t) = \infty$.

REMARK. It can be easily shown that, when $p^n > 2$, A_i's are not the only mutually distinct orthogonal arrays. For example, for $p^n = 3$, each A_i, $0 \leqslant i \leqslant 2$, generates 3 other orthogonal arrays by changing the first row with each of the other rows, giving a set of 4 distinct orthogonal arrays. Hence instead of 3 we could have used 12 distinct orthogonal arrays in the case of quadruple systems. When $p^n > 3$, one can also change the second row with each of the following rows. These facts may be utilized to improve upon the asymptotic behaviour of $D(t)$ and hence that of $I(t)$.

It is proposed to consider in a subsequent communication similar results for designs with parameters corresponding to points and lines in a finite Euclidean geometry.

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