Classification of 2F-modules, I

Robert M. Guralnick\textsuperscript{a,\ast,1} and Gunter Malle\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, University of Southern California, Los Angeles, CA 90089-1113, USA
\textsuperscript{b} FB Mathematik/Informatik, Universität Kassel, Heinrich-Plett-Str. 40, D-34132 Kassel, Germany

Received 1 March 2002
Communicated by Michel Broué
Dedicated to J.G. Thompson on his 70th birthday

Abstract
We classify the 2F-modules for nearly simple groups, excluding the case of modules for groups of Lie type in their defining characteristic. We also show that for all such modules there exists an offender with cubic action. These modules play a crucial role in the current revisions of the classification of the finite simple groups.

\textcopyright 2002 Elsevier Science (USA). All rights reserved.

1. Introduction

In this paper we study the following situation: Let $G$ be a finite group such that $F^\ast(G)$ is quasi-simple. For an absolutely irreducible faithful $FG$-modules $V$ over a finite field $F$ of characteristic $\ell$ and a subgroup $A \leq G$ of $G$ we let

\[ f(A) := f_{V}(A) := |A|^2 \cdot |C_{V}(A)|. \]

We classify those modules $V$ for which $G$ has an elementary abelian $\ell$-subgroup $A$ satisfying

\ast Corresponding author.

E-mail addresses: guralnic@math.usc.edu (R.M. Guralnick), malle@mathematik.uni-kassel.de (G. Malle).

1 The first author was partially supported by the National Science Foundation grant DMS-9970305.
\[ |V| \leq f(A). \] (1.1)

Variations on this theme have been studied for many years. Perhaps the first person to consider this situation was John Thompson [20] in studying the embeddings of a Sylow \( p \)-subgroup \( S \) in finite simple groups—especially considering the subgroups \( C(Z(S)) \) and \( N(J(S)) \). Glauberman also studied this—see [7–9] and [11]. They were interested in obtaining factorizations of groups and the existence of such modules is an obstruction (hence called failure of factorization modules by Glauberman). In [9], Glauberman showed that certain groups had no such modules (for \( \ell = 2 \)). In [10], he proved a factorization theorem for certain 2-constrained groups such that no chief factors were modules satisfying related conditions. In most instances, the case \( \ell = 2 \) has been the focus of investigations. Often there are extra or modified conditions on the modules—for example, the case where the exponent 2 above is replaced by 1 and/or additional conditions such as assuming that \([[[V, A], A], A] = 0\) have been considered.

More recently, these modules (again with some extra conditions) play a crucial role in the revision of the classification of finite simple groups. Aschbacher and Smith [1] will use the results of this paper (for \( \ell = 2 \)) and the sequel in the classification of quasithin finite simple groups. Their paper will be part of the revision program as well as provide the first published proof of the classification of quasithin groups. These modules (for all \( \ell \)) are also important in Meierfrankenfeld’s program of classifying simple groups which are of generic characteristic \( \ell \)-type, see [3] for example.

In most of these applications, one has extra conditions on the module—for example in [1], they assume that \( A \) has cubic action on \( V \)—i.e., \([[[V, A], A], A] = 0\). In this paper our approach is to use (1.1) only. It is then quite easy to decide which of the few remaining examples satisfy the additional hypotheses. We feel that this is intrinsically more pleasing. Moreover, if one wants to impose weaker conditions, then the theorem will still apply. As we have noted, many people have assumed that \( A \) acts quadratically on \( V \) (and used this property extensively). We always assume that \( V \) is irreducible and faithful—by considering extensions of the modules involved, one could determine the indecomposable examples as well.

Note that a special case of these modules is where \( G \) contains an element of order \( \ell \) that centralizes a subspace of codimension 2 on \( V \), and \( V \) is defined over \( \mathbb{F}_\ell \).

The more general case of classifying representations of groups containing ‘bireflections’ (in particular, pseudoreflections and transvections) is dealt with in [13]. We also note that the condition that \( A \) is an elementary \( \ell \)-group could be relaxed in a variety of ways.

An elementary lemma gives that under our conditions \( f(A) \leq f(G) = |G|^2 \). But now a simple argument shows that \( FG \)-modules satisfying (1.1) are rare: Assume for example that \( G = \mathcal{S}_n \) is the symmetric group and let \( \ell \) be a prime. Then \( \log_\ell(|G|) \sim n \log_\ell(n) \) by Stirling’s formula. It is well known that for large enough \( n \) only the heart of the permutation module and its product with the sign have dimension less than \( n^2/3 \). In particular, for large enough \( n \) only these two
modules can arise. (We will see that in fact even $\ell \leq 3$.) A similar reasoning deals with groups of Lie type defined over fields of characteristic prime to $\ell$, using the lower bounds of Landazuri–Seitz-type for dimensions of nontrivial irreducible representations.

We close the paper by determining in Section 6 for which of the triples $(G, V, A)$ satisfying condition (1.1) the action of $A$ is cubic, that is, the commutator $[[[V, A], A], A]$ is zero. Given the short list of examples from the first part, this can be done by straightforward verification. Thus we just have to make precise the rough sketch given above for the various families of quasi-simple groups. At each point we will explicitly state which properties of the known quasi-simple groups we use. In particular, we will only use properties of the group in question and its subgroups, thus making the result usable for being part of an inductive proof of the classification. The case of groups of Lie type in their defining characteristic will be considered in a sequel to this paper.

At present, we have to leave open the case of the smallest nontrivial module in characteristic 2 for the sporadic Conway groups $Co_1$ and $Co_2$, which we hope to solve in due course.

2. Notation and basic observations

Let $F$ be a finite field of characteristic $\ell$, $G$ a finite group and $V$ a finite-dimensional $FG$-module. For a subgroup $A$ of $G$ write

$$f(A) := f_V(A) := |A|^2 \cdot |C_V(A)|.$$  

$V$ is called a 2$F$-module for $G$ with offender $A$ if $\dim(V) > 1$ and $|V| \leq f(A)$.

The following well-known observation will be useful (see [4]):

**Lemma 2.1.** Let $A, B \leq G$ with $f(B)$ maximal among all subgroups of $B$ and let $C := \langle A, B \rangle$. Then $f(C) \geq f(A)$.

**Proof.** Note that $|C| \geq |A||B|/|A \cap B|$ and $|C_V(C)| \geq |C_V(A)||C_V(B)|/|C_V(A \cap B)|$. Thus we have

$$f(C) = |C|^2|C_V(C)|$$

$$\geq |A|^2|B|^2|C_V(A)||C_V(B)|/(|A \cap B|^2|C_V(A \cap B)|)$$

$$= f(A)f(B)/f(A \cap B) \geq f(A). \quad \Box$$

In particular, if $V$ is a 2$F$-module for $G$ with offender $A$ and $f(A) \geq f(B)$ for all $B \leq A$, then $f(A) \leq f(A^G)$. In particular, if $G$ is the normal closure of $A$ then $f(A) \leq f(G)$. Moreover, if $V$ is irreducible then $f(A) \leq |G|^2$. Also it follows that:
Lemma 2.2. Suppose $C \leq G$ is an elementary abelian $\ell$-subgroup containing $A$ and $B$, with $f(A) = f(B)$ maximal among all subgroups of $C$. Then $f(AB) = f(A)$.

This shows that when searching for examples satisfying (1.1) we may sometimes assume that $A$ is maximal elementary abelian.

The following can be used to obtain upper bounds on the centralizer of an $\ell$-subgroup:

Lemma 2.3. Let $B \leq A \leq G$ with $\dim CV(A) = \dim CV(B)$. Then

$$\dim CV\left(\left\{A^g \mid g \in C_G(B)\right\}\right) = \dim CV(B).$$

Proof. Let $g \in C_G(B)$. Then $CV(A) = CV(B) = CV(B^g) = CV(A^g)$. $\square$

We will often use this in the following form. Let $B$ be an elementary abelian $\ell$-subgroup, $g$ an $\ell$-element in $C_G(B) \setminus B$ and $A := \langle B, g \rangle$. If $\langle A^h \mid h \in C_G(B) \rangle$ has smaller centralizer in $V$ than $B$, then $A$ has smaller centralizer than $B$. For example, if $C_G(B)$ is an extension of $B$ by a simple group $H$, then all classes of $\ell$-elements in $C_G(B)$ outside $B$ generate $H$. But $H$ contains $\ell'$-elements and their centralizer can be read off from the Brauer character of $V$.

The following result is very useful for inductive arguments; it will be used in the second part.

Proposition 2.4. Let $V$ be a 2$F$-module for $G$. Let $H \leq G$ such that $H$ contains an offender. The either there exists an offender $A \leq U := O_\ell(H)$, or $CV(U)$ is a 2$F$-module for $H/U$.

Proof. Choose an offender $A \leq H$ such that all subgroups $A'$ of $A$ satisfy $f_V(A') \leq f_V(A)$. We may assume that $A$ is not contained in $U$. Let $W := CV(U)$. Then

$$CV(A) \cap CV(U) = CCV(U)(A) = CW(A),$$

so $|CW(A)| = |CV(A) \cap CV(U)| \geq |CV(A)||CV(U)|/|CV(A \cap U)|$. We obtain

$$|W| = |CV(U)| \leq \frac{|CW(A)||CV(A \cap U)|}{|CV(A)|} = \frac{|CW(A)||CV(A \cap U)||A|^2}{f_V(A)}.$$

Now let $\tilde{A} := AU/U = A/A \cap U$, so $|A| = |\tilde{A}||A \cap U|$. Then

$$|W| \leq |CW(A)||\tilde{A}|^2\frac{|A \cap U|^2|CV(A \cap U)|}{f_V(A)} = f_W(\tilde{A}) \frac{f_V(A \cap U)}{f_V(A)} \leq f_W(\tilde{A})$$

by the choice of $A$, hence $\tilde{A}$ is an offender for $H/U$ on $W$. $\square$
Let us make some easy observations. Assume that $G$ has $\ell$-rank $r$, then clearly $|A| \leq \ell^r$ for any elementary abelian $\ell$-subgroup $A$. Write $d := \dim(V)$ for the dimension over the smallest field of definition and let $d_\ell$ be the maximal dimension of the centralizer space in $V$ of a nontrivial $\ell$-element. Let $f$ be the degree over $\mathbb{F}_\ell$ of the smallest field over which the representation is defined. Then condition (1.1) certainly implies

$$f(d - d_\ell) \leq 2r. \quad (2.5)$$

For $x \in G$, let $k(x)$ be the minimal number of conjugates of $x$ which generate $\langle x^G \rangle$. Write

$$k_\ell := \max \{ k(x) \mid x \in G, x \neq 1 = x^\ell \}.$$

If we assume that every irreducible constituent of $V$ is faithful, then $\langle x^G \rangle$ has no fixed points on $V$ and so $(d - d_\ell)k_\ell \geq d$. Thus (1.1) implies

$$f d \leq 2rk_\ell. \quad (2.6)$$

This weaker condition will already rule out most possibilities for $G$.

The next result can be used when treating automorphism groups. The only interesting case for us is the one of outer automorphisms of order $\ell$ (since for $V$ irreducible and faithful, we can always restrict to $\langle A^G \rangle$).

**Proposition 2.7.** Let $G$ be a finite group with a normal subgroup $H \leq G$ of index $\ell$. Let $W$ be an absolutely irreducible $\mathbb{F}_\ell H$-module of dimension $\dim W \geq 2$ whose induction $V$ to $G$ is again irreducible. If $V$ is a $2F$-module for $G$ with offender $A$, then one of the following holds:

1. $W$ is a $2F$-module for $H$ with offender $A' := A \cap H$;
2. $\ell = f = 2$, $z := \dim C_W(A') = 1$, $\dim W \leq r$ where $|A| = 2^r$;
3. $|A| = \ell = 2$, $\dim W = 2$, $f \leq 2$.

**Proof.** If $A \leq H$ then clearly we arrive at case (i). Hence we may assume that $A' := A \cap H < A$, so $[A : A'] = \ell$. Write $|A| = \ell^r$, so $|A'| = \ell^{r-1}$. Furthermore, let $z := \dim C_W(A')$. Since $V$ is induced from $W$ and $A \nsubseteq H$ we then have $\dim V = \ell \dim W$ and $\dim C_V(A) = z$.

First assume that $V$ is not defined over a proper subfield of $\mathbb{F}_\ell$. Then the $2F$-condition for $V$ yields $f(d\ell - z) \leq 2r$, where $d := \dim W$. Hence

$$f(d - z) \leq (2r + fz)/\ell - fz \leq r - fz/2,$$

so $f(d - z) \leq r - 1$. Thus either $A'$ is an offender for $H$ on $W$ or $r = 1$. In the latter case $A' = 1$, $z = d$, and we obtain $f(d\ell - d) = f d(\ell - 1) \leq 2$. Then $d \geq 2$ forces $f = 1, d = \ell = 2$ as in case (iii).
It remains to consider the case where $V$ is defined over the subfield of index $\ell$ of $\mathbb{F}_d$. In particular $f = \ell f'$ with an integer $f'$. Here the 2F-property gives $f(d\ell - z)/\ell \leq 2r$, so $fd \leq 2r + fz/\ell$. Hence

$$f(d - z) \leq 2r + fz/\ell - fz = 2r - \ell f'z(1 - 1/\ell) = 2r - f'z(\ell - 1).$$

This shows that $A'$ is an offender for $H$ on $W$ unless $f'z(\ell - 1) \leq 1$. This can only happen if $\ell = 2$, $f' = z = 1$. Then the first inequality yields $2d \leq 2r + 1$, so $d \leq r$, and we are in case (ii). If $r = 1$ then $f'(d\ell - z) \leq 2$, and again we find $d = \ell = 2$, $f' = 1$, which is case (iii). $\square$

We fix the following notation. Let $G$ be a finite group such that $F^*(G)$ is quasi-simple. Write $S$ for the unique non-abelian simple composition factor of $G$. Let $\ell$ be a prime dividing $|G|$. We will consider the different possibilities for $S$ according to the classification of finite simple groups, excluding in this part the groups of Lie-type in their defining characteristic.

3. Alternating groups

To handle the alternating groups we will make use of the following properties: lower bounds for degrees of absolutely irreducible representations due to James [15] and Wagner [21], and the modular character tables of $A_n$ for $n \leq 10$ (if $\ell$ is odd) respectively $n \leq 12$ (if $\ell$ is even) [16].

**Proposition 3.1.** Let $S = A_n$ with $n \geq 5$, $(n, \ell) \neq (5, 2), (5, 5), (6, 3), (8, 2)$. Then the absolutely irreducible $FG$-modules $V$ satisfying (1.1) are given in Table 6.3.

**Proof.**

(1) We first consider the deleted permutation module. The commutator space for an $\ell$-element on the permutation module is at least $(\ell - 1)[n/\ell]$, while the $\ell$-rank of $S_n$ is $[n/\ell]$. By (2.5) this forces $\ell \leq 3$, and it is clear that in this case we get examples. Thus from now on we will assume that $V$ is not the deleted permutation module.

(2) We next work out the cases $n \leq 7$ from the known subgroup structure and the modular character tables [16]. For $n = 5$ we have to consider $\ell = 3$. Since $G = A_5$ has cyclic Sylow 3-subgroups and is generated by two 3-elements we have $d \leq 4$ by (2.6). Clearly $d = 3$ for $G$ and $d = 2$ for $2G$ give examples, while $d = 4$ for $G$ has $f = 2$.

For $n = 6$ and $\ell = 5$ we again have $d \leq 4$. The case $d = 3$ for $3A_6$ has $f = 2$, while $d = 4$ for $2A_6$ restricts to the irreducible 4-dimensional representation of $2A_5 = SL_2(5)$ in which 5-elements act indecomposably. Thus neither leads to an example. For $\ell = 2$ the rank is 2 and $k_2 = 3$, so $df \leq 12$. This leads to $d = 3$, $f = 2$ for $3S$ which gives an example with $|A| \in \{2, 4\}$ and 2-dimensional
centralizer. The group PGL$_2(9) = \mathfrak{A}_6.2_2$ also has 2-rank 2, and there is a unique class of four groups containing outer involutions. Two of these generate, forcing $d \leq 8$, so $d = 8$, $f = 1$ is the only possibility. Outer involutions invert elements of order 5, so have 4-dimensional commutator space. But the centralizer is $2 \times D_5$, and 5-elements have no fixed points, so the centralizer of a four group involving outer elements is too small. The group $M_{10} = \mathfrak{A}_6.2_3$ has no outer involutions. The symmetric group $S_6$ has 2-rank 3. Since $k_2 = 5$, $d \leq 10$ if $|A| = 2$. If $|A| \geq 4$ then it contains inner involutions and $k(\pi) = 3$ for an inner involution. Thus $d \leq 12$ if $|A| = 4$. If $|A| = 8$ then two conjugates of A generate, so $d \leq 12$ again. This leaves the deleted permutation modules and $d = 6$ for $3 \cdot S_6$, which occurs. Finally Aut($\mathfrak{A}_6$) cannot be generated by the conjugates of an elementary abelian 2-subgroup, since one of the cosets of $\mathfrak{A}_6$ does not contain involutions.

For $n = 7$ and $\ell = 5, 7$ we have $d \leq 4$ by (2.6). The possibility $d = 4$ for $2 \cdot \mathfrak{A}_7$ restricts irreducibly to $SL_2(\ell)$, so does not give an example. Also, $d = 3$ for $3 \cdot \mathfrak{A}_7$ with $\ell = 5$ has $f = 2$, so $\ell = 5, 7$ is not possible. If $\ell = 3$ then $d \leq 8$. Apart from the deleted permutation module we obtain $d = 4, 6$ for $2 \cdot \mathfrak{A}_7$, both with $f = 2$. This actually rules out $d = 6$. The 4-dimensional representation restricts irreducibly to $2 \cdot \mathfrak{A}_6 = SL_2(9)$, where the Sylow 3-subgroup acts indecomposably, hence is not an example either. For $\ell = 2$ we have rank 2 and three involutions generate, and moreover two four-groups generate, so $d \leq 8$ for the simple group. This implies $d = 4$ for $\mathfrak{A}_7$, which occurs with $|A| = 4$. For $S_7$ and $\ell = 2$ either $A$ is cyclic whence $d \leq 12$, or it contains an involution from $\mathfrak{A}_7$. If $A$ has rank 2 then $d \leq 12$ as above. Furthermore, two elementary abelian subgroups of order 8 generate, so $d \leq 12$ in the rank-3 case as well. This leads to $d \in \{8, 12\}$. Explicit computation shows that the 8-dimensional representation gives an example. Since the 12-dimensional representation of $G = 3 \cdot S_7$ is induced from $3 \cdot \mathfrak{A}_7$, outer involutions have 6-dimensional commutator space. In particular $|A| = 8$ and thus $A$ contains transpositions. The centralizer of a transposition in $S_7$ is $2 \times S_5$, which contains elements of order 5 with 8-dimensional commutator. Thus $A$ has at most 4-dimensional centralizer. This rules out $d = 12$.

(3) From now on let $n \geq 8$. First assume that $\ell$ is odd. The $\ell$-rank of $\mathfrak{A}_n$ equals $\lfloor n/\ell \rfloor$. By an easy induction argument it follows that $k_\ell \leq n/2$. We first consider representations of $\mathfrak{A}_n$. An application of a result of James [15] shows that for $n \geq 12$ any absolutely irreducible $F \mathfrak{S}_n$-module either has dimension $d \geq (n - 2)(n - 3)$ or lies in $R_n(2)$ [18, Proposition 2.2]. Here $R_n(2)$ denotes the set of those irreducible representations parametrized by an $\ell$-regular partition $\lambda$ such that $\lambda$ or its dual have a part at least equal to $n - 2$. Hence any absolutely irreducible $F \mathfrak{A}_n$-module either has dimension $d \geq (n - 2)(n - 3)/2$ or is a constituent of the restriction of a representation in $R_n(2)$. The first possibility violates (2.6). But the nontrivial $F \mathfrak{A}_n$-constituents of modules in $R_n(2)$ different from the deleted permutation module have dimension $d \geq (n^2 - 5n + 2)/2$ by [15], again contradicting (2.6), except when $n = 12$, $\ell = 3$ and $|A| = 3^4$. But
clearly $\mathfrak{A}_{12}$ is generated by two suitable conjugates of such a subgroup, which forces $d \leq 16 < 43 = (n^2 - 5n + 2)/2$, a contradiction.

For $8 \leq n \leq 11$ similar arguments may be used; we just sketch the most difficult case $\ell = 3$. If $|A| = 3$ then $d \leq n$, giving the deleted permutation module. If $|A| = 9$ then $d \leq 2n$, which leads to $n = 8, d = 13$. But for $n = 8$ two such subgroups generate, thus $d \leq 8$ in this case. If $|A| = 27$ then $n \geq 9$. Again two such groups generate if $n \leq 11$, hence $d \leq 12$ and no new cases arise.

(4) We still assume that $\ell$ is odd, and now consider the faithful representations of the 2-fold cover $G = 2 \cdot \mathfrak{A}_n$. According to Wagner [21] the smallest dimension is at least $2^{\lfloor \frac{n-2}{2} \rfloor}$, where $s$ denotes the number of 1’s in the 2-adic expansion of $n$. Comparison with the $\ell$-rank $[n/\ell]$ shows that $n \leq 15$.

If $n < 2\ell$ then the $\ell$-rank of $G$ is 1 and $k_{\ell} = 2$, hence $d \leq 4$, violating Wagner’s bound. For $\ell = 7, n = 14, 15$, or $\ell = 5, 10 \leq n \leq 13, k_{\ell} = 3$, so $d \leq 12$, which is only possible if $n = 10, \ell = 5$. But then $|A| = 25$, and two Sylow 5-subgroups generate, whence $d = 8$. On restriction to $2.\mathfrak{A}_5 = \text{SL}_2(5)$ we find two copies of the 4-dimensional representation in which elements of order 5 have a 3-dimensional commutator space. Thus this case is out. For $\ell = 5, n = 14, 15$, we find $d \leq 24$, a contradiction. Finally let $\ell = 3$. For $n = 10, 11, 12$, two conjugates of any 3-subgroup of rank 3 generate, whence $d \leq 12$, which is not possible. For $n = 13, 14, 15$, three conjugates of any rank-3 subgroup generate, giving $d \leq 18$, a contradiction. For $n = 8, 9$ the character tables show that $d = 8$. In the case $n = 8$ we need $|A| = 9$. Restriction to $2.\mathfrak{A}_5$ yields four 2-dimensional constituents which shows that 3-cycles have at least 4-dimensional commutator spaces. The centralizer of a 3-cycle $g \in A$ contains an $\mathfrak{A}_5$. If $A$ has 4-dimensional centralizer, then, since $\mathfrak{A}_5$ is generated by its unique class of 3-elements, $\mathfrak{A}_5$ has a 4-dimensional centralizer as well. But the character table reveals that elements of order 5 act fixed point freely, so this case is not possible. If $n = 9$ then restriction to $2.\mathfrak{A}_8$ (the case just treated) shows that $|A| = 27$. This gives an example.

(5) Now assume $\ell = 2$ and $G = \mathfrak{A}_n$ with $n \geq 9$ (since $\mathfrak{A}_8 = L_4(2)$). Then again $k_\ell \leq n/2$. For $n \geq 17$ again Proposition 2.2 in [18] shows that $V \in R_n(2)$, which only leaves the deleted permutation module by [15]. So now assume $n \leq 16$. If $A$ has rank at most 2 then $d \leq 2n$ by (2.6). For $9 \leq n \leq 11$ two suitable maximal rank-3 subgroups generate and the 2-rank is equal to 4. We deduce $d \leq 16$ in this case, and by the tables in [16] this leads to the deleted permutation module or $(n, d) \in \{(9, 8), (10, 16), (11, 16)\}$. The first gives an example with $|A| \in \{8, 16\}$ and centralizer of dimension 2, since it restricts irreducibly to the 8-dimensional representation of $\mathcal{S}_7$ considered above. The restriction of the 16-dimensional $\mathfrak{A}_{10}$-module to $\mathfrak{A}_9$ consists of two copies of the previously considered representation, thus the centralizer space is at most 4-dimensional, giving a contradiction. This also rules out $n = 11$. For $12 \leq n \leq 15$, two rank-5 subgroups, three rank-4 subgroups and four rank-3 subgroups generate. Thus in any case we get $d \leq 24$. Lemma 3 in [15] shows that the only possibility is
(n, d) = (12, 16). But this has f = 2, being defined over $\mathbb{F}_4$, which violates the bound. For n = 16 we obtain a contradiction from [15].

It remains to consider $\ell = 2$ and $G = \mathfrak{S}_n, n \geq 9$. If A is cyclic we obtain $d \leq 2(n-1)$ since $k_2 = n-1$. Otherwise A contains at least one element from $\mathfrak{A}_n$, hence with commutator space of dimension at least $2d/n$. In any case Theorem 7 in [15] shows $n \leq 14$ or $V$ restricted to $\mathfrak{A}_n$ is the deleted permutation module. Assume $n \leq 14$. The estimates from the $\mathfrak{A}_n$-case plus the tables in [16] show that we are in one of the following cases: $(n, d) \in \{(9, 16), (10, 16), (12, 32)\}$. Outer involutions invert 3-cycles, so by the character tables have commutator space of dimension at least 8, 8, 16 in the respective cases. This eliminates $n = 12$ and proves that $|A| \geq 16$ in the remaining cases. Suppose $n = 9$. The centralizer of a transposition is $2 \times \mathfrak{S}_7$, whose centralizer is less than 8-dimensional, so this case is out as well. Finally for the 16-dimensional representation of $\mathfrak{S}_{10}$ we must have $|A| = 32$ since the previous argument shows that the centralizer space is at most 7-dimensional. But the representation splits into two 8-dimensional ones upon restriction to $\mathfrak{A}_9$. There, the centralizer of a rank-4 subgroup is 2-dimensional (see above), hence it is at most 4-dimensional when $d = 16$. This rules out the last candidate.  

4. Groups of Lie type in nondefining characteristic

4.1. Linear groups

To deal with the linear groups in nondefining characteristic we make use of the Landazuri–Seitz–Zalesskii lower bounds for irreducible representations and the information in the Atlas and Modular Atlas for some small groups.

Proposition 4.1. Let $S = L_2(q)$ with $\ell \nmid q$, $q \neq 4, 5, 9$, $(q, \ell) \neq (7, 2)$. Then no absolutely irreducible $FG$-module satisfies (1.1).

Proof. The minimal degree of a nontrivial projective representation in nondefining characteristic for $S := L_2(q)$ is $(q - 1)/2$ if q is odd, respectively $q - 1$ if $q \geq 8$ is even. First assume that $\ell$ is odd. Then Aut(S) has $\ell$-rank at most 2, and $k_\ell \leq 3$ for any subgroup of Aut(S). Thus $d \leq 12$, so $q \leq 25$ (respectively $q \leq 13$ for even $q$). But for $q \leq 25$, $q \neq 8$, $|\text{Out}(S)|$ is even, hence the $\ell$-rank is 1, and $k_\ell = 2$. This gives the better bound $q \leq 9$ (respectively $q \leq 5$ or $(q, \ell) = (8, 3)$ for even $q$). Thus we are left with $q = 7, d \leq 4$, and $q = 8, d \leq 8$, both with $\ell = 3$. In the first case the relevant 3-modular representations, of degrees 3 and 4, are not defined over $\mathbb{F}_3$, thus no example arises. In the second case $S = L_2(8)$ the only possibility is $d = 7$. Since $k(x) = 2$ for $x$ an inner 3-element, their centralizer is at most 3-dimensional. Since they are third powers, their Jordan blocks have sizes 3, 2, 2. Outer 3-elements normalize 7-elements, so have Jordan blocks
of sizes 3, 3, 1. Thus the centralizer is at most 3-dimensional for $|A| = 3$, at most 2-dimensional for $|A| = 9$. This gives a contradiction.

Now consider $\ell = 2$, so $q$ is odd. The 2-rank of $S$ is 2 and $k_2 = 3$, thus in this case $d \leq 12$, $q \leq 25$. Furthermore there is a single class of involutions. Two of these can be chosen to generate a dihedral group of order $q + \epsilon$, $\epsilon \in \{\pm 1\}$, $q \equiv \epsilon \pmod{4}$. Since this is maximal for $q > 11$ we deduce that two suitable conjugates of any four-group in $\text{Aut}(S)$ containing inner elements generate at least $S$. Thus in fact $q \leq 17$ in this case. For $G = \text{PGL}_2(q)$ the minimal degree equals $q - 1$, the 2-rank is still 2 and four outer involutions generate, forcing $d \leq 8$, $q \leq 9$, so this case does not arise here. Finally, if $G$ involves field automorphisms, then either $A$ has rank 1, or $A$ contains an involution from $\text{PGL}_2(q)$. Since again four (conjugate) outer involutions generate, in either case $d \leq 12$, so $q \leq 25$.

For $L_2(25)$ we have $d = 12$. An inner involution inverts an element of order 13, which acts fixed point freely on the 12-dimensional module. Hence inner involutions have 6-dimensional commutator space, so $|A| = 8$. The centralizer of such an involution in $\text{PGL}_2(25)$ is a dihedral group of order 48. If all involutions in this centralizer have the same commutator space, the same is true for the elements of order 3 therein. But their centralizer on the 12-dimensional module is only 4-dimensional. This rules out the case $S = L_2(25)$. For $q = 19$, 23 there are no field automorphisms, so they cannot occur. Next, $L_2(17)$ contains a Frobenius group of order 17 · 8, so elements of order 8 have a single Jordan block on the 8-dimensional module. Thus elements of order 4 have two Jordan blocks and inner involutions have 4-dimensional commutator space. The involution centralizer is $D_4$, so is generated by its noncentral involutions. If all of these had the same 4-dimensional centralizer, the same would hold for elements of order 4 therein, contradiction. For $q = 13$ the 6-dimensional representation is not defined over $\mathbb{F}_2$, so it does not occur. For $q = 11$ the 5- and 12-dimensional representations are not defined over $\mathbb{F}_2$, so $df = 10$ with $f = 2$ respectively $f = 1$. Since elements of order 5 are inverted by involutions, the latter have commutator space of dimension 2 (respectively 4). If a second commuting involution has the same commutator space, then so has an element of order 3 in the centralizer of the first, which is not the case.

This completes the proof of Proposition 4.1.

Proposition 4.2. Let $S = L_n(q)$, $n \geq 3$, $(n, q) \notin \{(3, 2), (4, 2)\}$, with $\ell \nmid q$. Then the absolutely irreducible $FG$-modules $V$ satisfying (1.1) are given in Table 6.5.

Proof.

(1) We proceed as follows. By [19] a lower bound for the minimal degree of projective representations is given by $d(n, q) := (q^n - 1)/(q - 1) - n$ (unless $n \leq 4$, $q \leq 3$). On the other hand $f(A)$ is bounded above by $|\text{Aut}(G)|^2$. This severely restricts the possibilities for $(n, q, \ell)$. 

This completes the proof of Proposition 4.1.
(2) Let first $n = 3$, so $q \neq 2$. Then our estimate forces $q \leq 4$. First consider $S = L_3(4)$. The Sylow $\ell$-subgroups are cyclic for $\ell = 5, 7$ and two of them generate, so $d \leq 4$, which is not possible. For $\ell = 3$ the rank of $\operatorname{Aut}(S)$ equals 2, three elements generate, so $d \leq 12$. This leaves $d \in \{6, 10\}$ for $2L_3(4)$ and $d \in \{4, 8\}$ for the 4-fold coverings. The cases $d = 8, 10$ are out since they are not defined over $F_3$. The 6-dimensional representation extends to $2L_3(4).2_2$ from where it restricts irreducibly to $G_6$. The restriction to $A_6 = O_3(9)$ is now the direct sum of the natural module and its dual, so we get an example for $2L_3(4)$. The 4-dimensional representation has $f = 2$ and embeds $4_2L_3(4)$ into $SU_4(3)$. The fusion shows that elements of order 3 have Jordan blocks of sizes 3 and 1, and the Sylow 3-subgroup acts indecomposably. Thus no example arises. The group $S.3 = PGL_3(4)$ does not lead to examples.

For $S = L_3(3)$ only $\ell = 2, 13$ have to be considered. The Sylow 13-subgroups are cyclic and two of them generate, so this cannot occur. Now assume that $\ell = 2$. The 2-rank of $S$ is 2, for $\operatorname{Aut}(S)$ it equals 3. Since $k_2 = 3$, $d \leq 12$ respectively $d \leq 18$. From the character table we find that $d = 12$ is the only case. Inner involutions invert 3B-elements, so have at least 4-dimensional commutator. This forces $|A| \geq 4$. For a fixed involution $g \in A$ we can choose a further involution $h$ in its centralizer such that $\langle A, h \rangle = \langle g \rangle \times G_3$, in particular has at most 6-dimensional centralizer. This forces $|A| = 8$ and hence $G = \operatorname{Aut}(L_3(3))$. Let $g$ be an outer involution. Since it inverts an element of order 13, its commutator space is 6-dimensional. The centralizer of $g$ is $C = C_1 \times \langle g \rangle$ with $C_1 \cong G_4$. The involutions in $C_1$ not in the normal Klein four group generate $G_4$, so any $A$ containing one of these has centralizer of dimension at most 5, which contradicts (1.1). Now assume that $A$ is the normal subgroup of $C$ of order 8. Let $g'$ be a second outer involution in $A$. Then $C_1 \cap A$ is no longer normal in the centralizer of $g'$, since $C_1$ is the full normalizer of $C_1 \cap A$ in $S$. Thus we are reduced to the previous case.

(3) When $n \geq 4$ then comparison of our bounds shows that either $n = 4$, $q = 3$, or $S = L_5(2)$, $\ell = 3$. For $S = L_4(3)$ and $\ell = 5, 13$ the rank is 1 and five Sylow $\ell$-subgroups generate, so $d \leq 10$, which is impossible. For $\ell = 2$ we obtain the bound $d \leq 49$, so $d \in \{26, 38\}$. Involutions from both inner classes invert elements of order 5. From the character table we conclude that the commutator space of any inner involution is at least 10-dimensional for $d = 26$ (respectively 16-dimensional for $d = 38$). Thus $|A| \geq 32$ for $d = 26$, while $d = 38$ is not possible. Since $S$ has 2-rank four, some outer automorphism must be present when $d = 26$. The 26-dimensional representations only extend to $G.2_2$, thus $|A| = 32$, and $A$ must have centralizer of dimension 16. But the centralizer of an element $g$ from $2A$ is of the form $D_4 \times G_6$. If an elementary abelian subgroup of the $G_6$-factor of order 8 has the same centralizer as $g$, then the whole $G_6$ has the same centralizer, which is not the case by the trace of 5-elements. Hence no example arises.

The 3-rank of $S = L_5(2)$ is 2, and $k_3 \leq 6$, whence $d \leq 24$, a contradiction. \qed
4.2. Unitary groups

**Proposition 4.3.** Let $S = U_3(q)$ with $\ell \nmid q$, $q \neq 2$. Then the absolutely irreducible $FG$-modules $V$ satisfying (1.1) are given in Table 6.5.

**Proof.** We proceed as in the previous proof, using the lower bound $q^2 - q$ for degrees of nontrivial projective representations from [19]. If $\ell$ is odd and different from 3 then this yields $q \leq 4$. For $U_3(4)$ the $\ell$-rank is at most 2, and two subgroups of rank 2 generate, thus $d \leq 8$, smaller than the lower bound. For $U_3(3)$ we have $\ell = 7$, so the rank is one and two elements generate, again a contradiction.

For $\ell = 3$ we get $q \leq 5$. For $\text{Aut}(U_3(5))$ the 3-rank is equal to 2, and $k_2 \leq 4$, giving $d \leq 16$. For $U_3(4)$ the rank is 1, leading to no example.

Finally $S = U_3(3)$ has 2-rank 2 and $k_2 = 4$, so $d \leq 16$ for $S$. This gives $d \in \{6, 14\}$. The 6-dimensional representation extends to $\text{Aut}(S) = G_2(2)$ and the restriction to $L_3(2)$.2 remains irreducible. This restricts as the direct sum of the natural representation of $L_3(2)$ and its dual, hence we obtain an example. Since inner involutions invert a regular element of order 3, they have at least 5-dimensional commutator space in the 14-dimensional representation, thus this is out. Next, $G = \text{Aut}(U_3(3))$ has 2-rank 3, hence $d \leq 30$, which implies $d \in \{6, 14\}$. Clearly the case $d = 6$ yields an example. The case $d = 14$ is the adjoint representation of $G_2(2)$. Outer involutions have at least 6-dimensional commutator space, since they invert elements of order 7. Assume we have $A$ with 8-dimensional centralizer. Choose a further outer involution $g$ such that $H := \langle A, g \rangle$ contains elements of order 7. Then [5] shows that $H = G$. This contradicts the fact that both $A$ and $g$ have 8-dimensional centralizer.

**Proposition 4.4.** Let $S = U_n(q)$, $n \geq 4$, with $\ell \nmid q$, $(S, \ell) \neq (U_4(2), 3)$. Then the absolutely irreducible $FG$-modules $V$ satisfying (1.1) are given in Table 6.5.

**Proof.**

1. Let first $n = 4$. The lower bound for degrees of nontrivial projective representations is $(q^2 - 1)(q - 1)$ when $q \geq 4$. Our standard estimate shows that $q \leq 3$. Let first $\ell$ be odd. Then the $\ell$-rank of $U_4(3)$ is one, which gives no example.
For $U_4(2)$ we have $\ell = 5$, with rank 1 and $k_5 = 2$, so $d \leq 4$. But the 4-dimensional representation of $2.U_4(2)$ is not defined over $\mathbb{F}_5$.

For $\ell = 2$ we only have to treat $q = 3$. The group $S = U_4(3)$ has 2-rank 4 (see [12, 4.10.5]). Moreover $k_2 = 3$, so $d \leq 24$ in this case. This leaves $d = 20$ for $S$ and $(d, f) = (6, 2)$ for $3.S$. The latter gives an example with $|A| = 8$. Assume $d = 20$. Inner involutions have at least 8-dimensional commutator since they invert 5-elements, so $|A| = 16$. Let $C$ be the centralizer of an involution in $S$. The extra-special group $O_2(C) \cong 2^{1+4}$ does not contain an elementary abelian subgroup of order $2^4$. Thus inside $C$ there exist two conjugates of $A$ generating a subgroup containing 3-elements. But all 3-elements have at least 12-dimensional commutator, a contradiction. For $S.2_1$ the 2-rank of centralizers of outer involutions is at most 4, forcing $d \leq 24$. This leads to $d = 20$ for $S.2_1$ and $d = 12$ for $3_1.S.2_1$. In the first case, since outer involutions from 2C invert 3A-elements, they have at least 9-dimensional commutator, while for 2B-elements the centralizer $2 \times U_3(3)$ has only 2-rank 3, impossible. Since $d = 12$ for $G = 3_1.S.2_1$ is induced, outer involutions have 6-dimensional commutator. In particular $|A| \geq 8$. The module $V$ splits into a direct sum $V_1 \oplus V_2$ under $G' = 3_1.S$. Denote by $A'$ the intersection of $A$ with $G' = 3_1.S$. Then clearly $\dim(C_{V}(A)) = \dim((C_{V_1}(A'))$. Since inner involutions have 4-dimensional centralizer this shows $\dim(C_V(A)) \leq 4$ and hence $|A| = 16$. In particular $|A'| = 8$ has centralizer of dimension 4 on $V_1$, as any inner involution. But $O_2$ of the centralizer $C$ in $G'$ of an inner involution does not have a normal subgroup $2^3$. Thus we obtain a contradiction with Lemma 2.3.

For $S.2_2$ the centralizer of an outer involution in 2D is $2 \times U_4(2)$, hence $S.2_2$ has 2-rank 5. This leads to $d = 20$ for $S.2_2$ and $d = 6, 15$ with $f = 2$ for $3_1.S.2_2$. The case $d = 6$ is the reflection (transvection) representation of $3_1.S.2_2$, so gives an example. When $d = 15$ estimates for the centralizers of outer involutions show that $A$ has to contain inner involutions. Now 2A-involutions have at most 9-dimensional centralizer. Since $f = 2$ this forces $|A| \geq 2^6$, impossible. The 20-dimensional module is the exterior cube of the 6-dimensional representation of $S.2_2$ embedding it into $U_6(2)$. But the exterior cube of the natural module is not an example for $U_6(2)$, see the second part.

For $S.2_3$ the outer involution centralizer is $2 \times M_{10}$, so has 2-rank 3. This gives $d \leq 18$, so no example arises.

There are two possible groups $S.2^2$, namely $S.(2^2)_{122}$ and $S.(2^2)_{133}$ in Atlas notation. The 2-rank of $S.(2^2)_{122}$ equals 5, so $d \leq 40$. This leaves $d = 20, 34$ for $G = S.(2^2)_{122}$, and $d = 12, 30$ for $G = 3_1.S.(2^2)_{122}$. The 12-dimensional representation gives an example. For the 30-dimensional representation we have $|A| > 4$, so $A$ contains inner involutions. But 2A-involutions have centralizer dimension at most 18, whence $|A| \geq 2^6$, which is not possible. Similarly, in the 34-dimensional representation, $A$ necessarily contains inner involutions, but these have at most 20-dimensional centralizer. Finally, $d = 20$ is the restriction of the exterior cube of the natural module for $U_6(2)$, so does not give an example. The
2-rank of $S.(2^2)_{133}$ equals 4, so only $d = 20$ needs to be considered. Since 2E- and 2F-involutions have centralizer dimension at most 11 on this module, $A$ has to be 2A-pure, hence contained in $S$, contradiction.

(2) If $n = 5$ we arrive at $q \leq 3$. For $S = U_5(3)$ we have $\ell = 2$, with 2-rank bounded above by 5, and generated by 5 conjugates, so $d \leq 50$, impossible. For $G = \text{Aut}(S) = S.2$ there is a single class of outer involutions, with centralizer $2 \times O_5(3)$ [12, Table 4.5.2]. Thus the rank of elementary abelian 2-groups containing outer involutions is at most 5, and again no example arises. For $S = U_5(2)$ necessarily $\ell = 3$ and $d \leq 48$, so either $d = 10$ or $d = 44$ with $G = S$. By [2, E.5.8], any elementary abelian 3-group can be embedded into a maximal torus. By Lemma 2.2 we may assume that $A = 3^4$ is maximal, hence equal to a maximal torus. Thus it contains an element $g$ with centralizer $3 \times U_4(2)$. The 10-dimensional module has two 5-dimensional composition factors for $U_4(2)$, so $g$ has 5-dimensional centralizer space. Again by loc. cit. all maximal elementary abelian 3-subgroups of $U_4(2)$ are conjugate. The 5-dimensional representation identifies $U_4(2)$ with $O_5(3)$, and this clearly has an elementary abelian subgroup of order 27 with only 1-dimensional centralizer. Hence $A$ can only have 1-dimensional centralizer, ruling out the case $d = 10$. The 44-dimensional representation is the exterior square of the 10-dimensional one (up to a trivial composition factor), hence it occurs neither.

(3) For $n \geq 6$ the only possibilities are $q = 2$, $n \leq 7$ and $\ell = 3$. For $S = U_6(2)$ the 3-rank is 4 and six conjugates generate. This leads to $d = 21$. Since this representation extends to $\text{PGU}_6(2) = S.3$ we may argue there. As above we may assume that $A$ is a maximal torus of order $3^5$. Let $g$ be an element with centralizer $3 \times U_5(2)$. On restriction to $U_5(2)$ we find two copies of the 10-dimensional representation considered above, and we already saw that any $3^4$ has only 1-dimensional centralizer there. Hence $A$ has at most 3-dimensional centralizer, impossible. For $U_7(2)$, $\ell = 3$, we can reduce the question inductively to the case considered before. \(\Box\)

4.3. Symplectic and orthogonal groups

Proposition 4.5. Let $S$ be a finite simple symplectic or orthogonal group of Lie type in characteristic different from $\ell$ and in dimension at least 4, $S \neq S_4(2)', S_4(3)$. Then the absolutely irreducible $FG$-modules $V$ satisfying (1.1) are given in Table 6.5.

Proof.

(1) Let’s first consider the symplectic groups defined over fields of odd order. The minimal degree of nontrivial projective representations of $S_{2n}(q)$, $q$ odd, is given by $(q^n - 1)/2$. Let first $S = S_4(q)$. For odd $\ell$ the $\ell$-rank of $\text{Aut}(S)$ is bounded above by 3, and at most four conjugates generate, so $d \leq 24$. This
gives \( q \leq 7 \). The 2-rank is at most 6, and five conjugates generate. Moreover, three conjugates of elementary abelian subgroups of order 8 generate, so \( d \leq 36 \), and again \( q \leq 7 \). For \( S = S_4(7) \) a more precise study of the \( \ell \)-ranks shows that \( \ell = 2 \), with rank 5. This leads to \( d = 24 \). Identify \( S = O_5^+(7) \) and write 2A, 2B for the class of involutions with 4 respectively 2 eigenvalues \(-1\) in the natural 5-dimensional representation. From the embedding \( O_4^-(7) < O_5(7) \) we see that 2B-involutions invert elements of order 25, so have 12-dimensional commutator. But clearly there are no 2A-pure subgroups of order at least 4, so this case is out.

For \( S = S_4(5) \) and \( \ell = 3, 13 \) we get \( d \leq 12 \), so the only possible degree is \( d = 12 \) for 2.5. S, but in both cases \( f = 2 \). When \( \ell = 2 \) then \( d \leq 50 \), so \( d = 12, f = 2 \), or \( d = 40 \). Write 2A, 2B for the two classes of inner involutions as in the case \( S_4(7) \). Then 2B-involutions invert 5C-elements, hence have 6-dimensional respectively at least 16-dimensional commutator space when \( d = 12 \) respectively \( d = 40 \). Since 2A-pure subgroups have order 2 neither gives an example. The group \( S_4(3) \) is isomorphic to \( U_4(2) \) and has already been considered in Proposition 4.4.

For \( S = S_6(q) \) we arrive at \( q \leq 3 \) for odd \( \ell \) and \( q \leq 5 \) for \( \ell = 2 \). For \( S_6(5), \ell = 2 \), the only possibility would be \( d = 62 \), but there \( f = 2 \). For \( S_6(3) \) all odd primes \( \ell \neq 3 \) have rank 1, so \( \ell = 2 \). Then the rank is 7, seven conjugates generate, hence \( d \leq 98 \). This gives \( d = 13, f = 2 \), or \( d = 78 \). The latter is no example since both inner involutions invert elements from class 3G, hence have at least 26-dimensional commutator space. In the case of \( d = 13 \) the same argument shows that \( A \) has to be 2A-pure of order \( |A| \geq 2^4 \). The centralizer \( C \) of a 2A-involution has type \( 2(A_4 \times U_4(2)) \), so \( A \) cannot be contained in its \( O_2(C) \). Now \( U_4(2) \) has three 4-dimensional composition factors, hence any four group containing involutions outside \( O_2(C) \) has at most 7-dimensional centralizer. Thus \( |A| \geq 2^6 \), but \( U_4(2) \) does not contain an elementary abelian \( 2^3 \) with 3-dimensional centralizer on the natural module. This rules out the case \( d = 13 \). Outer involutions of \( S_6(3).2 \) have centralizer \( 2 \times L_3(3) : 2 \) of 2-rank 4. This gives \( d \leq 32 \), so \( d = 26 \). But this representation is induced, so the commutator space of outer involutions is 13-dimensional, impossible.

For \( S = S_8(q) \) only \( q = 3, \ell = 2 \) is possible, and leads to \( d = 40 \) with \( f = 2 \). The 40-dimensional Weil representations of \( S \) split into three copies of 13-dimensional Weil representations of \( S_6(3) \) plus a trivial constituent. From the character values it can be seen that any involution in \( S \) has at least 12-dimensional commutator space, but the 2-rank of \( \text{Aut}(S) \) is not larger than 10, hence we obtain no example.

For \( S_{2n}(q), n \geq 5 \), the generic estimate now gives \( S_{10}(3), \ell = 2 \) as the only other possibility. Here \( d = 121 \) with \( f = 2 \). Since \( S \) has 2-rank 10, this is not possible for \( S \), so \( G = S.2 \). But the outer involution has centralizer \( L_4(3) \times 2 \) by [12, Table 4.5.2], so the rank of \( A \) containing outer involutions is at most 5, ruling out this case as well.
(2) We are left with the orthogonal groups. Lower bounds for minimal degrees of nontrivial projective representations of $S$ are given in [19], and $f(A)$ is at most equal to $|G|^2$. For $S_{2n}(q)$ with $q$ even this leaves only the possibilities $S_4(4)$, $S_6(2)$ and $S_8(2)$. For $S_4(4)$ we get $d \leq 26$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S = S_6(2)$ the estimate yields $d \leq 27$. For $\ell = 5, 7$ the rank is 1 and 3 conjugates generate, hence there are no examples. For $S = S_8(2)$ the estimate yields $d \leq 12$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_4(4)$ we get $d \leq 26$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_6(2)$ the estimate yields $d \leq 27$. For $\ell = 5, 7$ the rank is 1 and 3 conjugates generate, hence there are no examples. For $S_8(2)$ the estimate yields $d \leq 12$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_4(4)$ we get $d \leq 26$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_6(2)$ the estimate yields $d \leq 27$. For $\ell = 5, 7$ the rank is 1 and 3 conjugates generate, hence there are no examples. For $S_8(2)$ the estimate yields $d \leq 12$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_4(4)$ we get $d \leq 26$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_6(2)$ the estimate yields $d \leq 27$. For $\ell = 5, 7$ the rank is 1 and 3 conjugates generate, hence there are no examples. For $S_8(2)$ the estimate yields $d \leq 12$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_4(4)$ we get $d \leq 26$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_6(2)$ the estimate yields $d \leq 27$. For $\ell = 5, 7$ the rank is 1 and 3 conjugates generate, hence there are no examples. For $S_8(2)$ the estimate yields $d \leq 12$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_4(4)$ we get $d \leq 26$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_6(2)$ the estimate yields $d \leq 27$. For $\ell = 5, 7$ the rank is 1 and 3 conjugates generate, hence there are no examples. For $S_8(2)$ the estimate yields $d \leq 12$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_4(4)$ we get $d \leq 26$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_6(2)$ the estimate yields $d \leq 27$. For $\ell = 5, 7$ the rank is 1 and 3 conjugates generate, hence there are no examples. For $S_8(2)$ the estimate yields $d \leq 12$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_4(4)$ we get $d \leq 26$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_6(2)$ the estimate yields $d \leq 27$. For $\ell = 5, 7$ the rank is 1 and 3 conjugates generate, hence there are no examples. For $S_8(2)$ the estimate yields $d \leq 12$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_4(4)$ we get $d \leq 26$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_6(2)$ the estimate yields $d \leq 27$. For $\ell = 5, 7$ the rank is 1 and 3 conjugates generate, hence there are no examples. For $S_8(2)$ the estimate yields $d \leq 12$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_4(4)$ we get $d \leq 26$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_6(2)$ the estimate yields $d \leq 27$. For $\ell = 5, 7$ the rank is 1 and 3 conjugates generate, hence there are no examples. For $S_8(2)$ the estimate yields $d \leq 12$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_4(4)$ we get $d \leq 26$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises. For $S_6(2)$ the estimate yields $d \leq 27$. For $\ell = 5, 7$ the rank is 1 and 3 conjugates generate, hence there are no examples. For $S_8(2)$ the estimate yields $d \leq 12$, but $k_\ell \leq 4$ and the $\ell$-rank is at most 2, hence no example arises.

(3) For the orthogonal groups over fields of odd order and in dimension at least 7 the only cases satisfying our basic inequality are $O_7(3)$ and $O_8^+(2)$. For $S = O_7(3)$ we necessarily have $\ell = 2$. Then $\log_2(|S.2|) \leq 70$, giving $d = 27$, $f = 2$ for $G = 3.S$. Involutions from class 2A have six equal eigenvalues on the natural module for $S$, so no 2A-pure four groups exist in $S$. Involutions from the other two classes invert elements from 3$G$, so have at least 9-dimensional commutator space for $d = 27$. Thus this cases is out.

For $S = O_8^+(2)$ we have $\ell = 3, 5$. For $\ell = 5$ the three classes of 5-elements are fused under the triality automorphism. Three elements from class 5A generate (via the maximal subgroup $A_9$), so the same is true for the other two classes. Consequently $d \leq 12$, so the only possibility is $d = 8$ for $G = 2.S$ with $A$ a Sylow 5-subgroup. This is the 5-modular reduction of the ordinary reflection representation of the Weyl group $W(E_8)'$. Let $K$ be (the intersection with 2$S$ of) a reflection subgroup $A_4 \times A_4$ (visible in the extended Dynkin diagram). Then the restriction to $K$ consists of two copies of the deleted permutation module for the two factors. In particular we see 5-elements with at least 6-dimensional commutator. When $\ell = 3$ then we either have $G = 2.S$, $d = 8$, or $G = S$, $d = 28$. Since $S$ has 3-rank 4, $d = 8$ gives an example for $\ell = 3$. When $d = 28$ we use that without loss $|A| = 3^4$ by Lemma 2.2. In particular $A$ contains elements from all classes of 3-elements. But one of these classes contains elements normalizing 7-elements, thus with at least 16-dimensional commutator. This yields a contradiction.

For $S = O_8^-(2)$ no examples arise. □
4.4. Exceptional groups

**Proposition 4.6.** Let $S$ be a finite simple exceptional group of Lie type in characteristic different from $\ell$, $S \neq 2G_2(3)', G_2(2)'$. Then no absolutely irreducible $FG$-module satisfies (1.1).

**Proof.** Lower bounds for minimal degrees of nontrivial projective representations of $S$ are given in [19], while an upper bound for $f(A)$ is $|\text{Aut}(S)|^2$. The resulting inequality is only satisfied for the groups $2B_2(8), G_2(3), G_2(4), F_4(2)$. For $2B_2(8)$ the $\ell$-rank is 1 and $k_\ell = 2$, so $d \leq 6$, which is impossible. For $S = G_2(3)$ and $\ell = 7, 13$, the rank is 1, $k_\ell \leq 3$, thus $d \leq 6$, a contradiction. For $\ell = 2$ the rank is 3 (respectively 4 for $S.2$), and $k_2 \leq 3$ (respectively 4), so $d \leq 18$ (respectively $d \leq 32$). Hence by [16] the only possibility is $d = 14$. Inner involutions invert elements from class 3C which have 12-dimensional commutator space, so $A$ has to have rank at least 3. The subgroup $U_3(3).2$ has odd index in $S$ and the 14-dimensional representation restricts irreducibly, so we must have $G = S.2$ by the proof of Proposition 4.3. Outer involutions have centralizer $2 \times L_2(8): 3$ and commutator space of dimension at least 6 on the 14-dimensional module. Under restriction to $L_2(8): 3$ this has two trivial composition factors and two 6-dimensional ones. The latter are induced from the natural module of $L_2(8)$, so involutions have 3-dimensional commutator. Clearly the centralizer space of an outer involution contains such a representation, hence any noninner Klein four group has at least 9-dimensional commutator, which violates the upper bound.

For $S = G_2(4)$ and $\ell = 7, 13$ the rank is 1 and two $\ell$-elements generate, so $d \leq 4$, which is not possible. The 5-rank equals 2 and three 5-elements respectively two Sylow 5-subgroups generate, so $d \leq 8$, again impossible.

The 3-rank also equals 2 and $k_3 \leq 3$, so $d \leq 12$, that is $d = 12$ and $G = 2.S$. Elements of class 3B act nontrivially on subgroups of order 13, which have no fixed points, so $A$ cannot contain 3B-elements. But the centralizer of a 3A-element is $\text{SL}_3(4).2_3$ and the fusion shows that it contains only two 3A-elements.

For $F_4(2)$ and $\ell = 5, 7, 13, 17$ the rank is at most 2 and $k_\ell \leq 4$, so $d \leq 16$, which is not possible. For $\ell = 3$ the rank is 4, and $k_3 \leq 5$, so $d \leq 40$, again too small. $\square$

5. Sporadic groups

In the case of sporadic groups at the moment we have to leave open two cases: the 22-dimensional respectively 24-dimensional $F_2G$-modules for $G = Co_2$ respectively $G = Co_1$. In both cases, there exist subgroups $A \leq G$ with $f(A)$ at most a factor 2 off $|V|$, e.g., $A$ maximal elementary abelian of order $2^{10}$ respectively $2^{11}$. 
Proposition 5.1. Let $S$ be a sporadic simple group. Then the absolutely irreducible $FG$-modules $V$ satisfying (1.1) are given in Table 6.7, or possibly the 22-dimensional $F_2\text{Co}_2$-module or the 24-dimensional $F_2\text{Co}_1$-module.

Proof. We refer to the table of $\ell$-ranks of sporadic groups in [12, Table 5.6.1], and to the table of lower bounds for degrees of faithful representations in [17, p. 137], plus for some groups like the Conway groups the more detailed information in [14]. The tables show that for the sporadic groups $J_1, He, ON, J_4, Fi_{223} Fi_{24}$, $HN, Ly, Th, B$ and $M$ no examples can occur. We deal with the remaining 15 groups case by case.

- For $S = M_{11}$ the basic inequality leaves $\ell = 3, d = 5$, clearly an example with $|A| = 3^2$, and $\ell = 2, d = 10$. In the latter case involutions have at least 4-dimensional commutator, so $|A| = 2^2$, but $O_2$ of the involution centralizer is a quaternion group, so the standard argument rules out this possibility.

- For $S = M_{12}$ we get $\ell \leq 3$. If $\ell = 3$ then only $d = 6$ for $2.S$ meets the bound, and we get an example with $|A| = 3^2$. For $\ell = 2$ we have to consider $d = 10$ for $S$ and for $S.2$. This is the deleted permutation module for the 5-fold transitive permutation representation of $S$, and explicit computation shows that this gives an example with $|A| \geq 8$.

- For $S = M_{22}$ we necessarily have $\ell \leq 3$. For $\ell = 3$ we get $d \leq 12$ which is not possible since the 10-dimensional representation of $2.S$ is not defined over $F_3$. Thus $\ell = 2$. The 2-rank of $S$ is 4 and $k_2 = 3$, so $d \leq 24$. Thus either $d = 10$ for $S$, or $d = 6, f = 2$ for $3.S$. Both give examples with $|A| \in \{8, 16\}$. The group $3.S$ has a subgroup $2^4(3.A_6)$, and upon restriction to $3.A_6$ the 6-dimensional module has two 3-dimensional composition factors, hence the $2^4$ has 3-dimensional centralizer. For $G = S.2$ the 2-rank is 5, and $k_2 \leq 4$, hence $d \leq 40$. This gives $d = 10, 34$ for $S.2, d = 12, 30$ for $3.S.2$. If $d = 34$ then $A$ contains inner involutions, but this is not possible since already three of these generate. Clearly $d = 10$ is an example. Since the 30-dimensional representation of $3.S.2$ is induced, outer involutions have fixed space of dimension 15, thus cannot lie in $A$. The 12-dimensional representation of $3.S.2$ is induced from $S.2$. Explicit computation in that representation shows that the centralizer space of an elementary abelian 2-subgroup $A$ involving outer involutions is of dimension at most $4, 3, 3, 1$ for $A$ of order $2^2, 2^3, 2^4, 2^5$ respectively. Hence we do not get an example.

- For $S = J_2$ it follows that $\ell \leq 5$. For $\ell = 5$ we have $d = 6$ for $2.S$. All 5-elements have at least 4-dimensional commutator space, since the restriction to $A_5 = L_2(5)$ containing 5AB-elements (respectively $SL_2(5)$ containing 5CD-elements) has only two constituents. The centralizer in $S$ of a 5A-element is $5 \times A_5$, thus if a Sylow 5-subgroup had more than 2-dimensional centralizer, the same were true for the $A_5$. But this contains elements from class 3A without fixed points on the module. Hence this case is out. For $\ell = 3$ the rank is 2. Then $k_3 \leq 3$ and two Sylow 3-subgroups generate, whence $d \leq 8$. Since the representation of
Since the centralizer of an outer involution is $2 \times d$ (respectively 8-dimensional) commutator space when $d = 11$, this case is out. On the other hand $d = 6$ gives an example. For $J_2.2$ we find $d = 12$. The centralizer of an outer involution $g$ is $\langle g \rangle \times \text{PGL}_2(7)$, so $|A| \leq 8$. On the other hand outer involutions have 6-dimensional centralizer. If $A$ is an example then all conjugates of $A$ by the centralizer of $g$ have to have the same centralizer. But they generate a group containing elements of order 7 with trivial centralizer, contradiction.

- For $S = M_{23}$ the only possibility is $\ell = 2$ with $d \leq 24$, so $d = 11$. This yields an example with $|A| \geq 8$.

- For $S = HS$ we find $\ell = 2$ and $d \leq 24$, respectively $d \leq 50$ for $S.2$, so $d = 20$. Let first $G = S$. Since 2B-elements invert 5A-elements, their commutator is 10-dimensional, so $A$ has to be 2A-pure. Furthermore, 2A-elements invert 5B-elements. Hence their commutator on the 20-dimensional module is at least 8-dimensional and necessarily $|A| = 2^4$. Let $C = O_2(C).\mathfrak{S}_5$ with $O_2(C) = D_4 \ast D_4 * 4$ [12, Table 5.3m] be the centralizer of a 2A-involution. As $A$ cannot be contained in $O_2(C)$ and $\mathfrak{A}_5$ is generated by its class of involutions, we deduce that $\mathfrak{A}_5$ has 12-dimensional centralizer. But this contradicts the character table. The group $G = S.2$ has 2-rank 5. Outer 2D-involutions have 10-dimensional commutator (inverting 11-elements) and centralizer $2^4.\mathfrak{S}_5$ [12, Table 5.3m], so $A$ cannot contain 2D elements by the standard argument. Since $A$ has to contain inner involutions, our considerations for $S$ show that $|A| \geq 2^4$. Clearly $A$ can be chosen in the centralizer $C = 2 \times \mathfrak{S}_8$ of a fixed 2C-involution $g$. The constituents of the restriction to $\mathfrak{S}_8$ are of dimensions 6, 6, 8: two times the deleted permutation module and the induced of the natural module for $L_4(2) = \mathfrak{A}_8$. Note that $|A| > 2^4$ implies that $A \cap \mathfrak{S}_8$ has to contain involutions not of permutation type $2^4$, hence with 4-dimensional commutator on the 8-dimensional constituent. But $g$ has to centralize this 8-dimensional module and has at least 6-dimensional commutator. So $A$ has at most 10-dimensional centralizer, forcing $|A| = 2^5$. But in fact $A \cap \mathfrak{S}_8$ clearly acts nontrivially on the two 6-dimensional composition factors, thus $A$ has at most 9-dimensional centralizer, contradiction.

- For $S = J_3$ either $\ell = 3$, $d \leq 18$, which is not possible since $d = 18$ has $f = 2$, or $\ell = 2$, $d \leq 30$. This leads to $d = 9$, $f = 2$, for $3.S$, or $d = 18$ for $3.S.2$. The inner involutions invert elements of order 17, so have 4-dimensional (respectively 8-dimensional) commutator space when $d = 9$ respectively $d = 18$. Since the centralizer of an outer involution is $2 \times L_2(17)$ with 2-rank 3, this rules out the second case. In the first case we get $|A| = 2^4$. The centralizer $C$ of an inner involution $g \in A$ is $(D_4 \ast Q_8).\mathfrak{A}_5$ (see [12, Table 5.3h]). Since $A$ cannot be contained in $O_2(C)$ and $g$ already has 4-dimensional commutator, the centralizer of $A$ is at most 4-dimensional, so we obtain no example.

- For $S = M_{24}$ we necessarily have $\ell = 2$. The rank is 6, and $k_2 \leq 4$, so $d \leq 48$. This leaves $d = 11$ or $d = 44$. The first clearly is an example since it is one upon
restriction to $M_{23}$. On the other hand inner involutions invert elements of order 5, so have at least 18-dimensional commutator when $d = 44$, a contradiction.

- For $S = M^{c}L$ we arrive at $\ell \leq 3$, when $\ell = 3$ $k_{3} \leq 5$, while the 3-rank is 4, so $d \leq 40$, which implies $d = 21$. The $(3A, 3A, 10A)$- and $(3B, 3B, 7A)$-structure constants in $S$ are nonzero, so elements from $3A$ (respectively $3B$) have at least 10-dimensional (respectively 12-dimensional) commutator. Thus this case does not occur. For $\ell = 2$ the rank is 4 and $k_{2} \leq 4$, so $d = 22$. Involutions invert elements from $5B$, so have at least 8-dimensional commutator space. Thus, $|A| = 16$ and $C_{V}(A)$ is 14-dimensional. Thus, $A$ is trivial on the fixed space of any involution in $A$. The centralizer of an involution is $2.\varphi_{8}$. Since this is generated by any class of elementary abelian subgroups of order 16 and contains elements of order 7 (which do not centralize a 14-dimensional subspace) we do not get an example. The rank of 2-subgroups of $\text{Aut}(S)$ involving outer involutions is 3 (since the centralizer is $2 \times M_{11}$), hence this gives no further example.

- For $S = Ru$ we arrive at $\ell = 2$, and since the 2-rank is 6 we have $d \leq 48$, so $d = 28$ has to be considered. Involutions from class $2B$ have 14-dimensional commutator space, since they invert elements of order 29, so $A$ has to be $2A$-pure. Elements from $2A$ invert elements of order 13, so have at least 12-dimensional commutator. In particular, to get an example we need that $2A$-elements have 16-dimensional centralizer, $|A| = 2^{6}$, and $C_{V}(g) = C_{V}(A)$ for all involutions $g \in A$. Since the center of the centralizer of a $2A$-involution only has order 2, there exist involutions in $A$ with distinct centralizer. But these centralizers are maximal subgroups, so two of them generate the group. Since both would also stabilize $C_{V}(A)$, this is a contradiction.

- For $S = Suz$ the Suzuki group the estimates give $\ell \leq 5$. For $\ell = 5$ the rank is 2 and $k_{5} \leq 5$. This forces $d \leq 20$, but $d = 12$ has $f = 2$, so no case arises. For $\ell = 3$ the rank is 5, thus $d \leq 50$, which gives $d = 12$ for $2S$. First assume that $A$ contains $3C$-elements. The commutator space of a $3C$-element is 8-dimensional, as can be seen by restricting to $L_{3}(3) : 2$, so $|A| \geq 3^{4}$. Since the centralizer has structure $3^{2} : 4 \times \varphi_{6}$, hence 3-rank 4, we even have $|A| = 3^{4}$. This is ruled out by the standard argument from the structure of the centralizer. Thus $A$ is $3C$-free. Now any $3B$-element $g$ is uniquely expressible as a product of two commuting $3A$-elements $h_{1}, h_{2}$, and thus any other 3-element commuting with $g$ also commutes with $h_{1}, h_{2}$. Hence any elementary abelian 3-subgroup of $S$ not containing $3C$-elements is contained in an elementary abelian 3-subgroup generated by $3A$-elements, with the same normalizer. So we may assume that $A$ contains $3A$-elements. Upon restriction to the centralizer $6_{2}.U_{4}(3)$ of a $3A$-element the 12-dimensional module has two 6-dimensional composition factors. Thus $3A$-elements have 6-dimensional centralizer and $|A| \geq 3^{3}$. Any 3-element of $2.U_{4}(3) = SO_{6}^{e}(3)$ has at least 2-dimensional commutator on this 6-dimensional centralizer, so $A$ has centralizer dimension at most 4, that is $|A| \geq 3^{4}$. But subgroups of order $3^{3}$ of $SO_{6}^{e}(3)$ have at most 3-dimensional centralizer on the natural module, hence $|A| = 3^{5}$. By [22, 2.2] there exists
a unique class of elementary abelian subgroups of that order in $G$, with normalizer $3^5:M_{11}$. The complement $M_{11}$ has composition factors of dimensions 1, 1, 5, 5, hence either $A$ has 1-dimensional or 6-dimensional centralizer. The first possibility violates the 2F-condition. In the second case, all 3-elements in $A$ would have the same centralizer. But this is impossible since 3A-centralizers are maximal and do not contain a central 3$^5$.

For $\ell = 2$ the rank is 6, so $d \leq 60$, which leads to $d = 12$ with $f = 2$ for $3.S$, or $d = 24$ for $3.S.2$. The two classes 2A, 2B of involutions of $S$ have at least 4- respectively 6-dimensional commutator space when $d = 12$. Let’s first assume that $A$ is 2A-pure. Then $|A| \geq 2^4$, and in fact $A = 2^4$ since by [22, 2.4] the largest 2A-pure 2-subgroup of $S$ has order $2^4$. So $C_V(A) = C_V(g)$ for all involutions $g \in A$. But 2A-centralizers $C \cong 2^{1+6}.O^-_6(2)$ are maximal in $S$ and do not contain a $2^4$ in their center, so this leads to a contradiction. Hence $A$ must contain 2B-elements, and $|A| = 2^6$. Since products of commuting 2A-elements lie again in class 2A, in the second case $A$ has to contain a 2B-pure four-group. According to [12, Table 5.3o] this has centralizer $(2^2 \times L_3(4)).2$. But $L_3(4)$ does not have 6-dimensional centralizer in this representation, hence the standard argument rules out this case. For $G = 3.S.2$ the outer involutions have 12-dimensional commutator since the representation with $d = 24$ is induced. On the other hand the centralizers are $2 \times M_{12}.2$ respectively $2 \times J_2.2$, both of 2-rank at most 5. This gives a contradiction.

- For $S = Co_3$ we obtain $\ell = 2, 3$ and $d = 22$. For $\ell = 3$ the rank is 5. The $(3C, 3C, 23A)$-structure constant does not vanish, so 3C-elements have at least 11-dimensional commutator space on the 22-dimensional module. Thus $A$ cannot contain 2B-elements, and $|A| = 2^6$. Since products of commuting 2A-elements lie again in class 2A, in the second case $A$ has to contain a 2B-pure four-group. According to [12, Table 5.3o] this has centralizer $(2^2 \times L_3(4)).2$. But $L_3(4)$ does not have 6-dimensional centralizer in this representation, hence the standard argument rules out this case. For $G = 3.S.2$ the outer involutions have 12-dimensional commutator since the representation with $d = 24$ is induced. On the other hand the centralizers are $2 \times M_{12}.2$ respectively $2 \times J_2.2$, both of 2-rank at most 5. This gives a contradiction.

- For $S = Co_2$ either $\ell = 3, d = 23$, or $\ell = 2, d = 22$. Let first $\ell = 3$. The subgroup $H = U_6(2)$ contains a full Sylow 3-subgroup of $S$, and the restriction of the 23-dimensional representation contains the 21-dimensional representation of $H$. But the latter is not an example by Proposition 4.4. The case $\ell = 2$ was excluded in the statement.

- For $S = Co_1$ the estimates give $\ell \leq 5$. For $\ell = 5$ the rank is 3. From the character table it follows that for $X \in \{A, B, C\}$ the $(5X, 5X, 3A)$-structure constants are nonzero. Since 3A-elements have trivial centralizer for $d = 24$ all 5-elements have at least 12-dimensional commutator space, which shows that $A$
cannot exist. For $\ell = 3$ the rank is 6, so $d \leq 60$, again only giving $d = 24$ for $2.S$. The $(3X, 3X, 5A)$-structure constant is nonzero for $X \in \{A, B, C, D\}$, and $5A$-elements act fixed point freely, so all 3-elements have at least 12-dimensional commutator. In particular $|A| = 3^6$. The centralizers of 3-elements are given in [12, Table 5.3l] and the standard argument shows that this case cannot occur.

For $\ell = 2$ the rank is 11, thus $d \leq 110$ whence $d = 24$ for $S$, which was excluded.

- For $S = Fi_{22}$ we arrive at $\ell = 2$, with rank 10, so $d \leq 154$. This leaves $d = 27$ with $f = 2$ for $3.S$ and $d = 78$ for $S$. Since all involutions invert elements from 3A, they have at least 21-dimensional commutator in the 78-dimensional representation. This violates the required bound. Now consider a 27-dimensional module $V$ for $3.S$. Elements from 2C have at least 12-dimensional commutator, so cannot lie in $A$. Elements from 2B have at least 10-dimensional commutator, so $|A| = 2^{10}$. But $O_2$ of the centralizer $(2 \times 2^{1+8}) : O_6^2(2).2$ of a 2B-involution [12, Table 5.3t] does not contain such an $A$, hence the standard argument forces $A$ to be 2A-pure. But the structure constants show that 2A-pure subgroups have order 2, while 2A-elements have at least 5-dimensional commutator. Thus this case is also impossible. In Aut$(S) = S.2$ we again arrive at $d = 78$ for $S.2$ or $d = 54$ for $3.S.2$. Since all outer involutions invert elements from 3C, their commutator space is at least 24-dimensional when $d = 78$ while clearly the 2-rank is at most 11. The 54-dimensional representation is induced, thus outer involutions have 27-dimensional commutator space, which is again impossible. □

6. Cubic action

In this section we show that all the examples $(G, V)$ classified in the previous sections have cubic action, that is, there exists an elementary abelian $\ell$-subgroup $A \leq G$ such that

$$[[[V, A], A], A] = 0 \quad \text{and} \quad |V/C_V(A)| \leq |A|^2. \quad (6.1)$$

We call these the 2$F$-modules with cubic offender.

In the following tables, we list 2$F$-modules and the size of some offender $A$. When there is an entry under Out, this indicates that the representation extends to the corresponding almost simple group. In that case, we give the size of some offender for the simple group and for the almost simple group.

**Theorem 6.2.** The 2$F$-modules with cubic offender of alternating groups $S = A_n$ with $n \geq 5$, $(n, \ell) \notin \{(5, 2), (5, 5), (6, 3), (8, 2)\}$, are given in Table 6.3.

**Proof.** By the result of Proposition 3.1 we only have to show that all cases in Table 6.3 admit $A$ with cubic action.
Table 6.3
2F-modules for alternating groups

| G      | Out | d   | f   | ℓ   | |A|   |
|--------|-----|-----|-----|-----|-----|
| \A_n   | 2   | n−2 | 1   | 2   | 2|n | 2/2  |
| \A_n   | 2   | n−1 | 1   | 2   | 2|n | 2/2  |
| 3.\A_6 | 3   | 2   | 2   |     |     | 2   |
| 3.\S_6 | 6   | 1   | 2   |     |     | 4   |
| \A_7   | 4   | 1   | 2   |     |     | 2   |
| \S_7   | 8   | 1   | 2   |     |     | 8   |
| \A_9   | 8   | 1   | 2   |     |     | 8   |

\A_n

| n−2 | 1  | 3     | 3|n | 3 |
| n−1 | 1  | 3     | 3|n | 3 |
| 2.\A_5 | 2 | 2 | 3   | 3 |
| 2.\A_9 | 8 | 1 | 3   | 27 |

The four infinite families corresponding to the deleted permutation module are examples since double-transpositions respectively 3-cycles have centralizer space of codimension 2 and clearly have cubic action. In particular \(|A| = ℓ\) in these cases. For \S_n on the deleted permutation module in characteristic 2 we can take \(A\) generated by a transposition.

The case 3.\A_6 is a transvection group, so \(|A| = 2\) works. The 6-dimensional representation of 3.\S_6 splits into the two 3-dimensional 3.\A_6-modules considered before. Thus a transvection of 3.\A_6 has centralizer dimension 4 on the 6-dimensional module, and on this a commuting outer involution has 2-dimensional centralizer. This gives an example with \(|A| = 4\).

The group \A_7 \leq L_4(2) is a bi-transvection group. Explicit calculation with representing matrices shows that the 8-dimensional representation of \S_7 also gives an example with \(|A| = 8\). The 8-dimensional representation of \A_9 restricts irreducibly to the subgroup \S_7, hence gives an example by the previous case.

The group 2.\A_5 \leq \SL_2(9) is a transvection group.

In the last case explicit computation shows that \(A\) of order 27 has cubic action. \(\Box\)

**Theorem 6.4.** The 2F-modules with cubic offender of simple groups of Lie type in nondefining characteristic are given in Table 6.5.

**Proof.** We have seen in Section 4 that the modules satisfying (1.1) are precisely those of Table 6.5. It remains to prove that we have cubic action in all these cases. The group U_3(3) in its 6-dimensional representation is generated by bi-transvections. The group 3_1.U_4(3).2_2 is a transvection group. The derived group S_6(2) of the Weyl group of type E_7 contains 3-elements with a single Jordan block of length 3 inside a reflection subgroup of type A_2. The same argument applies
Table 6.5
2F-modules for groups of Lie type in nondefining characteristic

| G             | Out | d | f | ℓ     | |A| |
|---------------|-----|---|---|-------|---|
| U₃(3)         | 2   | 6 | 1 | 2     | 2/4 |
| 3₁.U₄(3)      | 2   | 6 | 2 | 2     | 8/2 |
| 3₁.U₄(3).(2²)₁₂₂ | 12 | 1 | 2 |       | 32 |
| 2.L₂(4)       |     | 6 | 1 | 3     | 9  |
| S₆(2)         |     | 7 | 1 | 3     | 3  |
| 2.S₆(2)       |     | 8 | 1 | 3     | 27 |
| 2.O₈⁺(2)      |     | 8 | 1 | 3     | 3  |

to the derived group 2.O₈⁺(2) of the Weyl group of type E₈. The remaining cases can be checked by direct computation.

Theorem 6.6. The 2F-modules with cubic offender of sporadic simple groups, different from the 22- or 24-dimensional F₂-modules of the Conway group Co₂ respectively Co₁, are given in Table 6.7.

Table 6.7
2F-modules for sporadic groups

| G             | Out | d | f | ℓ | |A| |
|---------------|-----|---|---|---|---|
| M₁₂           | 2   | 10| 1 | 2 | 8/16 |
| M₂₂           | 2   | 10| 1 | 2 | 8/8  |
| 3.M₂₂         | 6   | 2 | 2 | 8 |     |
| J₂            | 6   | 2 | 2 | 16|     |
| M₂₃           | 11  | 1 | 2 | 8 |     |
| M₂₄           | 11  | 1 | 2 | 8 |     |
| M₁₁           | 5   | 1 | 3 | 9 |     |
| 2.M₁₂         | 6   | 1 | 3 | 9 |     |

Proof. By the result of Proposition 5.1 all 2F-modules of sporadic groups are listed in Table 6.7. It can now be checked by explicit computation in the corresponding representations that in all cases there exist offenders with cubic action, of the indicated orders.

Acknowledgment

We thank G. Stroth for pointing out an omission in a previous version.
References