ELSEVIER

Discrete Mathematics 131 (1994) 173–181

DISCRETE MATHEMATICS

Partitions of large Boolean lattices

Zbigniew Lonc

Institute of Mathematics, Warsaw University of Technology, 00-661 Warsaw, Poland

Received 7 June 1990; revised 18 November 1991

Abstract

Let 2^n_- be the ordered set obtained from the Boolean lattice 2^n by deleting both the greatest and the least elements. Define $f(n)$ to be the minimum number k such that there is a partition of 2^n_- into k antichains of the same size except for at most one antichain of a smaller size. In the paper we examine the asymptotic behavior of $f(n)$ and we show that $c_1 n \leqslant f(n) \leqslant c_2 n^2$ for some constants c_1 and c_2 and n sufficiently large. Moreover, we prove for all ordered sets P of size less than 5, a conjecture that for n sufficiently large there is a partition of 2^n_- into ordered sets isomorphic to P if and only if some obvious divisibility conditions are satisfied.

1. Introduction

Let 2^n be the ordered set of all subsets of $\{1, 2, \ldots, n\}$ ordered by set inclusion. It was conjectured by Sands [5] that 2^n can be partitioned into chains of size 2^k, for n sufficiently large given k. The conjecture was strengthened by Griggs [2] who suspected that for n sufficiently large given m, 2^n can be partitioned into chains of size m except for at most $m-1$ elements that also form a chain. This stronger conjecture was proved for $m=4$ (the simplest nontrivial case) by Griggs et al. [3] and for arbitrary m by Lonc [4].

In this paper we deal with a variation of the dual problem: (**Q**) Can, for n sufficiently large given m, the ordered set 2^n be partitioned into antichains of size m except for at most $m-1$ elements which also form an antichain?

An obvious answer to the above question is no (with exception of the trivial case $m=1$) because the greatest and the least elements of 2^n belong to 1-element antichains only. The answer is less trivial; however, when we delete both the greatest and the least elements from 2^n. Denote the ordered set obtained this way by 2^n_-. We show (see Theorem 2.1) that the answer to the question (Q) is yes this time. Even more turns out to be true (see Theorem 2.2), specifically m need not be a constant but we can let m grow exponentially with respect to n and the answer to (Q) still remains yes.

SSDI 0012-365X(92)00536-L

Denote by $m(n)$ the maximum integer m such that 2^n_- can be partitioned into antichains of size m except for at most $m-1$ elements which also form an antichain. Instead of asking about the behavior of the function $m(n)$ we can ask about the behavior of $f(n)$, where $f(n)$ is the minimum number f such that there is a partition of 2^n_- into f antichains of the same size except for at most one antichain of a smaller size. Clearly, $f(n)=\lceil (2^n-2)/m(n) \rceil$.

We prove (Corollary 2.5) that $f(n)\leqslant cn^2$ for every constant $c>1$ and n large. On the other hand, $f(n)\geqslant n-1$ because the longest chains in 2^n_- have $n-1$ elements. It is easy to improve this lower bound slightly but we are not able to disprove that there is a constant c_1 such that $f(n)\leqslant c_1 n$ for n large. Therefore, the following problem remains open.

Problem. Find the asymptotic behavior of $f(n)$.

Section 3 of this paper is devoted to supporting the following conjecture that generalizes some results on partitions of 2^n_- contained in both this paper and [4].

Conjecture 1.1 (Lonc [4]). Let P be an ordered set. For n sufficiently large the ground set of 2^n_- can be partitioned into subsets generating ordered sets isomorphic to P if and only if $2^n-2=0 \pmod{|P|}$.

If P is an antichain then the conjecture is true by our Theorem 2.1. Moreover, a very slight modification of the proof of the main theorem in [4] shows that the conjecture holds if P is a chain. In this paper we prove the conjecture for $P=2^k_-$, $k=1,2,\ldots$, and for all ordered sets P of size less than 5.

2. Antichains

Denote by L_i, $i=1,2,\ldots,n-1$, the ith level of 2^n_-, i.e. the set $\{A\in 2^n_- : |A|=i\}$.

Theorem 2.1. *For each positive integer m and $n\geqslant m+1$, the ordered set 2^n_- can be partitioned into antichains of size m except for at most $m-1$ elements which also form an antichain.*

Proof. Let X be the ground set of 2^n_-, i.e. $X=\{\emptyset\neq A\not\supseteq\{1,2,\ldots,n\}\}$. We introduce a lexicographic linear order $\leqslant_L$ on X. For $A,B\in X$, we define $A\leqslant_L B$ if $|A|<|B|$ or $|A|=|B|$ and $\max(A-B)\leqslant\max(B-A)$, where $\max C$ stands for the maximum number in $C\subseteq\{1,\ldots,n\}$.

We claim that if $A\subseteq B$ for some $A,B\in X$, $A\neq B$, then there are at least $n-2$ members of X between A and B in the linear order $\leqslant_L$. Note that

$A \in L_1 \cup \cdots \cup L_{n-2}$ when $A \subseteq B$ and $A \neq B$. To prove the claim we consider three cases.

Case 1: $A \in L_i$, $i = 2, \ldots, n-3$.

Suppose first that the greatest element in A is less than n. Then every $C \in L_i$ which contains n is after A in the order $\leqslant_L$ (and obviously before B because $B \in L_j$ for some $j > i$). There are exactly

$$\binom{n-1}{i-1} \geqslant n-2$$

such C's.

Thus, assume now that the greatest element in A is n. Then every $C \in L_{i+1}$ which does not contain n is before B in the order $\leqslant_L$ (and clearly after A) because $i+1 \leqslant j$. We have exactly

$$\binom{n-1}{i+1} \geqslant n-2$$

such C's so we are done in Case 1.

Case 2: $A \in L_1$.

Let $A = \{r\}$ for some $r = 1, \ldots, n$. Then by the definition of $\leqslant_L$, $A \leqslant_L \{1, r\} \leqslant_L B$. Our claim follows now from the observation that there are exactly $n - r + \binom{r-1}{2} \geqslant n-2$ members of X between A and $\{1, r\}$ in the order $\leqslant_L$.

Case 3: $A \in L_{n-2}$.

Then $B \in L_{n-1}$. This case is dual to the previous one because $A \leqslant_L B$ if and only if $\bar{B} \leqslant_L \bar{A}$, where $\bar{C}$ is the complement of C, for each $C \subseteq \{1, \ldots, n\}$. Therefore, it can be dealt with analogously.

It follows from the claim that each set of $m \leqslant n-1$ successive elements in the linear order $\leqslant_L$ forms an antichain in $2^{\underline{n}}$. Consequently, $2^{\underline{n}}$ can be partitioned into antichains of size m except for at most $m-1$ elements which also form an antichain. $\square$

In the next theorem we shall show that if n is large enough then some stronger statement than Theorem 2.1 is true. We will prove that the ordered set $2^{\underline{n}}$ can be partitioned into antichains of size m, except for at most one antichain of a smaller size, for each $m \leqslant A\,2^n/n^2$, where A is a constant. As a consequence we get an exponential lower bound for $m(n)$, namely $m(n) \geqslant A 2^n/n^2$. More precisely we shall prove the following theorem.

Theorem 2.2. *For every real number $c > 1$, there is an integer $n_0 = n_0(c)$ such that if $n \geqslant n_0$ then, for every $m \leqslant \lceil 2^n/cn^2 \rceil$, the ordered set $2^{\underline{n}}$ can be partitioned into antichains of size m except for at most $m-1$ elements which also form an antichain.*

Before proving Theorem 2.2 we shall need two auxiliary lemmas.

Lemma 2.3. *Let* $i=\lceil n/2-a\sqrt{n}\rceil$, *where* a *is a constant. Then*

$$|L_i|=\binom{n}{i}=\sqrt{\frac{2}{\pi}}\,\mathrm{e}^{-2a^2}2^n/\sqrt{n}(1+\mathrm{o}(1)),$$

as $n\to\infty$.

The proof of this lemma is based on Stirling's formula. We leave it to the reader.

Lemma 2.4. *Let* G *be a bipartite graph with vertex classes* X *and* Y *such that* $\deg_G v=x$ *for every* $v\in X$, $\deg_G v=y$ *for every* $v\in Y$ *and* $x<y$. *Then* G *has a factor whose every component is a star with either* $\lfloor y/x\rfloor$ or $\lceil y/x\rceil$ *leaves and with the center in a vertex of* Y.

Proof. The proof follows immediately from a much stronger theorem by de Werra [1]: *For every* k *a bipartite graph* G *is a union of* k *edge-disjoint graphs* $G_1,\ldots,G_k$ *such that* $\lfloor(\deg_G v)/k\rfloor\leqslant\deg_{G_i}v\leqslant\lceil(\deg_G v)/k\rceil$ *for each* $v\in V(G)$. We get the assertion by taking $k=x$. $\square$

Proof of Theorem 2.2. Let d be any real number such that $1<d<c$ and define a to be the real number such that

$$\int_0^{\sqrt{2}a}\mathrm{e}^{-x^2}\,\mathrm{d}x=\frac{\sqrt{\pi}}{2}\cdot\frac{d+1}{2d}.$$

Denote by l the smallest integer such that both the number $\binom{n-1}{l-1}$ of elements in L_l containing 1 and the number $\binom{n-1}{l}$ of elements in L_l not containing 1 are at least $2^n/cn^2$. By Lemma 2.3 l is well-defined and $l<\lceil n/2-a\sqrt{n}\rceil$ if n is sufficiently large.

Let $S_i^j=\{\{j\}\cup A\in L_i\colon A\subseteq\{j+1,j+2,\ldots,n\}\}$ for every $i,j=1,\ldots,n$. Clearly, the family $\{S_i^1,\ldots,S_i^{n-i+1}\}$ is a partition of L_i and, by the definition of l,

$$|S_i^j|<2^n/cn^2\quad\text{for } i=1,\ldots,l-1. \tag{1}$$

For $i=l,l+1,\ldots,n-l$, let R_i (or Q_i) be any subset of S_i^1 (or $L_i-S_i^1$) such that

$$|R_i|=|Q_i|=\lceil 2^n/cn^2\rceil. \tag{2}$$

Define a bipartite graph G_i whose vertex classes are L_1 and L_i, where $n/2-a\sqrt{n}\leqslant i<n/2$. A pair BC, $B\in L_1$, $C\in L_i$, is an edge in G_i if $B\nsubseteq C$. The degree of every vertex in L_1 (or in L_i) is $\binom{n-1}{i}$ (or $n-i$). Since $n-i<\binom{n-1}{i}$ for n sufficiently large, by Lemma 2.4 G_i has a factor whose every component is a star of size either

$$\left\lfloor\binom{n-1}{i}\Big/(n-i)\right\rfloor\quad\text{or}\quad\left\lceil\binom{n-1}{i}\Big/(n-i)\right\rceil$$

and with the center in a vertex of L_1. Denote by M_i^j the set of leaves of the star with the center in $\{j\}$, $j=1,\dots,n$. Clearly, the family $\{M_i^1,\dots,M_i^n\}$ is a partition of L_i. Define

$$q=\left\lceil \frac{5\sqrt{2/\pi}\,ad}{d-1}\right\rceil$$

and let $p=p(i)$, $1\leqslant p\leqslant q$, be the unique integer such that $n/2-a\sqrt{n}p/q\leqslant i<n/2-a\sqrt{n}(p-1)/q$. By Lemma 2.3

$$\begin{aligned}|M_i^j|&\geqslant\left\lfloor\binom{n-1}{i}\Big/(n-i)\right\rfloor\geqslant\left\lfloor\binom{n-1}{\lceil n/2-a\sqrt{n}p/q\rceil}\Big/(n/2+a\sqrt{n})\right\rfloor\\&=\left\lfloor\frac{n-\lceil n/2-a\sqrt{n}p/q\rceil}{n}\binom{n}{\lceil n/2-a\sqrt{n}p/q\rceil}\Big/(n/2+a\sqrt{n})\right\rfloor\\&=\sqrt{\frac{2}{\pi}}\mathrm{e}^{-2(ap/q)^2}2^n/n\sqrt{n}(1+\mathrm{o}(1)).\end{aligned}\tag{3}$$

Define $\overline{M_i^j}=M_i^j-(R_i\cup Q_i)$. Divide each set $\overline{M_i^j}$ into $m_i=\lceil\sqrt{\frac{2}{\pi}}\mathrm{e}^{-2(ap/q)^2}d\sqrt{n}\rceil$ almost equal parts (i.e. into parts whose sizes differ by at most 1) and denote the resulting sets by $N_{i,1}^j,N_{i,2}^j,\dots,N_{i,m_i}^j$. Obviously, the sets $N_{i,k}^j$, $n/2-a\sqrt{n}\leqslant i<n/2$, $j=1,\dots,n$, $k=1,\dots,m_i$, are pairwise disjoint, $N_{i,k}^j\subseteq L_i$ and, by (2) and (3),

$$|N_{i,k}^j|\geqslant 2^n/dn^2(1+\mathrm{o}(1))\geqslant\lceil 2^n/cn^2\rceil\tag{4}$$

if n is sufficiently large.

For a fixed j, the number of the sets $N_{i,k}^j$ is equal to

$$\begin{aligned}\sum_{i=\lceil n/2-a\sqrt{n}\rceil}^{\lceil n/2\rceil-1}m_i&\geqslant\sum_{p=1}^{q}\left\lceil\sqrt{\frac{2}{\pi}}\mathrm{e}^{-2(ap/q)^2}d\sqrt{n}\right\rceil\left\lfloor\frac{a\sqrt{n}}{q}\right\rfloor\\&\geqslant\sqrt{\frac{2}{\pi}}d\sqrt{n}\left(\frac{a\sqrt{n}}{q}-1\right)\cdot\left(\left(\sum_{p=0}^{q}\mathrm{e}^{-2a^2p^2/q^2}\right)-1\right)\\&\geqslant\sqrt{\frac{2}{\pi}}\frac{adn}{q}\left(\int_0^q\mathrm{e}^{-2a^2x^2/q^2}\,\mathrm{d}x-1\right)(1-\mathrm{o}(1))\\&=\left(\frac{d+1}{4}n-\sqrt{\frac{2}{\pi}}adn/q\right)(1-\mathrm{o}(1))\\&\geqslant\left(\frac{d+1}{4}-\frac{d-1}{5}\right)n(1-\mathrm{o}(1))\\&=\left(\frac{d}{20}+\frac{9}{20}\right)n(1-\mathrm{o}(1)).\end{aligned}\tag{5}$$

Note that $S^j_{i_1} \cup N^j_{i_2,k}$, for every $i_1 = 1, \ldots, l-1, j = 1, \ldots, n$, $n/2 - a\sqrt{n} \leqslant i_2 < n/2$ and $k = 1, \ldots, m_{i_2}$, is an antichain in 2^n_- because j belongs to each set in $S^j_{i_1}$, j belongs to no set in $N^j_{i_2,k}$ and $i_1 < i_2$. Since $l - 1 < n/2 - a\sqrt{n} < ((d/20) + (9/20))n(1 - o(1))$, for n large, according to (5) we can choose for every $i_1 = 1, \ldots, l-1$ and $j = 1, \ldots, n$ a set $N^j_{i_2(i_1),k(i_1)}$ such that the chosen sets are pairwise different. Let $D^j_i = S^j_i \cup N^j_{i_2(i),k(i)}$, for $i = 1, \ldots, l-1$, $j = 1, \ldots, n$. Define C^j_i to be any subset of the antichain D^j_i containing S^j_i whose size is a multiple of m. It exists since $m \leqslant \lceil 2^n/cn^2 \rceil$ and, by (4),

$$|D^j_i| \geqslant |S^j_i| + \lceil 2^n/cn^2 \rceil,$$

for n sufficiently large. Let $C = \bigcup_{i=1}^{l-1} \bigcup_{j=1}^{n} C^j_i$. Obviously,

$$\bigcup_{i=1}^{l-1} L_i \subseteq \bigcup_{i=1}^{l-1} \bigcup_{j=1}^{n} S^j_i \subseteq C \subseteq \bigcup_{i=1}^{l-1} \bigcup_{j=1}^{n} D^j_i \subseteq \bigcup_{i=1}^{\lceil n/2 \rceil - 1} L_i.$$

By the self-duality of 2^n_-, there exist pairwise disjoint antichains $\overline{C^j_i}$, for $i = n-l+1, \ldots, n-1$ and $j = 1, \ldots, n$ of sizes that are multiplies of m and

$$\bigcup_{i=n-l+1}^{n-1} L_i \subseteq \bar{C} \subseteq \bigcup_{i=\lfloor n/2 \rfloor + 1}^{n-1} L_i,$$

where $\bar{C} = \bigcup_{i=n-l+1}^{n-1} \bigcup_{j=1}^{n} \overline{C^j_i}$.

To prove the theorem it suffices to show that the ordered set $A_n = 2^n_- - (C \cup \bar{C})$ can be partitioned into antichains of size m except for at most $m-1$ elements which also form an antichain. Note that $A_n \subseteq \bigcup_{i=l}^{n-l} L_i$ and $R_i \cup Q_i \subseteq L_i \cap A_n$ for every $i = l, l+1, \ldots, n-l$.

Let P be a permutation of A_n such that for $i_1 < i_2$ the elements of $A_n \cap L_{i_1}$ appear in P before the elements of $A_n \cap L_{i_2}$, the elements of Q_i appear before all the remaining elements of $A_n \cap L_i$ and the elements of R_i appear after all the remaining elements of $A_n \cap L_i$. By the definitions of R_i and Q_i, $R_i \cup Q_{i+1}$ is an antichain for $i = l, l+1, \ldots, n-l-1$. Thus, by (2), each subsequence of $\lceil 2^n/cn^2 \rceil$ successive terms in P forms an antichain. Consequently, for every $m \leqslant \lceil 2^n/cn^2 \rceil$, A_n can be partitioned into antichains of size m except for at most $m-1$ elements which also form an antichain. □

The following corollary is an immediate consequence of Theorem 2.2.

Corollary 2.5. *For every real number $c > 1$ there is an integer $n_1 = n_1(c)$ such that for $n \geqslant n_1$ the ordered set 2^n_- can be partitioned into not more than cn^2 antichains of the same size except for at most one antichain of a smaller size.*

3. Small ordered sets

In this section we shall prove the Conjecture formulated in the introduction for some specific ordered sets P. As was mentioned in the introduction the conjecture holds for antichains and chains. Moreover, the condition $2^n-2\equiv 0 \pmod{||P|}$ excludes the case $|P|=4$ so the only ordered sets P that we have to consider in order to show that conjecture holds for $|P|<5$ are the ordered sets V, V^D and I depicted in Fig. 1.

Let us introduce some more notation. For every ordered set B isomorphic to 2^n we denote by B_- the ground set of the ordered set obtained from B by deleting both the maximum and the minimum elements. We identify the elements of 2^n with 0–1 sequences of length n. Every sequence s of length n consisting of the symbols 0, 1 and X denotes the set of all sequences of length n with 0's and 1's at the same places as in s. For example 0X1X stands for the set $\{0010, 0011, 0110, 0111\}$. Let P, Q and R be ordered sets. We write $P=Q+R$ to denote that the ground set of P can be partitioned into two subsets inducing the ordered sets Q and R in P.

Lemma 3.1. *For every positive integers k and n, $k\leqslant n$,*

$$2^n_- = 2^{n-k}\times 2^k_- + 2^{n-k+1}_-.$$

Proof. Note that

$$2^n = 2^{n-k}\times 2^k = 2^{n-k}\times(2^k_- + 2^1) = 2^{n-k}\times 2^k_- + 2^{n-k+1}.$$

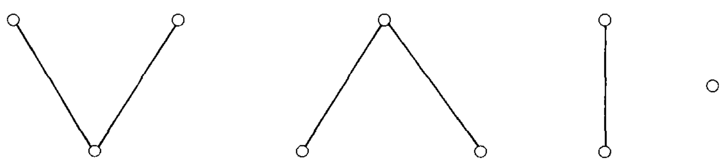
Clearly the greatest element (or the least element) of 2^n must belong to the ordered set isomorphic to 2^{n-k+1} because the set $2^{n-k}\times 2^k_-$ does not have a greatest (or least) element. Thus, $2^n_- = 2^{n-k}\times 2^k_- + 2^{n-k+1}_-$. $\square$

Theorem 3.2. *Let P be one of the ordered sets V, V^D, I and 2^m_- for some positive integer m. Then there exists a partition of 2^n_- into ordered sets isomorphic to P if and only if*

$$2^n-2\equiv 0 \pmod{|P|} \quad \text{when } P=I,\ 2^m_- \text{ and}$$

$$2^n-2\equiv 0 \pmod{|P|} \quad \text{and } n\neq 3 \text{ when } P=V,\ V^D.$$

Proof. Clearly, it suffices to consider the cases of P equal to V, I and 2^m_- only because V and V^D are dual. Necessity is trivial. Note that for $P=V$, I the condition $2^n-2\equiv 0$



V V^D I

Fig. 1.

(mod $|P|$) is equivalent to $n \equiv 1$ (mod 2) and for $P = 2^{\underline{m}}$ it is equivalent to $n \equiv 1$ (mod$(m-1)$). We shall divide the proof of sufficiency into three parts.

(1) $P = I$.

Note that $\{\{100, 011, 110\}, \{001, 101, 010\}\}$ is the required partition of $2^{\underline{3}}$. Since by Lemma 3.1, $2^{\underline{n}} = 2^{n-3} \times 2^{\underline{3}} + 2^{\underline{n-2}}$, the theorem follows for $P = I$ by induction.

(2) $P = V$.

One can readily check that the set

$$\begin{aligned}\mathcal{T} = \{&\{10000, 10100, 10010\}, \{00100, 11100, 10110\}, \{01110, 11110, 01111\},\\ &\{11000, 11010, 11101\}, \{01000, 01100, 01001\}, \{00010, 00110, 01010\},\\ &\{10001, 11001, 10101\}, \{00101, 01101, 00111\}, \{00011, 01011, 10011\},\\ &\{00001, 11011, 10111\}\}\end{aligned}$$

is the partition of the ground set of the ordered set $2^{\underline{5}}$ into subsets generating ordered sets isomorphic to V.

The ground set of $2^{\underline{7}}$ can be partitioned into the sets

$$A_1 = (\text{XXXXX}00)_-, \qquad A_2 = (\text{XXXXX}01)_-,$$

$$A_3 = (\text{XXXXX}10)_-, \qquad A_4 = (\text{XXXXX}11)_-$$

and

$$A_5 = \{1111100, 0000001, 1111101, 0000010, 1111110, 0000011\}.$$

Since A_1, A_2 and A_3 generate ordered sets isomorphic to $2^{\underline{5}}$, they can be partitioned into ordered sets isomorphic to V. Let $\mathcal{T}'$ be the partition of A_4 obtained from $\mathcal{T}$ by adding '11' after each sequence in each class of the partition $\mathcal{T}$. It is easy to verify that

$$\begin{aligned}\mathcal{T}' - &\{\{1000011, 1010011, 1001011\}, \{0010011, 1110011, 1011011\}\}\\ \cup &\{\{1111100, 1111101, 1111110\}, \{0000001, 1000011, 0010011\},\\ &\{0000010, 1010011, 1001011\}, \{0000011, 1110011, 1011011\}\}\end{aligned}$$

is a partition of $A_4 \cup A_5$ into subsets inducing V.

The theorem for $P = V$ follows now by induction because by Lemma 3.1, $2^{\underline{n}} = 2^{n-5} \times 2^{\underline{5}} + 2^{\underline{n-4}}$.

(3). $P = 2^{\underline{m}}$.

The theorem holds trivially for $n = m$. Since $2^{\underline{n}} = 2^{n-m} \times 2^{\underline{m}} + 2^{\underline{n-(m-1)}}$ by Lemma 3.1, we are done again by induction. $\square$

Acknowledgement

I would like to express my gratitude to an anonymous referee for a very careful reading of the first version of the paper. Her (or his) remarks helped greatly in improving this article.

References

[1] D. de Werra, Equitable colorations of graphs, R.I.R.O. R-3 (1971) 3–8.
[2] J.R. Griggs, Problems on chain partitions, Discrete Math. 72 (1988) 157–162.
[3] J.R. Griggs, R.K.C.Yeh and C.M. Grinstead, Partitioning Boolean lattices into chains of subsets, Order 4 (1987) 65–67.
[4] Z. Lonc, Proof of a conjecture on partition of a Boolean lattice, Order 8 (1991) 17–27.
[5] B. Sands, Problem session, Colloquium on Ordered Sets (Szeged, Hungary, August 1985).