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The Full Periodicity Kernel for σ Maps

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Let σ be the topological space formed by the points (x, y) of $\mathbb{R}^2$ such that either $x^2 + y^2 = 1$, or $0 \leqslant x \leqslant 2$ and $y = 1$. A σ map f is a continuous self-map of σ having the branching point $(0, 1)$ as a fixed point. We denote by $\operatorname{Per}(f)$ the set of periods of all periodic points of f, and by $\mathbb{N}$ the set of positive integers. We prove that if f is a σ map and $\{2, 3, 4, 5, 7\} \subseteq \operatorname{Per}(f)$, then $\operatorname{Per}(f) = \mathbb{N}$. Conversely, if $S \subseteq \mathbb{N}$ is a set such that for every σ map f, $S \subseteq \operatorname{Per}(f)$ implies $\operatorname{Per}(f) = \mathbb{N}$, then $\{2, 3, 4, 5, 7\} \subseteq S$.

1. INTRODUCTION

A connected finite regular graph (or *graph* for short) is a pair consisting of a connected Hausdorff space G and a finite subspace V (points of V are called *vertices*) such that the following conditions hold:

(1) $G \backslash V$ is the disjoint union of a finite number of open subsets $e_1, \ldots, e_k$ called *edges*. Each e_i is homeomorphic to an open interval of the real line.

(2) The boundary, $\operatorname{cl}(e_i) \backslash e_i$, of the edge e_i consists of two distinct vertices, and the pair $(\operatorname{cl}(e_i), e_i)$ is homeomorphic to the pair $([0, 1], (0, 1))$.

A vertex v which belongs to the boundary of at least three different edges is called a *branching point* of G.

A *G map* f is a continuous self-map of G having all branching points of G as fixed points. Note that all the continuous self-maps in a graph G

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considered in this paper are special in the sense that they have all the branching points of G fixed.

A point x of G is called *periodic* with respect to f of *period n* if n is the smallest positive integer such that $f^n(x) = x$. The set $\{x, f(x), ..., f^{n-1}(x)\}$ is called the *periodic orbit* of x. We denote by $\text{Per}(f)$ the set of periods of all periodic points of f and by $\mathbb{N}$ the set of positive integers.

A G map f has *full periodicity* if $\text{Per}(f) = \mathbb{N}$. The set $K \subseteq \mathbb{N}$ is a *full periodicity kernel of G* if it satisfies the following two conditions:

(1) If f is a G map and $K \subseteq \text{Per}(f)$, then $\text{Per}(f) = \mathbb{N}$.

(2) If $S \subseteq \mathbb{N}$ is a set such that for every G map f, $S \subseteq \text{Per}(f)$ implies $\text{Per}(f) = \mathbb{N}$, then $K \subseteq S$.

Note that for a given G if there is a full periodicity kernel, then it is unique.

The full periodicity kernel has been computed for the closed interval, the circle, and the Y; more precisely,

(I) Let I be the closed interval $[0, 1]$. Then the set $\{3\}$ is the full periodicity kernel of I (see [11] and [9]).

(S) Let S^1 be the circle. Then the set $\{1, 2, 3\}$ is the full periodicity kernel of S^1 (see [5] and [8]).

(Y) Set $Y = \{z \in \mathbb{C} : z^3 \in [0, 1]\}$. The set $\{2, 3, 4, 5, 7\}$ is the full periodicity kernel of Y (see [10] and [1]).

In this paper we characterize the full periodicity kernel of σ, where σ is the topological space formed by the points (x, y) of $\mathbb{R}^2$ such that either $x^2 + y^2 = 1$, or $0 \leqslant x \leqslant 2$ and $y = 1$. Then our main result is the following.

THEOREM 1.1. *The set* $\{2, 3, 4, 5, 7\}$ *is the full periodicity kernel of* σ.

The rest of this paper is dedicated to the proof of Theorem 1.1.

The *n-star* is the subspace of the plane which is most easily described as the set of all complex numbers z such that z^n is in the unit interval $[0, 1]$. Recently, the full periodicity kernel of the n-star has been studied in [3] and [4]. Also, in [7] the full periodicity kernel of the "circle with two whiskers" and of the "figure eight" have been computed.

The reader is advised to draw figures when reading most of the proofs, especially in Sections 6 and 7.

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2. First Reduction: Periods Forced by Periods 5 and 7

Since the Y is homeomorphic to $\{(x, y) \in \sigma : y \geqslant 0\}$, in this section we consider $Y = \{(x, y) \in \sigma : y \geqslant 0\} \subseteq \sigma$. Let f be an Y map ; we extend f to a σ map $\bar{f}$ as follows. We define $\bar{f}(p) = f(p)$ if $p \in Y$, and $\bar{f}|_{\sigma \setminus Y}$ as any homeomorphism between $\mathrm{cl}(\sigma \setminus Y)$ and the unique closed arc in Y having $f(1, 0)$ and $f(-1, 0)$ as endpoints such that $\bar{f}(1, 0) = f(1, 0)$ and $\bar{f}(-1, 0) = f(-1, 0)$. Of course, $\mathrm{Per}(f) = \mathrm{Per}(\bar{f})$.

From (Y) of Section 1 we know that if $S \subseteq \mathbb{N}$ is a set such that for every Y map f, $S \subseteq \mathrm{Per}(f)$ implies $\mathrm{Per}(f) = \mathbb{N}$, then $\{2, 3, 4, 5, 7\} \subseteq S$. So, the same is true for the σ maps $\bar{f}$ which are an extension of some Y map f. Hence, from the definition of a full periodicity kernel, to prove Theorem 1.1 it is sufficient to show the following.

Theorem 2.1. *If f is a σ map and $\{2, 3, 4, 5, 7\} \subset \mathrm{Per}(f)$, then* $\mathrm{Per}(f) = \mathbb{N}$.

In fact, Theorem 2.1 is a corollary of the following two propositions:

Proposition 2.2. *If f is a σ map and $5 \in \mathrm{Per}(f)$, then* $\mathrm{Per}(f) \supseteq \mathbb{N} \setminus \{2, 3, 4, 7, 10\}$.

Proposition 2.3. *If f is a σ map and $7 \in \mathrm{Per}(f)$, then* $10 \in \mathrm{Per}(f)$.

Propositions 2.2 and 2.3 are proved in the remaining sections of this paper.

3. Second Reduction: P'-Linear Maps

Let p_0 be the unique branching point of σ. The closures of the two components of $\sigma \setminus \{p_0\}$ are called the *circle* (of σ) and the *whiskers* (of σ) according to whether they are homeomorphic to a circle or to a closed interval, respectively.

Let f be a σ map (Fig. 1a), and suppose that $P = \{p_1, ..., p_k\}$ is a periodic orbit of period $k > 1$. Set $P' = \{p_0, p_1, ..., p_k\}$. A *basic interval* $[p_i, p_j]$ is the closure of the component (p_i, p_j) of $\sigma \setminus P'$ such that $p_i \neq p_j$ and $\{p_i, p_j\}$ is the boundary of (p_i, p_j). If P is contained in the whiskers of σ then there are exactly k basic intervals. Otherwise there are exactly $k+1$ basic intervals. Let B be the set of all basic intervals. The f-graph relation $\rightarrow$ on B is defined as follows. If I and J are basic intervals, then we say I f-covers J or $I \rightarrow J$ if there exists a closed subinterval K of I such that $f(K) = J$. A *path of length m* is any sequence $J_0 \rightarrow J_1 \rightarrow \cdots \rightarrow J_{m-1} \rightarrow J_m$, where $J_0, J_1, ..., J_m$ are all basic intervals. Furthermore, if $J_m = J_0$ then this

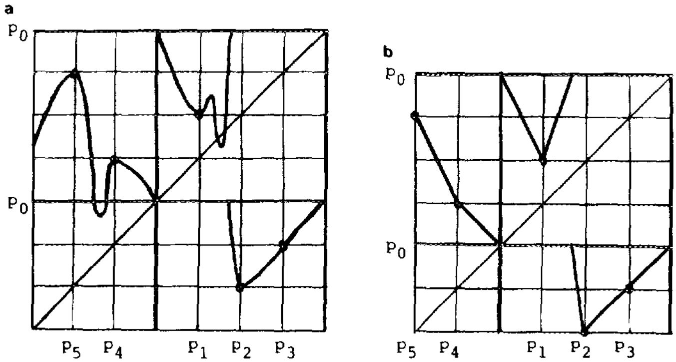

FIG. 1. (a) The graph of f. (b) The graph of g.

path is called a *loop of length m*. Such a loop is called *non-repetitive* if it cannot be written as a single smaller loop repeated an integer number of times.

The following three results are well known basic tools for obtaining periodic points of f from the f-graph, and minor variations of them have appeared on many previous occasions in the study of one-dimensional topological spaces different from σ (see, e.g., [12, 6, 1, 2, 4, ...]).

LEMMA 3.1. *Let $J_0 \to J_1 \to \cdots \to J_{m-1} \to J_0$ be a non-repetitive loop of length m in the f-graph such that at least one J_i does not contain p_0. Then f has a periodic point of period m.*

The proof of Lemma 3.1 is the usual one.

If I is a basic interval of σ, we denote by $\sigma_{I,f}$ the maximal subgraph formed by the union of the basic intervals contained in $f(I) \subseteq \sigma$. Then $f|_I$ is *linear* if $f(I) = \sigma_{I,f}$, $\sigma_{I,f}$ is a tree, and $f|_I$ is linear with respect to the taxicab metric of I and $\sigma_{I,f}$. The taxicab metric d on a tree satisfies the property that if z is in the interval $[x, y]$, then $d(x, y) = d(x, z) + d(z, y)$; of course, the closed interval $[x, y]$ in a tree is defined to be the unique minimal connected closed subset of the tree containing $\{x, y\}$.

For the remainder of this section, let g be a σ map defined by $g|_{P'} = f|_{P'}$, $g|_I$ linear for each basic interval I, and g constant on each component of the complement of the union of all basic intervals. Such a map will be called a *P'-linear σ map* (Fig. 1b).

Note that in general $g|_I$ is not homotopic to $f|_I$; roughly speaking, this is due to the fact that $g|_I$ cancels out the circles in the image of $f|_I$, so that every basic interval g-covers any basic interval at most once. Of course, it follows immediately that the g-graph on B is a subgraph of the f-graph on B.

In what follows, when we work with P'-linear σ maps we consider without loss of generality that the union of all basic intervals is σ.

LEMMA 3.2. *If g has a periodic point of period m, where m is neither 1 nor k, then there is a non-repetitive loop of length m through the g-graph such that at least one basic interval of the loop does not contain the point p_0.*

Proof. Let x be a such a point. Then the orbit of x misses P', so for each i, $0 \leqslant i \leqslant m$, there is a unique basic interval J_i containing $g^i(x)$, and since g is P'-linear, $J_0 \to J_1 \to \cdots \to J_{m-1} \to J_m = J_0$ forms a loop in the g-graph. First, we need to show that the loop is non-repetitive. Since g is linear on each J_i, we can define (by backwards induction on i) subintervals K_i of J_i such that $g: K_i \to K_{i+1}$ is one-to-one and onto and $K_m = J_m = J_0$.

Suppose the loop is repetitive. Then there exists $0 < s < m$, m divisible by s, such that $J_i = J_{i+s}$ for $0 \leqslant i \leqslant m-s$. Then it is easy to see (again by backwards induction on i and piecewise linearity) that if $0 \leqslant i \leqslant m-s$ then K_{i+s} contains K_i. Thus g^s has a fixed point y in K_0. Then since m is divisible by s, $g^m(y) = y$. Note that x and y must be different, since x has period m and y has period $s < m$. Thus the map $g^m: K_0 \to K_m$ is linear and has at least two fixed points. Therefore $g^m|_{K_0}$ must be the identity map, so $K_0 = K_s = K_m = J_0$. If p_0 is an endpoint of J_0, then $k = m \in \{2, 3\}$, and this is a contradiction because $k \neq m$. If p_0 is not an endpoint of J_0, then p_0 is not an endpoint of K_i for $i = 0, 1, ..., m-1$. So $k = m$ and $m = 2s$, again a contradiction. Thus, the loop $J_0 \to J_1 \to \cdots \to J_{m-1} \to J_m = J_0$ is non-repetitive.

Suppose that all the basic intervals of the non-repetitive loop of length m contain the point p_0. Let z be a point of the orbit of x such that the interval $[z, p_0]$ or $[p_0, z]$ satisfies $(z, p_0) \cap P' = \emptyset$ or $(p_0, z) \cap P' = \emptyset$, respectively. Then $g^m|_{[z, p_0]}$ or $g^m|_{[p_0, z]}$ is the identity map because g^m has two fixed points, p_0 and z, and is linear. Then $m = k \in \{2, 3\}$, in contradiction to $m \neq k$. Hence, the lemma is proved. ∎

THEOREM 3.3. *If g has a periodic point of period m, then so does f.*

Proof. Both f and g have points of period 1 and k. If m is neither 1 nor k and g has a periodic point of period m, then by Lemma 3.2 there is a non-repetitive loop in the g-graph on B of length m such that at least one of its basic intervals does not contain p_0. Therefore, by Lemma 3.1, f has a periodic point of period m. ∎

Remark 3.4. From this theorem it follows immediately that it is sufficient to prove Propositions 2.2 and 2.3 for P'-linear σ maps with k equal to 7 and 5, respectively.

4. Third Reduction: Each Basic Interval is f-covered by Some Basic Interval Different from Itself

We need the next two lemmas. The first one follows immediately from the Sharkovskii Theorem [11], Theorem A of [5], and the Main Theorem of [1].

LEMMA 4.1. *Let f be either an I map, an S^1 map having a fixed point, or a Y map. Then the following statements hold.*

(1) *If* $7 \in \mathrm{Per}(f)$, *then* $10 \in \mathrm{Per}(f)$.

(2) *If* $5 \in \mathrm{Per}(f)$, *then* $\mathrm{Per}(f) \supseteq \mathbb{N} \setminus \{2, 3, 4, 7, 10\}$.

In what follows we use the notation introduced in Section 3.

LEMMA 4.2. *Let f be a P'-linear σ map with k equal to 5 or 7. If the periodic orbit P is contained in the circle or the whiskers of σ, then either Proposition 2.2 or Proposition 2.3 holds according to whether $k=5$ or $k=7$, respectively.*

Proof. If P is contained in the whiskers of σ, then f is an I map. If P is contained in the circle of σ, then f is a S^1 map having p_0 as a fixed point. Hence, from Lemma 4.1 we are done. ∎

Remark 4.3. From Remark 3.4 and Lemma 4.2, it is sufficient to prove Propositions 2.2 and 2.3 for a P'-linear σ map f with k equal to 5 or 7 and such that the periodic orbit P is contained neither in the circle of σ nor in the whiskers of σ. So, in what follows we only consider such P'-linear σ maps.

PROPOSITION 4.4. *Let f be a P'-linear σ map as in Remark 4.3. Then at least one of the following statements hold.*

(1) *Each basic interval is f-covered by some basic interval different from itself.*

(2) $k=5$ *and f satisfies Proposition* 2.2.

(3) $k=7$ *and f satisfies Proposition* 2.3.

Proof. Let p_i be the endpoint of the whiskers of σ different from p_0. Since p_i belongs to the periodic orbit P, p_0 is a fixed point, f is P'-linear and $f(\sigma)$ is connected, if follows that each basic interval J contained in the whiskers of σ is f-covered by some basic interval I. We claim that I can be chosen so that $I \neq J$.

To prove the claim suppose that $J \to J$ and $J=[p_j, p_k]$ with $p_i \leqslant p_j < p_k \leqslant p_0$ (since the interval $[p_i, p_0]$ is the whiskers of σ we can consider on it a total ordering $<$ such that p_0 is the largest element and p_i the smallest one). Now, since f is a P'-linear σ map we can consider two cases.

Case 1. Either $p_i \leqslant f(p_j) < p_j < p_k \leqslant f(p_k)$, or $p_i \leqslant f(p_j) < p_j < p_k$ and $f(p_k) \notin [p_i, p_0]$. If there is no basic interval $I \neq J$ such that $I \to J$, then $f(P \cap [p_i, p_j]) \subseteq P \cap [p_i, p_j]$ with $P \cap [p_i, p_j] \neq \emptyset$. This is a contradiction because P is a periodic orbit not contained in the whiskers of σ.

Case 2. Either $p_i \leqslant f(p_k) \leqslant p_j < p_k \leqslant f(p_j)$, or $p_i \leqslant f(p_k) \leqslant p_j < p_k$ and $f(p_j) \notin [p_i, p_0]$. Then $p_k < p_0$, and clearly $f([p_k, p_0]) \supseteq [f(p_k), f(p_0)] \supseteq [f(p_k), p_0] \supseteq [p_j, p_0] = J$. Therefore there is a basic interval $I \subseteq [p_k, p_0]$ which f-covers J, and of course $I \neq J$. Thus, the claim is proved.

Now let J denote a basic interval contained in the circle of σ, and suppose that there is no basic interval $I \neq J$ which f-covers J. Since f is P'-linear, $f|_{\sigma \backslash \mathrm{Int}(J)}: \sigma \backslash \mathrm{Int}(J) \to \sigma \backslash \mathrm{Int}(J)$ is either a Y map or an I map such that $\mathrm{Per}(f) = \mathrm{Per}(f_{\sigma \backslash \mathrm{Int}(J)})$. Then, from Lemma 4.1 it follows that either (2) or (3) holds. Otherwise, there is a basic interval $I \neq J$ such that $I \to J$. Since we can repeat this argument for each basic interval J contained in the circle σ, the proposition is proved. ∎

Remark 4.5. From Remark 4.3 and Proposition 4.4, it is sufficient to prove Propositions 2.2 and 2.3 for a P'-linear σ map f with k equal to 5 and 7, respectively, such that the periodic orbit P is contained neither in the circle of σ nor in the whiskers of σ, and *each basic interval is f-covered by some basic interval different from itself.* Hence, in the rest of the paper we only consider such P'-linear σ maps.

5. Fourth Reduction: Any Basic Interval Does not f-cover itself

Since the periodic orbit P has points on the circle of σ and on the whiskers of σ, there are exactly three different basic intervals which have p_0 as endpoint. These basic intervals will be denoted by A, B, and C. We assume that A is contained in the whiskers of σ.

The goal of this section is to show the following result.

Proposition 5.1. *Let f be a P'-linear σ map as in Remark* 4.5. *Suppose that there is some basic interval which f-covers itself. Then* $\mathrm{Per}(f) \supseteq \{n \in \mathbb{N} : n \geqslant k\}$.

Proof. Let I be a basic interval such that $I \to I$. Since f is P'-linear, the connected set $f^i(I)$ is formed by some union of basic intervals and $f^i(I) \subseteq f^{i+1}(I)$ for $i = 1, 2, \ldots$. Since $P \subseteq f^k(I)$, it follows that there is at most one basic interval not contained in $f^k(I)$, but if there is one, it is f-covered by some other basic interval, so $f^{k+1}(I)=\sigma$. Let r be the smallest integer such that $f^r(I)=\sigma$.

Thus we have $I \subsetneq f(I) \subsetneq f^2(I) \subsetneq \cdots \subsetneq f^r(I)=\sigma$ with $r \leqslant k+1$, and each basic interval L of $f^j(I) \setminus f^{j-1}(I)$ is f-covered by a basic interval M of $f^{j-1}(I) \setminus f^{j-2}(I)$. From Remark 4.5 there is a basic interval $J \neq I$ such that $J \to I$. Let s be the smallest positive integer such that $J \subseteq f^s(I)$, clearly $1 \leqslant s \leqslant r$. Then we have a non-repetitive loop $I=I_1 \to I_2 \to \cdots \to I_s = J \to I$. First, suppose that some of the basic intervals of the loop do not contain p_0. Therefore, by using Lemma 3.1 and the loop $I \to I$, it follows that $\mathrm{Per}(f) \supseteq \{n \in \mathbb{N} : n \geqslant s\}$. Since $s \leqslant r \leqslant k+1$, we have that $\mathrm{Per}(f) \supseteq \{n \in \mathbb{N} : n \geqslant k+1\}$. Since $k \in \mathrm{Per}(f)$, the proposition is proved.

Now, we can assume that all the basic intervals of all the loops containing I have p_0 as an endpoint. Such loops have length 2 or 3, and all their basic intervals must be A, B, or C. Suppose that $I=A$. Then, since $I \to I$ and f is P'-linear, $f(A)$ is contained in the whiskers of σ, and A cannot f-cover B and C. This is in contradiction with the fact that we have a non-repetitive loop of length 2 or 3 such that all its basic intervals are in $\{A, B, C\}$. Now, without loss of generality, we can assume that $I=B$. Then, since $I \to I$, f is P'-linear, and I must f-cover A or C, it follows that $f(B)$ contains the circle of σ, in contradiction with the definition of P'-linear. Thus the proposition is proved. ∎

In fact, since the proof of Proposition 5.1 works for arbitrary $k>3$, it is not necessary that $k=5$ or $k=7$.

REMARK 5.2. *Proposition* 5.1 *shows that if there is some basic interval which f-covers itself, then Propositions* 2.2 *and* 2.3 *hold. So from now on we add the following assumption to the hypotheses from Remark* 4.5: *any basic interval does not f-cover itself.*

6. FIFTH REDUCTION: UNDIRECTED ASSUMPTION

When the f-graph contains the loop $A \to B \to C \to A$ or $A \to C \to B \to A$ we say f satisfies the *directed assumption*; otherwise, f satisfies the *undirected assumption.*

The goal of this section is to prove Propositions 2.2 and 2.3, for σ maps satisfying the hypotheses from Remarks 4.5 and 5.2, and the undirected assumption.

PROPOSITION 6.1. *Let f be a P'-linear σ map satisfying the hypotheses from Remarks* 4.5 *and* 5.2, *and the undirected assumption. Then* $\mathrm{Per}(f) \supseteq \{n \in \mathbb{N} : n \geqslant k\}$.

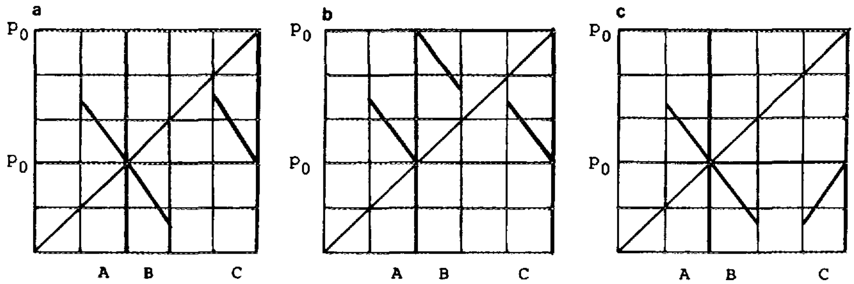

FIGURE 2

Proof. Since f is P'-linear and any basic interval does not f-cover itself, p_0 is the unique fixed point of f. On the other hand, since the intervals A, B, and C have the fixed point p_0 as endpoint and any basic interval does not f-cover itself, each element of the set $\{A, B, C\}$ f-covers one and only one of the other two elements. Therefore, from the undirected assumption and interchanging the names of the intervals B and C on the circle of σ (if necessary), it follows that $A \rightleftarrows B \leftarrow C$, $A \rightarrow B \rightleftarrows C$, or $C \rightarrow A \rightleftarrows B$.

Case 1. $A \rightleftarrows B \leftarrow C$. From the continuity of f, the existence of the periodic orbit P, the fact that the unique fixed point of f is p_0, and Fig. 2a it follows immediately that there is a basic interval J contained in the circle of σ such that $J \notin \{B, C\}$ and either $A \leftarrow J \rightarrow B$ or $C \leftarrow J \rightarrow B$.

Since f is P'-linear, $A \rightleftarrows B$, and $A \cup B$ is connected, the connected set $f^i(A \cup B)$ is formed by some union of basic intervals and $f^i(A \cup B) \subseteq f^{i+1}(A \cup B)$ for $i = 1, 2, \ldots$. Since $P \subseteq f^{k-1}(A \cup B)$, it follows that there is at most one basic interval not contained in $f^{k-1}(A \cup B)$, but if there is one, it is f-covered by some other basic interval, so $f^k(A \cup B) = \sigma$. Let r be the smallest integer such that $f^r(A \cup B) = \sigma$.

Thus we have $A \cup B \subsetneq f(A \cup B) \subsetneq f^2(A \cup B) \subsetneq \cdots \subsetneq f^r(A \cup B) = \sigma$ with $r \leqslant k$, and each basic interval L of $f^j(A \cup B) \setminus f^{j-1}(A \cup B)$ is f-covered by a basic interval M of $f^{j-1}(A \cup B) \setminus f^{j-2}(A \cup B)$. Then there is a path $I_1 \rightarrow I_2 \rightarrow \cdots \rightarrow I_{r-1} \rightarrow J$ with I_1 equal to either A or B. Therefore, this path of length $r-1$, together with the subgraphs $A \rightleftarrows B \leftarrow C$ and either $A \leftarrow J \rightarrow B$ or $C \leftarrow J \rightarrow B$, implies that $\mathrm{Per}(f) \supseteq \{r+1, r+2, \ldots\}$. Hence, since $k \in \mathrm{Per}(f)$ and $r \leqslant k$, we have that $\mathrm{Per}(f) \supseteq \{n \in \mathbb{N} : n \geqslant k\}$. So, the proposition is proved in Case 1.

Case 2. $A \rightarrow B \rightleftarrows C$. From the continuity of f, the existence of the periodic orbit P, the fact that the unique fixed point of f is p_0, and Fig. 2b it follows immediately that three is a basic interval J contained in the circle of σ such that $J \notin \{B, C\}$ and either $A \leftarrow J \rightarrow B$ or $C \leftarrow J \rightarrow B$. Now, the proof of the proposition in this case uses the same arguments found in Case 1.

Case 3. $C \to A \rightleftarrows B$. Since k is odd, the periodic orbit P has more points either in the circle or in the whiskers of σ. Therefore there is a basic interval J contained in the circle of σ if $\text{Card}(P \cap \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}) > \text{Card}(P \cap \{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 2 \text{ and } y = 1\})$, or contained in the whiskers of σ otherwise, such that $J \notin \{A, B, C\}$ and either $A \leftarrow J \to B$ or $A \leftarrow J \to C$ (see Fig. 2c). Again, by repeating the same arguments found in Case 1, the proposition follows. ∎

In fact, Proposition 6.1 is true for arbitrary $k > 3$ odd; i.e., it is not necessary that $k = 5$ or $k = 7$.

7. Sixth Reduction: Directed Assumption

The goal of this section is to prove Propositions 2.2 and 2.3 for σ maps satisfying the hypotheses from Remarks 4.5 and 5.2, and the directed assumption. Without loss of generality we can assume that $A \to B \to C \to A$. Take in account that since f is P'-linear and any basic interval does not f-cover itself, p_0 is the unique fixed point of f.

The basic intervals $I_1, ..., I_{k-2}$ different from A, B, C and the position of the periodic points $p_1, ..., p_k$ of P in σ are defined in Fig. 3a.

From the reductions of the previous sections and the following result we obtain Proposition 2.2.

PROPOSITION 7.1. *Let f be a P'-linear σ map satisfying the hypotheses from Remarks* 4.5 *and* 5.2. *If $k = 5$ and f satisfies the directed assumption, then* $\text{Per}(f) \supseteq \{n \in \mathbb{N} : n \geqslant 5\}$.

Proof. We separate the proof into four cases.

Case 1. $\text{Card}(P \cap \{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 2 \text{ and } y = 1\}) = 1$. Then, from Fig. 3a with $k = 5$ and $r = 4$, it follows that $f(p_4) = p_5$.

Suppose $f(p_1) \neq p_4$. Then $f(p_1) \in \{p_2, p_3\}$. So $B \to I_3 \to A$. We may assume that $f(p_3) = p_4$; otherwise, $I_3 \to B$; and from the three loops $A \to B \to C \to A$, $B \to I_3 \to B$, and $A \to B \to I_3 \to A$ and Lemma 3.1 we obtain $\text{Per}(f) \supseteq \{n \in \mathbb{N} : n \geqslant 5\}$, and the proposition is proved. Also $f(p_5) \in \{p_2, p_3\}$; otherwise, $f(p_5) = p_1$; and the basic interval I_1 would not be f-covered by any basic interval in contradiction with the hypotheses. Therefore $f(p_2) = p_1$. Consequently $f(p_1) = p_3$ and $f(p_5) = p_2$. Now, drawing the P'-linear graph of Fig. 3b we see that the interval I_2 is not f-covered, a contradiction to Remark 4.5.

Assume $f(p_1) = p_4$. We have $f(p_5) = p_3$; otherwise, either $f(p_5) = p_1$ and $\{p_5, p_1, p_4\}$ would be a periodic orbit of period 3, or $f(p_5) = p_2$ and the basic interval I_2 would not be f-covered by any basic interval. But now the

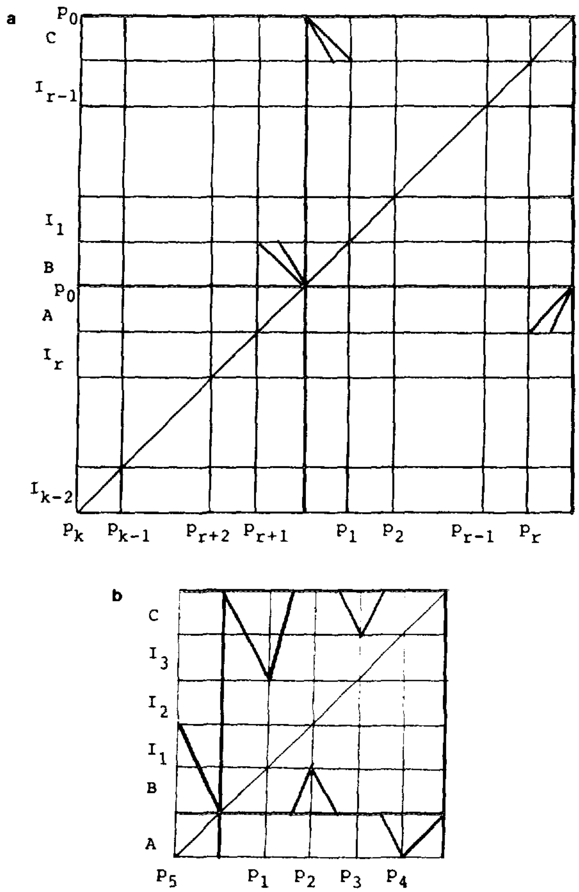

FIG. 3. Definition of the basic intervals $I_1, ..., I_{k-2}$ and of the points $p_1, ..., p_k$ of P.

basic interval I_3 is not f-covered by any basic interval. This completes the proof of the proposition in Case 1.

Case 2. $\text{Card}(P \cap \{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 2 \text{ and } y = 1\}) = 2$. Since f is P'-linear, $f(p_1) = p_2$; otherwise, $f(p_1) = p_3$, and the basic interval I_2 would not be f-covered by any basic interval. We have either $f(p_4) = p_3$ or $f(p_5) = p_3$; otherwise, the basic interval I_1 would not be f-covered by any basic interval. Since $f(p_3) \neq p_1$ (see Fig. 3a with $k = 5$ and $r = 3$), $f(p_2) \neq p_1$ (otherwise, $\{p_1, p_2\}$ would be a periodic orbit of period 2), either $f(p_4) = p_1$ or $f(p_5) = p_1$. Therefore $f(\{p_4, p_5\}) = \{p_1, p_3\}$, and consequently $f(\{p_2, p_3\}) = \{p_4, p_5\}$.

Suppose $f(p_5) = p_1$. Then, from the three loops $A \to B \to C \to A$, $A \to I_1 \to A$, and $A \to I_1 \to C \to A$ and Lemma 3.1, we obtain $\text{Per}(f) \supseteq$

$\{n \in \mathbb{N} : n \geqslant 5\}$. Hence, we can assume $f(p_4) = p_1$. Therefore, $f(p_5) = p_3$, $f(p_3) = p_4$, and $f(p_2) = p_5$. Consequently, from the three loops $I_1 \to I_3 \to I_1$, $I_2 \to I_3 \to I_1 \to I_2$, $I_3 \to I_2 \to I_3$ and Lemma 3.1, we obtain $\text{Per}(f) \supseteq \{n \in \mathbb{N} : n \geqslant 5\}$. This completes the proof of the proposition in Case 2.

Case 3. $\text{Card}(P \cap \{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 2 \text{ and } y = 1\}) = 3$. From Fig. 3a with $k = 5$ and $r = 2$ it follows that $f(p_1) = p_2$ and $f(p_3) = p_1$. Furthermore, $f(p_2) = p_5$; otherwise, f would have a fixed point different from p_0. Now the images under f of p_4 and p_5 must be in $\{p_3, p_4\}$. So $f(p_4) = p_3$ and $f(p_5) = p_4$. Then the graph of f is completely determined except on the interval I_2. Since each basic interval is f-covered by some different basic interval, we have that I_2 f-covers A, C, and I_1. Then from the three loops $C \to A \to B \to C$, $C \to I_i \to I_1 \to C$, and $I_1 \to I_i \to I_1$ of the f-graph and Lemma 3.1, we obtain $\text{Per}(f) \supseteq \{n \in \mathbb{N} : n \geqslant 5\}$. So the proposition is proved in Case 3.

Case 4. $\text{Card}(P \cap \{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 2 \text{ and } y = 1\}) = 4$. This case is incompatible with the existence of the loop $A \to B \to C \to A$ (see Fig. 3a with $k = 5$ and $r = 1$). So the proposition is proved. ∎

LEMMA 7.2. *Let f be a P'-linear σ map and let J_0 be a basic interval contained in the whiskers of σ. Then there is a loop of the f-graph of length k containing J_0.*

Proof. Let $J_0 = [x, y]$. For each i, $0 < i \leqslant k$, we define J_i recursively as an interval or arc with endpoints $f^i(x)$ and $f^i(y)$ and such that $J_{i-1} \to J_i$. (Note that, in general, the intervals J_i are not basic. So, here we say that J_{i-1} f-covers J_i or $J_{i-1} \to J_i$ if there exists a closed subinterval L of J_{i-1} such that $f(L) = J_i$.) Then $J_k = J_0$. Define the basic interval K_i for $0 \leqslant i \leqslant k$ by backwards induction on i as follows. Let $K_k = J_k$, and if K_{i+1} has been defined and is a basic interval of J_{i+1}, then let K_i be a basic interval of J_i such that $K_i \to K_{i+1}$. Then $J_0 = K_0 \to K_1 \to \cdots \to K_k = J_0$ is the required loop. ∎

Note that the loop of Lemma 7.2 can be repetitive or non-repetitive.

From the reductions of the previous sections and the following result we obtain Proposition 2.3.

PROPOSITION 7.3. *Let f be a P'-linear σ map satisfying the hypotheses from Remarks* 4.5 *and* 5.2. *If $k = 7$ and f satisfies the directed assumption, then* $10 \in \text{Per}(f)$.

Proof. Since A is a basic interval contained in the whiskers of σ, by Lemma 7.2 there is a loop of length 7 containing A. Since 7 is prime and

A does not f-cover A, this loop is non-repetitive. Then, this loop together with the loop $A \to B \to C \to A$ gives a loop of length 10. If this loop of length 10 is non-repetitive by Lemma 3.1 we are done. If it is repetitive, then it must be repetition of a loop of length 5 (it cannot be repetition of a loop of length 2 because it contains the loop $A \to B \to C \to A$). Then, since the loop of length 10 ends with $A \to B \to C \to A$, the unique possibility for the loop of length 5 is $A \to J \to A \to B \to C \to A$. Hence, we can consider the following new loop of length 10

$$A \to J \to A \to J \to A \to B \to C \to A \to B \to C \to A,$$

which clearly is non-repetitive, and again we are done. ∎

REFERENCES

1. L. Alsedà, J. Llibre, and M. Misiurewicz, Periods orbits of maps of Y, *Trans. Amer. Math. Soc.* **313** (1989), 475–538.
2. L. Alsedà, J. Llibre, and M. Misiurewicz, Combinatorial dynamics and entropy in dimension one, *in* "Advanced Series in Non-linear Dynamics," Vol. 5, World Scientific, Singapore, 1993.
3. L. Alsedà and M. Moreno, Linear orderings and the full periodicity kernel for the n-star, *J. Math. Anal. Appl.* **180** (1993), 599–616.
4. S. Baldwin, An extension of Sharkovskii's Theorem to the n-od, *Ergodic Theory Dynamical Systems* **11** (1991), 249–271.
5. L. Block, Periods of periodic points of maps of the circle which have a fixed point, *Proc. Amer. Math. Soc.* **82** (1981), 481–486.
6. L. Block, J. Guckenheimer, M. Misiurewicz, and L. S. Young, Periodic points and topological entropy of one dimensional maps, *in* "Global Theory of Dynamical Systems," Lecture Notes in Math. Vol. 819, pp. 18–34, Spinger, New York/Berlin, 1980.
7. C. Leseduarte and J. Llibre, The full periodicity kernels in one-dimensional spaces, manuscript in preparation.
8. J. Llibre and R. Reventós, Sur le nombre d'orbites périodiques d'une application continue du cercle un lui-même, *C. R. Acad. Sci. Paris, Sér. I Math.* **294** (1982), 52–54.
9. T. Li and J. Yorke, Period three implies chaos, *Amer. Math. Monthly* **82** (1975), 985–992.
10. P. Mumbrú, "Periodes 1, 2, 3, 4, 5, i 7 equivalen a caos," Master's thesis. Universitat Autònoma de Barcelona, 1982. [in Catalan]
11. A. N. Sharkovskii, Co-existence of the cycles of a continuous mapping of the line into itself, *Ukrain. Math. Zh.* **16** (1964), 61–71. [in Russian]
12. P. D. Straffin, Periodic points of continuous functions. *Math. Mag.* **51** (1978), 94–105.