Perturbation of Embedded Eigenvalues by Operators of Finite Rank

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INTRODUCTION

In this classical paper of 1948, K. O. Friedrichs considered an example of a self-adjoint operator $T_0$ having a point eigenvalue embedded in its continuous spectrum which disappears under a perturbation $\epsilon V$ of rank 2 [1, Sections 6-8 and Section 10]. Friedrichs analyzed the perturbed operator $T_\epsilon = T_0 + \epsilon V$, and showed that although there was no eigenvalue of $T_\epsilon$ the density function of the spectral measure of $T_\epsilon$ was peaked, for small positive $\epsilon$, near the solution $\lambda_\epsilon$ of the formal perturbation equations—an example of the “spectral concentration” phenomenon, which has been discussed by several authors for isolated eigenvalues. (See [2], [3, Chapt. VIII] and [4].)

In the present paper, as a first step toward a general analysis of the problem, we shall consider the case of a perturbation $V$ of arbitrary finite rank. First, we shall give sufficient conditions under which an embedded eigenvalue $\lambda_0$ of $T_0$ of simple multiplicity vanishes under perturbation by $V$ (Theorem 1). These conditions state essentially that (a) except for the point mass at $\lambda_0$, the spectral measure of $T_0$ is rather smooth near $\lambda_0$, and (b) there is, in a certain sense, no “degeneracy” of $V$ at $\lambda_0$. Under these conditions, the part of $T_0$ on the orthogonal complement of the eigenspace of $\lambda_0$ is absolutely continuous near $\lambda_0$ and is unitarily equivalent near $\lambda_0$ to the perturbed operator $T = T_0 + V$. The main tool in the proof of this, and the subsequent theorem is the formula (1.6) for the inverse of the Weinstein-Aronszajn matrix $W(z)$. We do not employ the contour integral techniques used by Friedrichs [1].

Secondly, we shall consider the asymptotic properties of the family of operators $T_\epsilon = T_0 + \epsilon V$ as $\epsilon \to 0^+$. We shall give sufficient conditions that Theorem 1 apply for all sufficiently small positive $\epsilon$ (Theorem 1), and

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then under additional conditions, examine the spectral measure of\( T \) near\( \lambda_0 \) (Theorem 3) and the spectral concentration at\( \lambda_0 \) (Theorem 4). Extensions of the present work to perturbations of infinite rank will be discussed in a subsequent publication.

1. Preliminaries

A general reference for this section is the article [3] of Kuroda. If \( G \) is a subset of the real axis, the phrase "for \( z \) near \( G \)" will mean for every non-real \( z \) in a neighborhood of \( G \) in the complex plane.

Let \( T_0 \) be a self-adjoint operator on a Hilbert space \( \mathcal{H} \), with domain \( \mathcal{D}(T_0) \), and \( V \) a bounded self-adjoint operator of finite rank \( r \). \( V \) may then be written as \( V = \sum_{i=1}^{r} c_i (\cdot, \phi_i) \phi_i \), where \( c_1, \ldots, c_r \) are nonvanishing real numbers and \( \{\phi_1, \ldots, \phi_r\} \) is a finite orthonormal set spanning the range \( \mathcal{R}(V) \) of \( V \). It follows that the operator \( T - T_0 \vdash V \) with \( \mathcal{D}(T) = \mathcal{D}(T_0) \) is also self-adjoint. Let

\[
R(z) = (T - z)^{-1} \quad \text{and} \quad R_0(z) = (T_0 - z)^{-1}
\]

be the resolvents of \( T \) and \( T_0 \), and \( E(\lambda) \) and \( E_0(\lambda) \) the corresponding spectral resolutions. If \( S \) is a Borel set, we define \( E[S] = \int_S dE(\lambda) \), and similarly for \( E_0[S] \).

For \( \Im z \neq 0 \), define \( Q(z) = I - VR(z) \) and \( Q_0(z) = I + VR_0(z) \). \( Q(z) \) and \( Q_0(z) \) are then analytic operator-valued functions for \( \Im z \neq 0 \) and satisfy the following equations:

\[
Q(z)Q_0(z) = Q_0(z)Q(z) = I
\]

\[
R(z) = R_0(z)Q(z)
\]

and

\[
R(z) - R(\bar{z}) = Q^*(z) [R_0(z) - R_0(\bar{z})] Q(z),
\]

where \( Q^*(z) \) is the adjoint of \( Q(z) \). These facts follow easily from the expressions \( Q(z) = (T - z) R(z) \) and \( Q_0(z) = (T - z) R_0(z) \) and the resolvent equations [5, Section 1.1]. If we define, for \( \Im z \neq 0 \),

\[
E'(z) = \frac{1}{2\pi i} [R(z) - R(\bar{z})]
\]

and similarly for \( E'_0(z) \), then (1.3) can be written

\[
E'(z) = Q^*(z) E'_0(z) Q(z).
\]

Observe that the finite dimensional space \( \mathcal{R}(V) \) is invariant under both \( Q(z) \) and \( Q_0(z) \). We shall denote by \( W(z) \) the restriction of \( Q_0(z) \) to \( \mathcal{R}(V) \).
Then $W^{-1}(z)$ exists for $\text{Im } z \neq 0$ and is the restriction of $Q(z)$ to $\mathcal{H}(V)$. Note that with respect to the basis $\psi_1, \ldots, \psi_r$, $W(z)$ has the matrix

$$
\delta_{ij} - c_j(R_0(z) \phi_i, \phi_j)
$$

so that $\det W(z)$ is the Weinstein-Aronszajn determinant [3, Chapt. IV, Section 6].

Assume that $\lambda_0$ is an eigenvalue of $T_0$ of simple multiplicity, and let $\psi_0$ be the corresponding normalized eigenvector. Let $P_0 = (\cdot, \psi_0) \psi_0$ and $P = I - P_0$, and define $W_a(z)$ to be the restriction to $\mathcal{H}(V)$ of the operator $I + VR_0(z)P$, then

$$
W(z) = W_a(z) + (\lambda_0 - z)^{-1} (\cdot, \psi_0) V\psi_0.
$$

**Lemma 1.1.** $W_a^{-1}(z)$ exists for $\text{Im } z \neq 0$, and

$$
W^{-1}(z) = W_a^{-1}(z) [I - \Delta(z) (W_a^{-1}(z) \cdot, \psi_0) V\psi_0],
$$

where

$$
\Delta(z) = \epsilon[\lambda_0 - z + (W_a^{-1}(z) V\psi_0, \psi_0)]^{-1}.
$$

**Proof.** Suppose that for some $x \in \mathcal{H}(V)$,

$$
W_a(z) x = x + VR_0(z)Px = 0
$$

Multiplying on the left by $P$, we find that

$$
y + V_1 R_0(z) Py = 0,
$$

where $y = Px$ and $V_1 = VVP$. But $R_0(z)P$ is the resolvent of $T_0P$ on the space $P\mathcal{S}$, and hence, by a suitable version of (1.1), $P + V_1 R_0(z)P$ has a bounded inverse on $P\mathcal{S}$ given by $P - V_1 S(z)$ where $S(z)$ is the resolvent of $T_0P + V_1$ on $P\mathcal{S}$. Thus $y = 0$ and hence by (1.8) $x = 0$. Thus $W_a^{-1}(z)$ exists.

Recall now the formula.

$$
[I + (\cdot, x)y]^{-1} = I - [1 + (y, x)]^{-1} (\cdot, x)y
$$

for the inverse of the sum of the identity and an operator of rank one. If we define $F(z) = W(z) - W_a(z)$, then $F(z)$ has rank one, and we have

$$
W^{-1}(z) = (W_a(z) + F(z))^{-1} = W_a^{-1}(z) (I + F(z) W_a^{-1}(z))^{-1}.
$$

If we apply (1.9) to the rank one operator $FW_a^{-1}$, then we obtain (1.6) from (1.10).

For simplicity in the sequel, we shall define

$$
\gamma(x, z) = (W_a^{-1}(z) x, \psi_0)
$$

for every $x$ in $\mathcal{H}(V)$.
2. Spectral Theory

If $T = \int \lambda \, dE(\lambda)$ is a self-adjoint operator on $\mathcal{H}$ and $G$ is a Borel subset of the reals, then the part of $T$ in $G$ is the operator $TE[G]$, considered as an operator on $E[G] \mathcal{H}$. $T$ is absolutely continuous on $G$ iff the part of $T$ in $G$ is absolutely continuous, or equivalently iff the measure $d(E(\lambda) E[G] x, x)$ is absolutely continuous with respect to Lebesgue measure for every $x$ in $\mathcal{H}$.

Let $\mathcal{H}_a(G, T)$ denote the set of all $x$ in $\mathcal{H}$ such that the restriction of $d(E(\lambda) x, x)$ to $G$ is absolutely continuous.

**Lemma 2.1.** $\mathcal{H}_a(G, T)$ is a closed subspace of $\mathcal{H}$ which reduces $T$.

**Proof.** Let $E[G] \mathcal{H} = \mathcal{H}_a \oplus \mathcal{H}_s$ be the decomposition of $E[G] \mathcal{H}$ into absolutely continuous and singular subspaces relative to $E[G] T$ [3, Chapt. X, Section 1.2]. The lemma follows if we show that $\mathcal{H}_a(G, T) = \mathcal{H}_a$. But if $x = x_1 \cdot x_2 \cdot x_3$ is the decomposition of $x$ with respect to the direct sum

$$\mathcal{H} = E[G'] \mathcal{H} \oplus \mathcal{H}_a \oplus \mathcal{H}_s,$$

where $G'$ is the complement of $G$, then

$$d(E(\lambda) x, x) = d(E(\lambda) x_1, x_1) - d(E(\lambda) x_2, x_2) + d(E(\lambda) x_3, x_3).$$

The first measure on the right side vanishes identically on $G$, while the second is absolutely continuous. The third is purely singular with support in $G$, and vanishes iff $x_3 = 0$. Hence, $x \in \mathcal{H}_a(G, T)$ iff $x_3 = 0$.

We also define $\mathcal{M}(\phi_1, ..., \phi_r; T_0)$ to be the smallest subspace of $\mathcal{H}$ which contains $\phi_1, ..., \phi_r$ and reduces $T_0$. It follows [6] that $\mathcal{M}(\phi_1, ..., \phi_r; T_0)$ also reduces $T$ and that

$$\mathcal{M}(\phi_1, ..., \phi_r; T_0) = \mathcal{M}(\phi_1, ..., \phi_r; T).$$

Moreover, $T = T_0$ on the orthogonal complement of $\mathcal{M}(\phi_1, ..., \phi_r; T_0)$.

The following is the main result on spectral theory.

**Theorem 1.** Let $G$ be an open subset of the reals and $\lambda_0 \in G$ be an eigenvalue of $T_0$ of simple multiplicity. Suppose that

(a) $W_\lambda(z)$ and $W_\lambda^{-1}(z)$ are bounded near $G$.

(b) $\text{Im}(W_\lambda^{-1}(z) V \psi_0, \psi_0)$ is bounded away from zero near $G$.

Then

(1) The restrictions of $T$ and $T_0$ to $\mathcal{M}(\phi_1, ..., \phi_r; T_0)$ are absolutely continuous on $G$ and $G \sim \{\lambda_0\}$ respectively.

(2) The part of $T$ in $G$ is unitarily equivalent to the part of $T_0$ in $G \sim \{\lambda_0\}$. 
PROOF. Since the parts of $T$ and $T_0$ on $\mathcal{M}(\phi_1, \ldots, \phi_r; T_0)$ are identical, we may assume that $\mathcal{S} = \mathcal{M}(\phi_1, \ldots, \phi_r; T_0)$. By the theorem of Rosenblum and Kato [3, Theorem 4.4, p. 540], the absolutely continuous parts of $T$ and $T_0$ are unitarily equivalent, so that (2) follows from (1). By an analogue of (1.5), boundedness of $W_\delta(z)$ near $G$ implies boundedness near $G$ of $(R_0(z) P \phi_i, \phi_i)$ for $i = 1, \ldots, r$. Since

$$(R_0(z) \phi_i, \phi_i) = (R_0(z) P \phi_i, \phi_i) - (\lambda_0 - z)^{-1} \cdot (\phi_i, \psi_0)^2,$$

it follows that $(R_0(z) \phi_i, \phi_i)$ is bounded locally near $G \sim \{\lambda_0\}$, and hence [7] that $\sigma(E_0(\lambda) \phi_i, \phi_i)$ is absolutely continuous on $G \sim \{\lambda_0\}$. Therefore $\mathcal{S}_\delta(G \sim \{\lambda_0\}; T_0)$ contains $\phi_1, \ldots, \phi_r$ and reduces $T_0$, and must equal $\mathcal{S}$ by our assumption. $T_0$ is thus absolutely continuous on $G \sim \{\lambda_0\}$. From (1.7) and condition (b), it is immediate that $\Delta(z)$ is bounded near $G$. It therefore follows from (1.6) and condition (a) that $W_\delta^{-1}(z)$, and hence by (1.5) $(R(z) \phi_i, \phi_i)$, are bounded near $G$. By the above argument, $T$ is absolutely continuous on $G$.

REMARKS. (1) It is clear from the statement of Theorem 1, that it suffices to assume that (a) and (b) hold for every compact subset of $G$, rather than for $G$ itself.

(2) Since the unitary equivalence was obtained by the Rosenblum-Kato theorem, it follows that the connecting operators may be taken as the wave operators

$$W_{\pm} = \lim_{t \to \pm \infty} \frac{1}{\hbar} e^{iT_0 t} e^{-iT t} E[G].$$

(See [3].)

(3) Observe that in Theorem 1, $\lambda_0$ can never be an isolated eigenvalue of $T_0$. For if we assume that $\sigma(T_0) \cap G \sim \{\lambda_0\}$, then $P = 0$ and $W_\delta(z) = I$. Hence $\text{Im}(W_\delta^{-1}(z) V \psi_0, \psi_0) \neq \text{Im}(V \psi_0, \psi_0) = 0$ since $V$ is self-adjoint.

(4) Theorem 1 can be easily applied to the example of Friedrichs mentioned in the Introduction.

3. SMALL PERTURBATIONS

In this and the following sections, we shall be considering the properties of the operator $T_\epsilon = T_0 + \epsilon V$ for sufficiently small positive values of $\epsilon$. All of the previous notation will be taken over simply by indicating the dependence on $\epsilon$ of quantities connected with the perturbed operator. For example, $R_\epsilon(z) = (T_\epsilon - z)^{-1}$ and $E_\epsilon(\lambda)$ is the spectral resolution of $T_\epsilon$.

The following theorem gives sufficient conditions for application of Theorem 1 to the operator $T_\epsilon$ for sufficiently small $\epsilon$. Note that the hypotheses do not involve $W_\epsilon^{-1}(z; \epsilon)$. 
Theorem 2. Let $G$ be an open subset of the reals and $\lambda_0 \in G$ an eigenvalue of $T_0$ of simple multiplicity. Suppose that

(a) \((R_0(\lambda) P\phi_i, \phi_j)\) is bounded near $G$, for $i, j = 1, \ldots, r$.

(b) $d(E_0(\lambda) PV\psi_0, V\psi_0) d\lambda$ is essentially bounded away from zero on $G$.

Then $T_\epsilon$ satisfies the hypotheses of Theorem 1 for all sufficiently small positive $\epsilon$.

Proof. If $K(\lambda)$ denotes the restriction of $VR_0(\lambda) P$ to $\Re(V)$, then it follows from (a) that $K(\lambda)$ is bounded near $G$ in operator norm; say $\|K(\lambda)\| \leq C$ for $\lambda$ near $G$. Hence, if $0 < \epsilon < 1$, it follows from the Neumann series that $W^{-1}_a(\lambda, \epsilon) = [I + \epsilon K(\lambda)]^{-1}$ is bounded near $G$ by $(1 - \epsilon)^{-1}$. Moreover, $W^{-1}_a(\lambda, \epsilon) = I - \epsilon K(\lambda) + O(\epsilon^2)$ so that a brief calculation yields,

for $\Im \lambda > 0$,

$$\epsilon \Im(W^{-1}_a(\lambda, \epsilon) V\psi_0, \psi_0) = -\pi \epsilon^2 (E_0(\lambda) \{1 + O(\epsilon)\}).$$

But the first term on the right is the Poisson integral of $\epsilon^2 d(E_0(\lambda) PV\psi_0, V\psi_0)$ and is therefore bounded away from zero near any compact subset of $G$, by (b).

4. Spectral Asymptotics

In this section, an asymptotic result for the spectral density $d(E_0(\lambda) x, y)/d\lambda$ is obtained for $x, y$ in $\Re(V)$. The result is analogous to the formula of Weisskopf and Wigner discussed by Friedrichs [1]. The necessary additional assumption is stated in the following lemma.

Lemma 4.1. Let the hypotheses of Theorem 2 hold. Suppose that

\begin{equation}
(4.1) \frac{A}{\Im \lambda_0} (K(\lambda) V\psi_0, \psi_0) \end{equation}

exists, and let $\Gamma_\epsilon = -\epsilon^2 \Im A$, $\lambda_\epsilon = \lambda_0 + \epsilon (V\psi_0, \psi_0) - \epsilon^2 \Re A$ and $A_\epsilon = \lambda_\epsilon - i\Gamma_\epsilon$. Then $\Gamma_\epsilon > 0$ and

\begin{equation}
(4.2) \frac{\epsilon}{\eta(\lambda, \epsilon; V\psi_0) = \lambda_\epsilon \{1 + \eta(\lambda, \epsilon)\} + O(\epsilon^2)} \text{ uniformly in } \lambda, \text{ where } \eta(\lambda, \epsilon) \to 0 \text{ as } \lambda \to \lambda_0, \text{ uniformly in } \epsilon.
\end{equation}

Moreover, as $\lambda \to \lambda_0$ and $\epsilon \to 0+$, we have the asymptotic formulae

\begin{equation}
(4.3) \Delta(\lambda, \epsilon) = \epsilon [\lambda - i\Gamma_\epsilon - \lambda]^2 \{1 + o(1)\}.
\end{equation}

and, provided that $0 < \Im \lambda \leq \Gamma_\epsilon/2$,

\begin{equation}
(4.4) W^{-1}(\lambda, \epsilon) x = -\epsilon [\lambda - i\Gamma_\epsilon - \lambda]^{-1} (x, \psi_0) \{1 + o(1)\} V\psi_0 + O(1),
\end{equation}

for every $x$ in $\Re(V)$. 
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Proof: \( \Gamma_e \) is positive by (3.1). If \( y(x, \epsilon; V\psi_0) \) is approximated by taking the zeroth and first order terms of the Newmann series of \( W^{-1}_a(x, \epsilon) \), then (4.2) results from the assumption (4.1) by a simple calculation. By (4.2) and (1.7), we have

\[
\Delta(x, \epsilon) = \epsilon[\lambda_e - i\Gamma_e - z]^{-1} [1 + (\lambda_e - i\Gamma_e - z)^{-1} o(\epsilon^2)]^{-1}.
\]

Since \( |\lambda_e - i\Gamma_e - z| \geq \Gamma_e/2 = \epsilon \) when \( 0 < \text{Im} \, z \leq \Gamma_e/2 \), we may expand the second factor in a binomial series to obtain (4.3). (4.4) now follows from (1.6) and (4.3) by a simple calculation.

The following theorem is the main asymptotic result.

**Theorem 3.** Let \( \{J_\epsilon : \epsilon > 0\} \) be a family of intervals containing \( \lambda_0 \) whose length tends to zero as \( \epsilon \to 0 + \). Then under the hypotheses of Lemma 4.1, we have for every \( x, y \) in \( \mathfrak{R}(V) \)

\[
\frac{d(E_\epsilon(\lambda) x, y)}{d\lambda} = (P_0 x, y) \frac{\Gamma_e}{\pi} [(\lambda - \lambda_0)^2 + \Gamma_e^2]^{-1} [1 + o(1)] + O(\epsilon^{-1})
\]

uniformly in \( \lambda \), for a.e. \( \lambda \) in \( J_\epsilon \).

Notice that since \( I_\epsilon = O(\epsilon^2) \), the first term of (4.5) is \( O(\epsilon^{-2}) \) at \( \lambda = \lambda_e \). Thus, at least very near \( \lambda_e \), the first term dominates the second.

**Proof.** For a.e. \( \lambda \) in \( J_\epsilon \), the left side of (4.5) is the limit as \( \eta \to 0 + \) of

\[
(E_\epsilon(\lambda + i\eta) x, y),
\]

which by (1.4) is equal to

\[
(E_0(\lambda + i\eta) PW^{-1}(\lambda + i\eta) x, W^{-1}(\lambda + i\eta) y).
\]

(4.6)

The factor \( P \) has been inserted, since the term \( E_0(\lambda + i\eta) P_0 \) tends to zero in operator norm as \( \eta \to 0 + \), for \( \lambda \neq \lambda_0 \). Using the asymptotic result (4.4) and boundedness of \( E_0(x) P \) near \( G \), we find after a simple calculation that (4.6) is equal to

\[
(x, \psi_0)(\psi_0, y) [(\lambda - \lambda_e)^2 + \Gamma_e^2]^{-1} \epsilon^2(E_0(\lambda + i\eta) PV\psi_0, V\psi_0)\{1 + o(1)\}
\]

\[
+ O(\epsilon^{-1}).
\]

(4.7)

However, by (3.1) and (4.1)

\[
\epsilon^2(E_0(\lambda + i\eta) PV\psi_0, V\psi_0) = \frac{\Gamma_e}{\pi}\{1 + o(1)\}.
\]

Making this replacement in (4.7), and noting that \( (P_0 x, y) = (x, \psi_0)(\psi_0, y) \) we obtain (4.5).
5. Spectral Concentration

Let \( \{T_\epsilon : \epsilon > 0\} \) be a family of self-adjoint operators. Following [2], we say that the spectrum of \( T_\epsilon \) is concentrated to order \( p \) at \( \lambda_0 \) as \( \epsilon \to 0^+ \), iff there exists a family \( \{J_\epsilon : \epsilon > 0\} \) of intervals such that

\[
E_\epsilon[J_\epsilon] \to E_0[\{\lambda_0\}]
\]

and \( |J_\epsilon| = o(\epsilon^p) \) as \( \epsilon \to 0^+ \), where \( |J_\epsilon| \) is the Lebesgue measure of \( J_\epsilon \). Note that concentration is an asymptotic property of the family \( \{T_\epsilon\} \) and not a property of any single operator \( T_\epsilon \).

We also say that \( T_\epsilon \) tends to \( T_0 \) strongly in the generalized sense, and write \( T_\epsilon \to T_0 \) (sgs) if \( R_\epsilon(z) \to R_0(z) \) strongly for some nonreal \( z \). (See [3], Chapt. VIII, Section 1, esp. Corollary 1.4 on p. 429). Let \( C_0^\infty \) denote the space of continuous complex valued functions on the real line with compact support, and \( \chi_S \) be the characteristic function of \( S \).

**Lemma 5.1.** If \( T_\epsilon \to T_0 \) (sgs), then \( f(T_\epsilon) \to f(T_0) \) strongly for every \( f \) in \( C_0^\infty \).

**Proof.** For any \( \delta > 0 \), there exists a simple function \( s(\epsilon) = \sum_{i=1}^{n} a_i \chi_{S(i)}(\epsilon) \) such that (1) each \( S(i) \) is an interval whose end points are not discontinuities of \( E_\epsilon(\lambda) \) and (2) \( \sup |s(\epsilon) - f(\epsilon)| < \delta \). Since \( s(T_\epsilon) - f(T_\epsilon) | \leq \delta \) for all \( \epsilon > 0 \), and since

\[
s(T_\epsilon) = \sum_{i=1}^{n} a_i E_\epsilon[S(i)] \to s(T_0)
\]

strongly by [3, p. 432, Theorem 1.15], the result follows easily.

The following is our main result on spectral concentration.

**Theorem 4.** Let \( \{J_\epsilon : \epsilon > 0\} \) be a family of intervals, symmetric about \( \lambda_0 \), such that \( |J_\epsilon| = o(\epsilon) \) as \( \epsilon \to 0^+ \). If the hypotheses of Lemma 4.1 hold, then

(a) \( \beta = \lim |J_\epsilon|/(2T_\epsilon) \) exists as \( \epsilon \to 0^+ \), \( 0 < \beta < \infty \),

then

\[
E_\epsilon[J_\epsilon] \to \frac{2}{\pi} \arctan(\beta) P_0 \text{ weakly as } \epsilon \to 0^+.
\]

(b) \( E_\epsilon[J_\epsilon] \to P_0 \text{ strongly iff } \lim |J_\epsilon|/e^2 = \infty \text{ as } \epsilon \to 0^+ \).

As an immediate consequence of Theorem 4 and the definitions, we have:

**Corollary 5.2.** The spectrum of \( T_\epsilon \) is concentrated at \( \lambda_0 \) to order \( p \), for \( 0 < p < 2 \), but not \( p \geq 2 \).
Proof of Theorem 4. Define $P_\varepsilon = E_\varepsilon [J\varepsilon]$ and let $x, y \in \mathfrak{M}(V)$ and $f, g \in \mathcal{C}^2_\varepsilon (\mathbb{R})$. Since for any fixed non-real $z$, the Neumann series for $Q_\varepsilon(z) = [I + \varepsilon VR_0(z)]^{-1}$ converges for sufficiently small $\varepsilon$, it follows from (1.2) that $R_\varepsilon(z) \rightarrow R_0(z)$ in operator norm, and hence, a fortiori, $T_\varepsilon \rightarrow T_0$ (sgs). Hence

$$(P_\varepsilon f(T_0)x, g(T_0)y) - (P_\varepsilon f(T_\varepsilon)x, g(T_\varepsilon)y) \rightarrow 0$$

(5.1)
as $\varepsilon \rightarrow 0 +$ by Lemma 5.1. On the other hand, we have

$$(P_\varepsilon f(T_\varepsilon)x, g(T_\varepsilon)y)$$

$$= \frac{1}{\pi} \langle P_\varepsilon x, y \rangle \left(1 + o(1)\right) \int_{\mathbb{R}} \frac{f(\lambda) g(\lambda)}{\left(\lambda - \lambda_\varepsilon\right)^2 + I_{\varepsilon}^2} d\lambda + O\left(\frac{J_{\varepsilon}}{\varepsilon}\right)$$

(5.2)

by Theorem 3. If $t = (\lambda - \lambda_\varepsilon)/\Gamma_\varepsilon$, the right side of (5.2) becomes

$$\frac{1}{\pi} \langle P_\varepsilon x, y \rangle \int_{\mathbb{R}} (t^2 + 1)^{-1} f(\lambda_\varepsilon + \Gamma_\varepsilon t) g(\lambda_\varepsilon + \Gamma_\varepsilon t) dt + o(1),$$

(5.3)

where $\beta(\varepsilon) = \|f\|/(2\Gamma_\varepsilon)$. But $f(\lambda_\varepsilon + \Gamma_\varepsilon t) g(\lambda_\varepsilon + \Gamma_\varepsilon t) \rightarrow f(\lambda_0) g(\lambda_0)$ uniformly on compact subsets so that if $\beta = \lim \beta(\varepsilon)$ exists, then (5.3) converges to

$$\frac{2}{\pi} \arctan(\beta) f(\lambda_0) g(\lambda_0) \langle P_\varepsilon x, y \rangle = \frac{2}{\pi} \arctan(\beta) (P_\varepsilon f(T_0)x, g(T_0)y)$$

as $\varepsilon \rightarrow 0 +$. 

Now let $D$ be the space of all finite linear combinations of elements of the form $f(T_0)x$, where $f \in \mathcal{C}^2_\varepsilon (\mathbb{R})$ and $x \in \mathfrak{M}(V)$. It follows from the above equations that for every $u, v \in D$

$$(P_\varepsilon u, v) \rightarrow \frac{2}{\pi} \arctan(\beta) (P_\varepsilon u, v) \quad \text{as} \quad \varepsilon \rightarrow 0 +.$$ 

(5.4)

However, [8, Section 3] $D$ is dense in $\mathfrak{H} = \mathfrak{M}(\phi_1, \ldots, \phi_r, T_\varepsilon)$, so that (5.4) implies weak convergence since $\| P_\varepsilon \| \leq 1$. This proves (a).

For (b), observe that if $\beta = \infty$, the limit operator $P_0$ is a projection, so that weak convergence implies strong convergence. (The same is true if $\beta = 0$.) Conversely, if $\beta_\varepsilon$ does not converge to $\infty$, then some subsequence $\beta_n$ of $\beta_\varepsilon$ must converge to a point $\beta_0 < \infty$. By the above arguments, the corresponding subsequence $P_n$ of $P_0$ must converge weakly to $(2/\pi) \arctan (\beta_0) \cdot P_0 \neq P_0$. 

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