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On the existence of generically smooth components for moduli spaces of rank 2 stable reflexive sheaves on $\boldsymbol{P}^3$ ☆

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Abstract

The goal of this work is to prove that for almost all possible triples $(c_1, c_2, c_3) \in \mathbb{Z}^3$ the moduli scheme $M(2; c_1, c_2, c_3)$, which parametrizes isomorphism classes of rank 2 stable reflexive sheaves on $\boldsymbol{P}^3$ with Chern classes c_1, c_2 and c_3, has a generically smooth component. In order to obtain these results we construct a wide range of non-obstructed, m-normal curves with suitable degree and genus. We conclude this paper by adding some examples and remarks.

0. Introduction

It is well known that the set of isomorphism classes of rank 2 stable reflexive sheaves $\mathscr{E}$ with Chern classes c_1, c_2, c_3 is parametrized by a coarse moduli scheme $M(2; c_1, c_2, c_3)$ which is a separated k-scheme of finite type [10, 11]. This moduli scheme has turned out to be an extremely complicated object; the most natural questions concerning the number of components, irreducibility, smoothness, ..., are far from being answered. In [12], the second author determined the set of triples $\{(c_1, c_2, c_3) \in \mathbb{Z}^3$ such that the moduli scheme $M = M(2; c_1, c_2, c_3)$ is non-empty. To be more precise, let us recall the following:

0.1. Theorem. *A triple of integers* (c_1, c_2, c_3) *are the Chern classes of a normalized rank 2 stable reflexive sheaf* $\mathscr{E}$ *on* $\boldsymbol{P}^3$ *if and only if the following conditions hold*:

(M1) $c_1 = 0$ *or* -1.

(M2) $c_2 > 0$ [6, *Propositions* 3.3 *and* 9.7].

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(M3) $c_1 c_2 \equiv c_3 \pmod 2$ [6, *Corollary* 2.4].

(M4) *We have*

$$0 \le c_3 \le \begin{cases} c_2^2 & \text{if } c_1 = -1, \\ c_2^2 - c_2 + 2 & \text{if } c_1 = 0, \end{cases} \quad [6, \text{ Theorem } 8.2].$$

(M5) *If* $c_1 = -1$, *then* $c_3 \in [0, c_2^2] \setminus \bigcup_{r=1}^{b(-1,c_2)} (c_2^2 - 2rc_2 + 2r(r+1), c_2^2 - 2(r-1)c_2)$, *where* $b(-1,c_2) = E[\frac{1}{2}(-1 + \sqrt{4c_2 - 7})]$.

If $c_1 = 0$, *then* $c_3 \in [0, c_2^2 - c_2 + 2] \setminus \bigcup_{r=2}^{b(0,c_2)} (c_2^2 - (2r-1)c_2 + 2r^2, c_2^2 - (2r-3)c_2)$, *where* $b(0,c_2) = E[\sqrt{c_2 - 2}]$ [12, *Theorems* A *and* B].

The goal of this work is to prove that for almost all possible triples $(c_1, c_2, c_3) \in \mathbb{Z}^3$ the moduli scheme $M(2; c_1, c_2, c_3)$ has a generically smooth component. More precisely, if for each pair of integers (c_1, c_2) we consider the following sets:

$A_{c_1}(c_2) := \{c_3 \in \mathbb{Z} \mid M(2; c_1, c_2, c_3) \neq \emptyset\}$,

$B_{c_1}(c_2) := \{c_3 \in A_{c_1}(c_2) \mid M(2; c_1, c_2, c_3)$ has a generically smooth component$\}$,

then our results can be summarized in the following theorem (see Section 7):

Theorem. *For every integer* $c_1 \in \mathbb{Z}$, *we have*

$$\lim_{c_2 \to \infty} \frac{\# B_{c_1}(c_2)}{\# A_{c_1}(c_2)} = 1.$$

The above theorem is a consequence of the existence of generically smooth components of $M = M(2; c_1, c_2, c_3)$. The existence of such components is proved in Section 6. As a main tool we use Serre's correspondence between rank 2 stable reflexive sheaves on $\boldsymbol{P}^3$ and space curves, together with Kleppe's results [8, Section 2] which give conditions in order to assure that the local ring $\mathcal{O}_{M,\mathscr{E}}$ of the moduli scheme M at the point $[\mathscr{E}]$ is regular, provided that the local ring $\mathcal{O}_{H,Y}$ of the Hilbert scheme $H = Hilb(\boldsymbol{P}^3)$ at the point $[Y] \in H$, where Y is the curve associated to $\mathscr{E}$, is regular.

In order to apply Kleppe's results we need to construct non-obstructed, m-normal curves of appropriate degree and genus. This is done in Sections 2–5.

As a general reference on reflexive sheaves the reader may consult [6].

We conclude this paper by adding some examples and remarks.

Conventions. Throughout this paper we work over an algebraically closed field k of characteristic 0, $S = k[x, y, z, t]$, and $\boldsymbol{P}^3 = Proj(S)$. For a coherent sheaf $\mathscr{E}$ on $\boldsymbol{P}^3$ we will often write $H^i\mathscr{E}$ (resp. $h^i\mathscr{E}$) for $H^i(\boldsymbol{P}^3, \mathscr{E})$ (resp. $\dim_k H^i(\boldsymbol{P}^3, \mathscr{E})$). The dual of $\mathscr{E}$ is written $\mathscr{E}^\vee =: \mathrm{Hom}(\mathscr{E}, \mathcal{O}_{\boldsymbol{P}^3})$. A coherent sheaf $\mathscr{E}$ on $\boldsymbol{P}^3$ is *reflexive* if the natural map $\mathscr{E} \to \mathscr{E}^{\vee\vee}$ is an isomorphism.

By a curve we mean a closed, locally Cohen–Macaulay, one-dimensional subscheme of $\boldsymbol{P}^3$ which is generically locally complete intersection. Given two subschemes $Y_1, Y_2 \subset \boldsymbol{P}^3$ defined by the sheaves of ideals $\mathscr{I}_{Y_1}$ and $\mathscr{I}_{Y_2}$, respectively, we denote by

$Y_1 \cup Y_2$ the subscheme of $\boldsymbol{P}^3$ defined by $\mathscr{I}_{Y_1} \cap \mathscr{I}_{Y_2}$, and by $Y_1 \cap Y_2$ the subscheme of $\boldsymbol{P}^3$ defined by $\mathscr{I}_{Y_1} + \mathscr{I}_{Y_2}$.

The cardinality of a set S will be denoted $\sharp S$, and given any real number x we define $E[x] := \max\{n \in \mathbb{N} \mid n \leq x\}$.

We will write $Hilb^{d,g}(\boldsymbol{P}^3)$ (or simply $Hilb(\boldsymbol{P}^3)$ when there is no risk of confusion) to denote the Hilbert scheme of curves Y in $\boldsymbol{P}^3$ of degree d and arithmetic genus $g = p_a(Y)$.

1. Preliminaries

In this section we begin recalling some basic facts which will be used later and we prove two results (Proposition 1.4 and Corollary 1.7) which will be our basic tools to show, in Section 6, the existence of smooth components of $M(2; c_1, c_2, c_3)$.

1.1. Definition. Let $X \subset \boldsymbol{P}^3$ be a curve. We will say that X is *non-obstructed* if and only if the corresponding point $[X]$ of $Hilb(\boldsymbol{P}^3)$ is non-singular. Otherwise, we will say that X is *obstructed*.

1.2. Definition. Let $[\mathscr{E}] \in M(2; c_1, c_2, c_3)$ denote the closed point parametrizing the sheaf $\mathscr{E}$. We say that a stable rank 2 reflexive sheaf $\mathscr{E}$ on $\boldsymbol{P}^3$ is *non-obstructed* (resp. *obstructed*) if $[\mathscr{E}]$ is a non-singular (resp. singular) point of $M(2; c_1, c_2, c_3)$.

One knows from deformation theory that the Zariski tangent space to the moduli scheme $M(2; c_1, c_2, c_3)$ at the point corresponding to a stable sheaf $\mathscr{F}$ is $\mathrm{Ext}^1(\mathscr{F}, \mathscr{F})$, and that the obstructions to extending an infinitesimal deformation lie in $\mathrm{Ext}^2(\mathscr{F}, \mathscr{F})$ [10]. However, necessary and sufficient conditions for a rank 2 stable reflexive sheaf on $\boldsymbol{P}^3$ to be non-obstructed are not known. Partial results can be found, for instance, in [1, 2, 6, 13, 14], where examples of non-singular moduli spaces $M(2; c_1, c_2, c_3)$ are described. Examples of rank 2 stable obstructed reflexive sheaves on $\boldsymbol{P}^3$ can be found in [3, 8].

1.3. Recall that, for every integer c_1, there is a one-to-one correspondence between pairs $(\mathscr{F}, s)$, where $\mathscr{F}$ is a rank 2 reflexive sheaf on $\boldsymbol{P}^3$ with $c_1(\mathscr{F}) = c_1$, and $s \in H^0\mathscr{F}$ a global section whose zero set has codimension 2, and pairs (Y, ξ) where Y is a Cohen–Macaulay curve in $\boldsymbol{P}^3$, generically complete intersection and $0 \neq \xi \in H^0\omega_Y(4 - c_1)$ is a global section which generates the sheaf $\omega_Y(4 - c_1)$ except at finitely many points (see [6, Theorem 4.1]). Furthermore, there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\boldsymbol{P}^3} \longrightarrow F \longrightarrow I_Y(c_1) \longrightarrow 0,$$

$c_2 F = d$ and $c_3 F = 2p_a - 2 + d(4 - c_1)$, where d and p_a are the degree and arithmetic genus of Y.

The following result will be one of our main tools for the construction of non-obstructed reflexive sheaves.

1.4. Proposition. *Let $\mathscr{E}$ be a stable rank 2 reflexive sheaf on $\boldsymbol{P}^3$ corresponding to a curve $Y \subset \boldsymbol{P}^3$ and with Chern classes c_1, c_2 and c_3. Assume that*

(A1) $H^1\mathcal{O}_Y = 0$,

(A2) $H^1\mathcal{O}_Y(c_1) = 0$,

(A2) $H^1\mathcal{N}_Y = 0$.

Then $\mathrm{Ext}^2(\mathscr{E},\mathscr{E}) = 0$. *Thus $\mathscr{E}$ is non-obstructed and the irreducible component of the moduli scheme $M(2; c_1, c_2, c_3)$ containing the point $[\mathscr{E}]$ has dimension* $\dim \mathrm{Ext}^1(\mathscr{E},\mathscr{E})$.

Proof. From the local–global *Ext*-sequence,

$$0 \to H^1(\mathrm{End}(\mathscr{E})) \to \mathrm{Ext}^1(\mathscr{E},\mathscr{E}) \to H^0(\mathrm{Ext}^1(\mathscr{E},\mathscr{E})) \to H^2(\mathrm{End}(\mathscr{E})) \to \mathrm{Ext}^2(\mathscr{E},\mathscr{E}) \to 0,$$

we see that it is enough to show that $h^2(\mathrm{End}(\mathscr{E})) = 0$. Since the sheaf $\mathscr{E}$ corresponds to the curve Y, we have an exact sequence

$$0 \longrightarrow \mathcal{O}(-c_1) \longrightarrow \mathscr{E}(-c_1) \longrightarrow \mathscr{I}_Y \longrightarrow 0, \qquad (*)$$

but $\mathscr{E}^\vee \cong \mathscr{E}(-c_1)$ and $\mathscr{E}^\vee \otimes \mathscr{E} \cong \mathrm{End}(\mathscr{E})$, so tensoring with $\mathscr{E}$ yields the following exact sequence:

$$\mathrm{Tor}(\mathscr{E}(-c_1)), \mathscr{E}) \to \mathrm{Tor}(\mathscr{I}_Y, \mathscr{E}) \to \mathscr{E}(-c_1) \to \mathrm{End}(\mathscr{E}) \to \mathscr{I}_Y \otimes \mathscr{E} \to 0,$$

which splits in short exact sequences

$$0 \longrightarrow \mathscr{M}_1 \longrightarrow \mathrm{End}(\mathscr{E}) \longrightarrow \mathscr{I}_Y \otimes \mathscr{E} \longrightarrow 0, \qquad (1)$$

$$0 \longrightarrow \mathscr{M}_2 \longrightarrow \mathscr{E}(-c_1) \longrightarrow \mathscr{M}_1 \longrightarrow 0. \qquad (2)$$

We will proceed in two steps.

Step 1: We show that $H^2(\mathscr{M}_1) = 0$. In order to do this, consider the long exact cohomology sequence obtained from (2):

$$\cdots \longrightarrow H^2\mathscr{M}_2 \longrightarrow H^2\mathscr{E}(-c_1) \longrightarrow H^2\mathscr{M}_1 \longrightarrow H^3\mathscr{M}_2 \longrightarrow \cdots.$$

Since $\mathscr{M}_2$ is supported on a set of points, we have $h^3\mathscr{M}_2 = 0$. On the other hand, $H^2\mathscr{E}(-c_1)$ fits in the exact sequence

$$0 \longrightarrow H^2\mathscr{E}(-c_1) \longrightarrow H^2\mathscr{I}_Y \longrightarrow H^3\mathcal{O}(-c_1) \longrightarrow H^3\mathscr{E}(-c_1) \longrightarrow 0$$

obtained from sequence (*). But $H^2\mathscr{I}_Y \cong H^1\mathcal{O}_Y$, which is zero by hypothesis (A1). Hence $H^2\mathscr{E}(-c_1) = 0$ and thus $H^2\mathscr{M}_1 = 0$.

Step 2: We show that $h^2(\mathscr{I}_Y \otimes \mathscr{E}) = 0$. First, when we tensor by $\mathscr{E}$ the sequence

$$0 \longrightarrow \mathscr{I}_Y \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Y \longrightarrow 0,$$

we obtain the exact sequence

$$0 \longrightarrow \mathrm{Tor}(\mathcal{O}_Y, \mathscr{E}) \longrightarrow \mathscr{I}_Y \otimes \mathscr{E} \longrightarrow \mathscr{E} \longrightarrow \mathcal{O}_Y \otimes \mathscr{E} \longrightarrow 0,$$

which splits into two short exact sequences

$$0 \longrightarrow \mathrm{Tor}(\mathcal{O}_Y, \mathcal{E}) \longrightarrow \mathcal{I}_Y \otimes \mathcal{E} \longrightarrow \mathcal{P} \longrightarrow 0, \tag{3}$$

$$0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Y \otimes \mathcal{E} \longrightarrow 0. \tag{4}$$

But $\mathcal{O}_Y \otimes \mathcal{E} \cong \mathcal{N}_Y$, because tensoring sequence (*) with $\mathcal{O}_Y$ gives an isomorphism

$$\mathcal{E}(-c_1) \otimes \mathcal{O}_Y \xrightarrow{\cong} \mathcal{I}_Y \otimes \mathcal{O}_Y \cong I_Y/I_Y{}^2$$

and thus $H^1(\mathcal{E} \otimes \mathcal{O}_Y) = H^1\mathcal{N}_Y$, which is zero by hypothesis (A3). This, together with the fact that

$$H^2\mathcal{E} \cong H^2 I_Y(c_1) \cong H^1\mathcal{O}_Y(c_1)$$

and $h^1\mathcal{O}_Y(c_1) = 0$ (A2), implies that, in the long exact sequence

$$\cdots \longrightarrow H^1(\mathcal{E} \otimes \mathcal{O}_Y) \longrightarrow H^2\mathcal{P} \longrightarrow H^2\mathcal{E} \longrightarrow \cdots$$

obtained from sequence (4), we must have $h^2\mathcal{P} = 0$. Hence, if we consider the sequence

$$\cdots \longrightarrow H^2(\mathrm{Tor}(\mathcal{O}_Y, \mathcal{E})) \longrightarrow H^2(\mathcal{I}_Y \otimes \mathcal{E}) \longrightarrow H^2\mathcal{P} \longrightarrow \cdots$$

we see that $h^2(\mathcal{I}_Y \otimes \mathcal{E}) = 0$ (since $\mathrm{Tor}(\mathcal{O}_Y, \mathcal{E})$ is supported on a zero-dimensional scheme and thus $h^2\mathrm{Tor}(\mathcal{O}_Y, \mathcal{E}) = 0$).

Thus, putting all together we have $H^2(\mathcal{M}_1) = H^2(\mathcal{I}_Y \otimes \mathcal{E}) = 0$, so from the long exact cohomology sequence obtained from (1) we must have $h^2(\mathrm{End}(\mathcal{E})) = 0$, hence the result. $\square$

1.5. In [8], Kleppe studies how deformations of a curve $Y \subset \boldsymbol{P}^3$ correspond to deformations of the associated sheaf $\mathcal{F}$ and, as an application, he finds a relationship between the local ring $\mathcal{O}_{H,Y}$ of the Hilbert scheme $H = Hilb^{d,g}(\boldsymbol{P}^3)$ at $[Y]$, and the local ring $\mathcal{O}_{M,\mathcal{F}}$ of $M = M(2; c_1F, c_2F, c_3F)$ at the point $[\mathcal{F}]$. In particular, he proves the following:

1.6. Lemma. *There exists a quasi-projective scheme D parametrizing equivalent pairs (C, ξ), where*

(1) C is an equidimensional Cohen–Macaulay curve and where

(2) the extension $\xi: 0 \to \mathcal{O}_{P^3} \to \mathcal{F} \to \mathcal{I}_C(c_1) \to 0$ is such that $\mathcal{F}$ is a rank 2 stable reflexive sheaf on $\boldsymbol{P}^3$.

Moreover, there are projective morphisms

$$\begin{array}{ccc} D & \xrightarrow{q} & Hilb^{d,g}(\boldsymbol{P}^3) \\ \Big\downarrow{\scriptstyle p} & & \\ M(2; c_1, c_2, c_3) & & \end{array}$$

defined by $p(\mathcal{F}_k, s_k) = \mathcal{F}_k$ and $q(C_k, \xi_k) = C_k$ for a geometric k-point (C_k, ξ_k) corresponding to $(\mathcal{F}_k, s_k)$, such that the fibers of p and q are smooth connected schemes.

Furthermore, p is smooth at $(\mathscr{F}_k, s_k)$ *provided* $H^1\mathscr{F}_k = 0$ *and q is smooth at* (C_k, ξ_k) *provided* $H^1 I_{C_k}(c_1 - 4) = 0$.

We must remark that although the above conditions are sufficient they are not necessary in general. In fact, let $\mathscr{E}$ be a stable rank 2 vector bundle on $\boldsymbol{P}^3$ defined as an extension

$$\xi: 0 \longrightarrow \mathscr{O} \longrightarrow \mathscr{E}(1) \longrightarrow \mathscr{I}_Y \longrightarrow 0, \quad 0 \neq \xi \in H^0\omega_Y(2),$$

where Y is the disjoint union of r skew lines ($r \geq 2$). Then $\mathscr{E}$ is non-obstructed, with Chern classes $c_1 = 0$ and $c_2 = r - 1$ (see [5, Example 4.3.1]) but $h^1\mathscr{E}(1) = h^1\mathscr{I}_Y(2) \neq 0$.

The following result shows that we can replace the vanishing of $H^1\mathscr{I}_C(c_1 - 4)$ under certain additional hypothesis:

1.7. Corollary. *Let* $\mathscr{F}$ *be a stable rank* 2 *reflexive sheaf on* $\boldsymbol{P}^3$ *constructed as an extension*

$$\xi: 0 \longrightarrow \mathscr{O}_{\boldsymbol{P}^3} \xrightarrow{s} \mathscr{F} \longrightarrow I_C(c_1) \longrightarrow 0,$$

where $\mathscr{I}_C$ *is the ideal sheaf of an equidimensional, locally Cohen–Macaulay curve C in* $\boldsymbol{P}^3$. *Assume that* $H^1\mathscr{F} = 0$ *and that C is the general curve of a generically smooth component of* $Hilb^{d,g}(\boldsymbol{P}^3)$. *If C is non-obstructed then* $\mathscr{F}$ *is non-obstructed.*

Proof. Let V be a generically smooth component of $Hilb^{d,g}(\boldsymbol{P}^3)$ with general curve C. Consider the restriction $q|w$:

$$\begin{array}{ccc} D & \xrightarrow{\quad q \quad} & Hilb^{d,g}(\boldsymbol{P}^3) \\ \cup\uparrow & & \uparrow\cup \\ W := q^{-1}(V) & \xrightarrow{\quad q/w \quad} & V \end{array}$$

Since $H^0(\omega_C(4 - c_1))$ contains a section which generates the sheaf $\omega_C(4 - c_1)$ except at a finite number of points, the map $q|w$ is dominating. Thus, by generic flatness, there exists an open set $U \subset Hilb^{d,g}(\boldsymbol{P}^3)$, $\emptyset \neq U \subset V$, $q^{-1}(U) \neq 0$, such that the map $\kappa := q|_{q^{-1}(U)}: q^{-1}(U) \to U$ is flat.

Since the fibers of q are smooth, the morphism κ is also smooth (see for example [15, p.2.10]). But the fact that the map κ is smooth implies that whenever C is non-obstructed, (C, ξ) must be non-obstructed, hence the result. $\square$

2. Construction of non-obstructed nodal curves in $\boldsymbol{P}^3$

The purpose of this section is to construct nodal curves $Y \subset \boldsymbol{P}^3$ which are non-obstructed. In order to do this we will need to generalize some of the smoothing results of [7].

2.1. Let X be a curve in $\boldsymbol{P}^3$ and consider the natural map $\varphi : \mathcal{T}_{\boldsymbol{P}^3} \otimes \mathcal{O}_X \to \mathcal{N}_X$, where $\mathcal{T}_{\boldsymbol{P}^3}$ is the tangent sheaf and $\mathcal{N}_X$ is the normal sheaf of X. The cokernel of φ is a coherent sheaf T^1_X which is supported at the set $S = \mathrm{Sing}(X)$ of singular points of X, and which parametrizes the local deformations of those points.

A curve X is *nodal* if it is reduced with only ordinary double points (nodes) as singularities. For nodal curves, T^1_X is isomorphic with its restriction T^1_S to S. In particular, at each node P the local deformation space T^1_P is a one-dimensional vector space with smooth total space, and a non-zero element of T^1_P corresponds to a deformation which smooths the double point.

Two curves X, Y are said to intersect *quasi-transversally* if their intersection $X \cap Y$ is a finite set of nodes of $X \cup Y$. We also say that $X \cup Y$ is the *nodal* union of X and Y.

2.2. Remark. We fix the following notation: if X is the nodal union of r smooth connected curves $X_1, \dots, X_r$, then we will write S_{ij} and S_i to denote the following sets of points:

$$S_{ij} := X_i \cap X_j \quad (i \neq j),$$

$$S_i := S_{i1} \cup \cdots \cup S_{i,i-1} \cup S_{i,i+1} \cup \cdots \cup S_{ir}.$$

The corresponding cardinalities will be denoted by $s_{ij} := \sharp S_{ij}$ and $s_i := \sharp S_i$.

In [7], the connection between the normal sheaf of a nodal curve and the normal sheaf of its normalization is given in terms of the elementary transformations, $elm^+_\Delta\, \mathcal{N}_X$ and $elm^-_\Delta\, \mathcal{N}_X$, of $\mathcal{N}_X$ over a finite set of points Δ. The next two propositions are generalizations of Corollaries 3.2 and 3.3 of [7]. They show the relation between the restriction $\mathcal{N}_{\cup X_i} |_{X_i}$ of the normal sheaf $\mathcal{N}_{\cup X_i}$ to a smooth irreducible component X_i and the normal sheaf $\mathcal{N}_{X_i}$ of this component.

2.3. Proposition. *Let X be the nodal union of r smooth connected curves $X_1, \dots, X_r$. Let Δ be the subset of $\mathcal{P}(\mathcal{N}_{X_1})_{S_1}$ defined by S_1. Then $\mathcal{N}_X|_{X_1}$ is isomorphic with $elm^+_\Delta\, \mathcal{N}_{X_1}$, and there is a natural exact sequence*

$$0 \longrightarrow \mathcal{N}_{X_1} \longrightarrow \mathcal{N}_X|_{X_1} \longrightarrow T^1_{S_1} \longrightarrow 0.$$

Proof. Let X' be the normalization of X (i.e., X' is the disjoint union of the irreducible components $X_1, \dots, X_r$). Let $v : X' \to \boldsymbol{P}^3$ be the associated morphism, and $S = \bigcup_{i \neq j} S_{ij}$ the singular locus of X. Then there is an exact sequence [7, Proposition 3.1]

$$0 \longrightarrow \mathcal{N}_{X'} \longrightarrow v^* \mathcal{N}_X \longrightarrow v^* T^1_S \longrightarrow 0,$$

which restricted to X_1 becomes

$$0 \longrightarrow \mathcal{N}_{X_1} \longrightarrow \mathcal{N}_X|_{X_1} \longrightarrow T^1_{S_1} \longrightarrow 0.$$

In order to apply [7, Proposition 2.3(b)] we only have to check that the subset H of $\mathbb{P}(\mathcal{N}_{X_1})$ defined by the map $\mu:\mathcal{N}_{X_1} \to im\,\mu$ is Δ. To this end, let P be a point in S_1. In particular $P \in X_1 \cap X_j$ for some j, $2 \le j \le r$. Let t be a local section of the tangent sheaf $\mathcal{T}_{X_1}$ such that $t(P)$ is tangent to X_j. Then $t(P)$ vanishes on $\mathcal{I}_X$ and thus $\mu(t) = 0$. Hence $\Delta \subset H$. But then equality must hold, since they are in bijective correspondence with S_1. □

2.4. Proposition. *In the situation of Proposition 2.3, suppose F is a non-singular surface containing X_1 and transversal to $X_2 \cup \cdots \cup X_r$. Then, for any subset Δ' of Δ with image S'_1 in S_1, there is an exact sequence*

$$0 \longrightarrow \mathcal{N}_{X_1/F} \longrightarrow elm^{+}_{\Delta'}\mathcal{N}_{X_1} \longrightarrow \mathcal{N}_F|_{X_1}(S'_1) \longrightarrow 0.$$

Proof. We have an exact sequence

$$0 \longrightarrow \mathcal{N}_{X_1/F} \longrightarrow \mathcal{N}_{X_1} \longrightarrow \mathcal{N}_F|_{X_1}(S'_1) \longrightarrow 0,$$

which, by [7, Proposition 2.2], yields the result. □

The next proposition together with Theorem 2.10 will be our main criteria for constructing non-obstructed nodal curves.

2.5. Proposition. *Let Z be the nodal union of two curves X and Y meeting in a set of point S. Assume*

(a) *X and Y are non-obstructed,*

(b) *the map $H^0\mathcal{N}_X \to H^0\mathcal{N}_Z|_X$ is bijective,*

(c) *the sequence $H^0\mathcal{N}_Y \to H^0\mathcal{N}_Z|_S \to H^0 T^1_S$ is exact,*

then there exists an exact sequence

$$0 \longrightarrow H^0\mathcal{N}_Z \longrightarrow H^0\mathcal{N}_X \oplus H^0\mathcal{N}_Y \longrightarrow H^0 R \to 0.$$

where $R := \ker(\mathcal{N}_Z|_S \to T^1_S)$. Moreover, Z is non-obstructed and non-smoothable.

Proof. It is implicitly contained in [7, Proposition 5.1 and Corollary 5.2]. □

2.6. Definition. (1) We will say that a pair of curves (X, Y) *satisfies the HH conditions* if it satisfies conditions (a)–(c) of Proposition 2.5.

(2) We will say that a pair of curves (X, Y) *satisfies the NN conditions,* if it satisfies the HH conditions and the sequence

$$H^0\mathcal{N}_X \to H^0 N_Z|_S \to H^0 T^1_S \text{ is exact.}$$

In particular, we have the following situations where the NN conditions hold:

2.7. Corollary. *Let C be a non-singular plane curve of degree $c \geq 1$, and let $L_1, \dots, L_n$ be n skew lines in $\mathbf{P}^3$, each meeting C in one point. Set*

$$Z = C \cup L_1 \cup \cdots \cup L_n.$$

(1) *If $n > \frac{1}{2}(c-2)(c-3)$, then Z is non-obstructed and smoothable.*

(2) *If $n \leq \frac{1}{2}(c-2)(c-3)$ and if the points $P_i = C \cap L_i$ are in general position in the plane containing C, then the pair $(C, \cup L_i)$ satisfies the NN conditions and hence, Z is non-obstructed and non-smoothable.*

Proof. Except for the explicit statement of the non-obstructedness, it follows from Proposition 5.3 of [7]. But for the case (1), applying [7, Theorem 4.1] we get $H^1 \mathcal{N}_Z = 0$, and thus Z is non-obstructed. On the other hand, if we are in case (2), then if $X = C$ and $Y = L_1 \cup \cdots \cup L_n$, the pair (X, Y) satisfies the NN conditions. □

2.8. Definition. Let C and D be two non-singular plane curves in $\mathbf{P}^3$ lying in distinct planes H and H', respectively. We will say that C and D meet *transversally at s points* if

(1) they intersect quasi-transversally at s points, and

(2) C is transversal to H' and D is transversal to H.

2.9. Corollary. *Let D and C be two non-singular plane curves of degrees d and c $(d \geq c)$ meeting transversally in s points, $1 \leq s \leq c$. If $d \geq s+3$ then the pair (D, C) satisfies the NN conditions and thus $Z = C \cup D$ is non-obstructed and non-smoothable.*

Proof. Except for the case $s = c = 1$ the result follows from [7, Proposition 5.5] and Proposition 2.5 above. If $c = s = 1$, Z is a plane curve of degree $d \geq 4$ with a line attached at one point, therefore it is non-obstructed (Corollary 2.7). □

The following theorem generalizes Proposition 2.5 and will be our main criterion for constructing non-obstructed nodal curves.

2.10. Theorem. *Let Z be the nodal union of three curves X_1, X_2* and X_3. Assume that:

(S1) *The pairs (X_1, X_2) and (X_2, X_3) satisfy the HH conditions, and the pair (X_1, X_3) satisfies the NN conditions.*

(S2) *The map $H^0 \mathcal{N}_{X_1} \to H^0 \mathcal{N}_Z|_{X_1}$ is bijective.*

Then there exists an exact sequence

$$0 \longrightarrow H^0 \mathcal{N}_Z \longrightarrow H^0 \mathcal{N}_{X_1} \oplus H^0 \mathcal{N}_{X_2} \oplus H^0 \mathcal{N}_{X_3} \longrightarrow H^0 R \longrightarrow 0,$$

where $R := \ker(\mathcal{N}_Z|_S \to T^1_S)$, $S = S_{12} \cup S_{13} \cup S_{23}$. Moreover, Z is non-obstructed and non-smoothable.

Proof. First of all, remark that hypothesis (S1) implies that the sequences

$$H^0\mathcal{N}_{X_1} \to H^0\mathcal{N}_{X_1\cup X_2}|_{S_{13}} \longrightarrow H^0T^1_{S_{13}},$$
$$H^0\mathcal{N}_{X_2} \to H^0\mathcal{N}_{X_1\cup X_2}|_{S_{12}} \longrightarrow H^0T^1_{S_{12}},$$
$$H^0\mathcal{N}_{X_3} \to H^0\mathcal{N}_{X_2\cup X_3}|_{S_{23}} \longrightarrow H^0T^1_{S_{23}}$$

are exact. Furthermore, from Proposition 2.5, we deduce that the curve $X_2\cup X_3$ is non-obstructed and that there exists an exact sequence

$$0 \longrightarrow H^0\mathcal{N}_{X_2\cup X_3} \longrightarrow H^0\mathcal{N}_{X_2} \oplus H^0\mathcal{N}_{X_3} \longrightarrow H^0R_{23} \longrightarrow 0, \tag{5}$$

where $R_{23} := \ker(\mathcal{N}_{X_2\cup X_3}|_{S_{23}} \to T^1_{S_{23}})$.

Let W (resp. W') be the generically smooth component of the Hilbert scheme containing the point associated with X_1 (resp. with $X_2\cup X_3$). We define T to be the locally closed subvariety in $W \times W'$ of couples (X, Y) of curves meeting quasi-transversally at s_1 points ($s_1 := \#S_1 = \#(S_{12}\cup S_{13})$). As in [7, Corollary 5.2], we see that T is at most s_1-codimensional in $W \times W'$. Hence Z belongs to an irreducible family of distinct singular curves of dimension $h^0\mathcal{N}_{X_1} + h^0\mathcal{N}_{X_2\cup X_3} - s_1$. Hence, to see that Z is non-obstructed and non-smoothable, it is enough to show that

$$h^0\mathcal{N}_Z = h^0\mathcal{N}_{X_1} + h^0\mathcal{N}_{X_2\cup X_3} - s_1$$

[7, Proposition 5.1]. From sequence (5), this is equivalent to showing that

$$h^0\mathcal{N}_Z = h^0\mathcal{N}_{X_1} + h^0\mathcal{N}_{X_2} + h^0\mathcal{N}_{X_3} - s, \tag{6}$$

where $s := \#S$. In order to prove (6), consider the exact sequence

$$0 \longrightarrow \mathcal{N}_Z \longrightarrow \mathcal{N}_Z|_{X_1} \oplus \mathcal{N}_Z|_{X_2} \oplus \mathcal{N}_Z|_{X_3} \longrightarrow \mathcal{N}_Z|_S \longrightarrow 0$$

and let $R = \ker(\mathcal{N}_Z|_S \to T^1_S)$. Since the sets S_{12}, S_{13} and S_{23} are disjoint, we have

$$\mathcal{N}_Z|_S \cong \mathcal{N}_Z|_{S_{12}} \oplus \mathcal{N}_Z|_{S_{13}} \oplus \mathcal{N}_Z|_{S_{23}}, \qquad T^1_S \cong T^1_{S_{12}} \oplus T^1_{S_{13}} \oplus T^1_{S_{23}}$$

and thus $R = R'_{12} \oplus R'_{13} \oplus R'_{23}$, where $R'_{ij} := \ker(\mathcal{N}_Z|_{S_{ij}} \to T^1_{S_{ij}})$, $1 \le i < j \le 3$.

On the other hand, from the exact sequences:

$$0 \longrightarrow \mathcal{N}_{X_2} \longrightarrow \mathcal{N}_Z|_{X_2} \longrightarrow T^1_{S_{12}} \oplus T^1_{S_{23}} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{N}_{X_3} \longrightarrow \mathcal{N}_Z|_{X_3} \longrightarrow T^1_{S_{13}} \oplus T^1_{S_{23}} \longrightarrow 0$$

(see Proposition 2.3) we obtain the sequence

$$H^0\mathcal{N}_{X_2} \oplus H^0\mathcal{N}_{X_3} \longrightarrow H^0\mathcal{N}_Z|_{X_2} \oplus H^0\mathcal{N}_Z|_{X_3} \longrightarrow H^0T^1_{S_{12}} \oplus H^0T^1_{S_{23}} \oplus H^0T^1_{S_{13}}. \tag{7}$$

Claim. *Sequence* (7) *is exact.*

Proof of the claim. We consider the diagram shown in Fig. 1, where the columns and the top and bottom rows are exact. Thus, in order to prove the claim, it only remains

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & H^0\mathcal{N}_{X_2\cup X_3} \arrow[r] \arrow[d] & H^0\mathcal{N}_{Z|X_2\cup X_3} \arrow[r] \arrow[d] & H^0T^1_{S_{12}} \oplus H^0T^1_{S_{13}} \arrow[d] \\
0 \arrow[r] & H^0\mathcal{N}_{X_2} \oplus H^0\mathcal{N}_{X_3} \arrow[r, "\nu_1"] \arrow[d] & H^0\mathcal{N}_{Z|X_2} \oplus H^0\mathcal{N}_{Z|X_3} \arrow[r, "\omega_1"] \arrow[d, "r_1"'] & H^0T^1_S \arrow[d, "r_2"'] \\
0 \arrow[r] & H^0R_{23} \arrow[r, "\nu_2"] \arrow[d] & H^0\mathcal{N}_{X_2\cup X_3|S_{23}} \arrow[r, "\omega_2"] & H^0T^1_{S_{23}} \arrow[d] \\
 & 0 & & 0
\end{tikzcd}$$

Fig. 1.

to show that $\ker(\omega_1)$ is contained in the image of ν_1. But, if $x \in \ker(\omega_1)$, then $\tau_1 x \in \ker(\omega_2) = im(\nu_2)$, and thus there exists an element $t \in H^0\mathcal{N}_{X_2} \oplus H^0\mathcal{N}_{X_3}$ such that $\nu_1 t = x$. □

Proof of Theorem 2.10 (*continued*). We now consider the commutative diagram as shown in Fig. 2. By the exactness of (7) and the hypothesis (S2), we have $im(\rho) = \ker(\gamma)$. Hence

$$im(\eta) = \ker(\beta) \subset \ker(\delta\beta) = \ker(\gamma) = im(\rho),$$

so the map η factors through $H^0\mathcal{N}_{X_1} \oplus H^0\mathcal{N}_{X_2} \oplus H^0\mathcal{N}_{X_3}$. That is, $H^0\mathcal{N}_Z = \ker(\sigma)$. By hypothesis, σ is surjective and so, we have an exact sequence

$$0 \longrightarrow H^0\mathcal{N}_X \longrightarrow H^0\mathcal{N}_{X_1} \oplus H^0\mathcal{N}_{X_2} \oplus H^0\mathcal{N}_{X_3} \longrightarrow H^0R \longrightarrow 0,$$

hence the result. □

Applying Theorem 2.10, we will now proceed to construct non-obstructed nodal curves.

2.11. Proposition. *Let C and D be two non-singular plane curves of respective degrees c and d $(1 \le c \le d)$ meeting transversally in s points, $1 \le s \le c$, and let $L_1, \ldots, L_n$ be n skew lines each meeting C in one point and D in another point. Assume that the points $\{q_i\}_{1\le i\le n} = D \cap [\bigcup_{i=1}^n L_i]$ (resp. $\{p_i\}_{1\le i\le n} = C \cap [\bigcup_{i=1}^n L_i]$) are in general position in the plane containing D (resp. C). If $d \ge s+3$ and $1 \le n \le \frac{1}{2}(c-2)(c-3)$ then the curve*

$$Z = D \cup C \cup L_1 \cup \cdots \cup L_n$$

is non-obstructed and non-smoothable.

$$\begin{tikzcd}
 & & & 0 \\
 & & H^0 T^1_{S_{12}} \oplus H^0 T^1_{S_{23}} \oplus H^0 T^1_{S_{13}} \arrow[r, "\cong"] & H^0 T^1_S \arrow[u] \\
0 \arrow[r] & H^0 \mathcal{N}_Z \arrow[r, "\eta"] & H^0 \mathcal{N}_{Z|X_1} \oplus H^0 \mathcal{N}_{Z|X_2} \oplus H^0 \mathcal{N}_{Z|X_3} \arrow[u, "\gamma"] \arrow[r, "\beta"] & H^0 \mathcal{N}_{Z|S} \arrow[u, "\delta"'] \\
 & & H^0 \mathcal{N}_{X_1} \oplus H^0 \mathcal{N}_{X_2} \oplus H^0 \mathcal{N}_{X_3} \arrow[u, "\rho"] \arrow[r, "\sigma"] & H^0 R \arrow[u] \\
 & & 0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

Fig. 2.

Proof. In order to apply Theorem 2.10, set $X_1 = D$, $X_2 = C$ and $X_3 = \bigcup_{i=1}^{n} L_i$. By Corollaries 2.7(2) and 2.9, condition (S1) is fulfilled. So it remains to prove that the map $H^0\mathcal{N}_{X_1} \to H^0\mathcal{N}_Z|_{X_1}$ is bijective. In order to do this, let H_i be the plane containing the curve X_i $(i = 1, 2)$. Since H_1 is transversal to $X_2 \cup X_3$, we have an exact sequence (see Propositions 2.3 and 2.4)

$$0 \longrightarrow \mathcal{O}_{X_1}(dH_1) \longrightarrow \mathcal{N}_Z|_{X_1} \longrightarrow \mathcal{O}_{X_1}(H_1 + S) \longrightarrow 0, \tag{8}$$

where $d = \deg(X_1)$, $S = S_{12} \cup S_{13}$ (see Remark 2.2 for the notation). But

$$H^0\mathcal{N}_{X_1} = H^0\mathcal{O}_{X_1}(dH_1) \oplus H^0\mathcal{O}_{X_1}(H_1)$$

(X_1 being a plane curve), and $H^1\mathcal{O}_{X_1}(dH_1) = 0$, so from the exact sequence (8), it is enough to show that

$$H^0\mathcal{O}_{X_1}(H_1 + S) = H^0\mathcal{O}_{X_1}(H_1)$$

or equivalently, that $H_1 + S \sim H_1$ (they belong to the same linear system). In order to do this, let

$$\Sigma = (X_1 \cap L) - S_{12},$$

where $L := H_1 \cap H_2$. By hypothesis, $\#\Sigma = d - s \geq 3$, hence

$$H_1 + S_{12} \sim 2H_1 - \Sigma \sim H_1$$

(any conic containing three collinear points contains the line joining them). Hence

$$H_1 + S \sim H_1 + S_{13}. \tag{9}$$

On the other hand, by Riemann–Roch,

$$h^0\mathcal{O}_{X_1}(H_1 + S_{13}) - h^1\mathcal{O}_{X_1}(H_1 + S_{13}) = d + n + 1 - g$$

(g being the arithmetic genus of X_1) and so, by Serre duality,

$$h^0\mathcal{O}_{X_1}(H_1 + S_{13}) = d + n + 1 - g + h^0\mathcal{O}_{X_1}((d-4)H_1 - S_{13})$$

but $h^0\mathcal{O}_{X_1}(d-4) = \frac{1}{2}(d-2)(d-3)$ and the points in S_{13} are in general position in H_1 (more precisely, they define independent linear forms on $H^0\mathcal{O}_{H_1}(d-4)$). Since the natural restriction map $H^0\mathcal{O}_{H_1}(d-4) \to H^0\mathcal{O}_{X_1}(d-4)$ is an isomorphism, we see that S_{13} also imposes independent conditions on $H^0\mathcal{O}_{X_1}(d-4)$. Thus

$$h^0\mathcal{O}_{X_1}((d-4)H_1 + S_{13}) = h^0\mathcal{O}_{X_1}(d-4) - n$$

and then,

$$h^0\mathcal{O}_{X_1}(H_1 + S_{13}) = h^0\mathcal{O}_{X_1}(H_1) = 3,$$

that is, $H_1 + S_{13} \sim H_1$, which together with (9) yields $H_1 + S \sim H_1$, which is what we needed. □

2.12. Proposition. *Let $Z = X_0 \cup X_1 \cup X_2$ be the nodal union of three non-singular plane curves of respective degrees d_0, d_1 and d_2 ($1 \le d_0 \le d_1 \le d_2$) and such that*

(1) *X_1 and X_2 meet transversally at s_{12} points, $1 \le s_{12} \le d_1$,*

(2) *X_0 meets transversally the curve $X_1 \cup X_2$ in s_0 points ($s_0 = s_{01} + s_{02}$), with $1 \le s_{01} \le d_0$ and $1 \le s_{02} \le d_0$.*

Assume that $d_2 \ge s_{12} + 3$, $d_2 \ge s_{02} + 3$ and $d_1 \ge s_{01} + 3$. Then the curve Z is non-obstructed and non-smoothable.

Proof. The case $d_0 = 1$ follows from Proposition 2.11. Assume $d_0 \ge 2$. By Corollary 2.9, the pairs (X_2, X_1), (X_2, X_0) and (X_1, X_0) satisfy the NN conditions, so in order to apply Theorem 2.10, it remains to show that the map $H^0\mathcal{N}_{X_2} \to H^0\mathcal{N}_Z|_{X_2}$ is bijective. Let H_i denote the plane containing the curve X_i. Since the plane H_2 is transversal to $X_0 \cup X_1$ we have an exact sequence (see Propositions 2.3 and 2.4)

$$0 \longrightarrow \mathcal{O}_{X_2}(d_2H_2) \longrightarrow \mathcal{N}_Z|_{X_2} \longrightarrow \mathcal{O}_{X_2}(H_2 + S) \longrightarrow 0. \tag{10}$$

where $d = \deg(X_2)$, $S = S_{02} \cup S_{12}$. But $H^0\mathcal{N}_{X_2} = H^0\mathcal{O}_{X_2}(d_2H_2) \oplus H^0\mathcal{O}_{X_2}(H_2)$ and $H^1\mathcal{O}_{X_2}(d_2H_2) = 0$, so from the exact sequence (10), it is enough to show that

$$H^0\mathcal{O}_{X_2}(H_2 + S) = H^0\mathcal{O}_{X_2}(H_2).$$

Let $\Sigma_0 = (X_2 \cap L_0) - S_{02}$, where $L_0 := H_0 \cap H_2$. By hypothesis, $\sharp\Sigma_0 = d_2 - s_{02} \ge 3$, hence

$$H_2 + S_{02} \sim 2H_2 - \Sigma_0 \sim H_2.$$

In a similar way, if we consider $\Sigma_1 = (X_2 \cap L_1) - S_{12}$ where $L_1 := H_1 \cap H_2$ then, by hypothesis, $\sharp\Sigma_1 = d_2 - s_{12} \geq 3$, hence

$$H_2 + S_{12} \sim H_2.$$

But this implies that $H_2 + S \sim H_2$, which is what we needed. $\square$

For the next construction we will need the following easy lemma.

2.13. Lemma. *Let C be a non-singular plane conic and let $S = \{P_1, \dots, P_s\}$ be a general set of points in C. If $s \leq 2d + 1$, then S imposes independent conditions on $H^0\mathcal{O}_H(d)$.*

2.14. Proposition. *Let X be a smooth plane curve of degree $d > 3$ and let Y be a non-singular curve of type (a, b), $2 \leq a \leq b$, on a non-singular quadric Q. Assume that X and Y meet quasi-transversally in $1 \leq s \leq 2d - 7$ points and that X is transversal to Q. Then the pair (Y, X) satisfies the conditions HH and thus the curve $Z = X \cup Y$ is non-obstructed and non-smoothable.*

Proof. We want to apply Proposition 2.5. Condition (a) is fulfilled for any plane curve as well as for curves on smooth quadric surfaces. Let us prove that condition (b) holds. Since the plane H containing X is transversal to Y, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(dH) \longrightarrow \mathcal{N}_Z|_X \longrightarrow \mathcal{O}_X(H + S) \longrightarrow 0, \tag{11}$$

where $S = X \cap Y$. But $H^0\mathcal{N}_X = H^0\mathcal{O}_X(dH) \oplus H^0\mathcal{O}_X(H)$ and $H^1\mathcal{O}_X(dH) = 0$, so from the exact sequence (11) we see it is enough to check that

$$H^0\mathcal{O}_X(H + S) = H^0\mathcal{O}_X(H).$$

By Riemann–Roch and Serre duality,

$$h^0\mathcal{O}_X(H + S) = d + s + 1 - g + h^0\mathcal{O}_X((d - 4)H - S)$$

(where g stands for the arithmetic genus of X). Since the natural restriction map

$$H^0\mathcal{O}_H(d - 4) \longrightarrow H^0\mathcal{O}_X((d - 4)H)$$

is an isomorphism and the points in S impose independent conditions on $H^0\mathcal{O}_H(d - 4)$ (see Lemma 2.13), we get

$$h^0(\mathcal{O}_X(d - 4)H - S) = h^0(\mathcal{O}_X(d - 4)H) - S$$

and thus $h^0\mathcal{O}_X(H + S) = h^0\mathcal{O}_X(H) = 3$.

We now turn to condition (c) of Proposition 2.5. Set $R := \ker(\mathcal{N}_Y|_S \to T^1_S)$. It is enough to see that the map $H^0\mathcal{N}_Y \to H^0R \cong H^0\mathcal{O}_S$ is surjective. We consider the exact sequence

$$0 \longrightarrow \mathcal{N}_Y|_Q = \mathcal{O}_Y(a, b) \longrightarrow \mathcal{N}_Y \longrightarrow \mathcal{N}_Q|_Y = \mathcal{O}_Y(2, 2) \longrightarrow 0.$$

Since $h^1\mathcal{O}_Y(a,b)=0$, we have $H^0\mathcal{N}_Y = H^0\mathcal{O}_Y(a,b)\oplus H^0\mathcal{O}_Y(2,2)$, and the result follows from the following claim:

Claim. *The map $H^0\mathcal{O}_Y(a,b)\to H^0\mathcal{O}_S$ is surjective.*

Proof of the claim. It is enough to see that the map $\varphi: H^0\mathcal{O}_Q(a,b)\to H^0\mathcal{O}_S$ is surjective. Since the points of S lie in the conic $C=H\cap Q$ we can factorize φ:

$$\begin{tikzcd} H^0\mathcal{O}_Q(a,b) \arrow[r, "\varphi"] \arrow[d, "\varphi_1"'] & H^0\mathcal{O}_S \\ H^0\mathcal{O}_C(a,b) \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

Since $H^0\mathcal{O}_C(a,b)\cong H^0\mathcal{O}_{P^1}(2(a+b))$ is generated by $2(a+b)+1$ homogeneous forms of degree $2(a+b)$ and $s\le a+b=\#(Y\cap H)$, we get that φ_2 is surjective. From the exact sequence

$$0\longrightarrow\mathcal{O}_Q(a-1,b-1)\longrightarrow\mathcal{O}_Q(a,b)\longrightarrow\mathcal{O}_C(a,b)\longrightarrow 0$$

(C is a curve of type $(1,1)$ on Q) we get

$$H^0\mathcal{O}_Q(a,b)\xrightarrow{\varphi_1}H^0\mathcal{O}_C(a,b)\longrightarrow H^1\mathcal{O}_Q(a-1,b-1).$$

By Serre duality, $H^1\mathcal{O}_Q(a-1,b-1)\cong H^1\mathcal{O}_Q(-a-1,\ -b-1)=0$ [4, Ch. III, Example 5.6] and thus φ_1 is surjective. $\square$

3. Construction of m-normal, nodal curves in P^3 ($m=1,2$)

Having in mind the construction of nodal non-obstructed curves given in the preceding section, we now turn to the problem of showing that, under certain additional conditions, these constructions yield curves Z with $H^1(\mathcal{I}_Z(m))=0$ for $m=1,2$. As before, all curves are assumed to be in P^3.

3.1. Definition. A curve Y is called m-normal if $H^1\mathcal{I}_Y(m)=0$, where $\mathcal{I}_Y$ is the ideal sheaf of the curve. In particular, 1-normal (resp. 2-normal) curves are usually called *linearly normal* (resp. *quadratically normal*).

3.2. Proposition. *Let $m=1$ or 2. Let X_1 and X_2 be two plane curves of degrees d_1 and d_2 ($d_1\le d_2$) which meet quasi-transversally at s different points, $m+1\le s\le d_1$. Then $Z=X_1\cup X_2$ is m-normal.*

Proof. Consider the exact sequence

$$0\longrightarrow H^0\mathcal{I}_Z(m)\longrightarrow H^0\mathcal{O}_{P^3}(m)\longrightarrow H^0\mathcal{O}_Z(m)\longrightarrow H^1\mathcal{I}_Z(m)\longrightarrow 0. \tag{12}$$

Since Z is not contained in any plane but it is contained in one quadric then, for $m = 1, 2$, we have $h^0\mathscr{I}_Z(m) = m - 1$. Hence, it is enough to show that, for $m = 1, 2$,

$$h^0\mathcal{O}_Z(m) \leq h^0\mathcal{O}_{P^3}(m) - (m - 1)$$

i.e.,

$$h^0\mathcal{O}_Z(m) \leq \begin{cases} 4 & \text{if } m = 1, \\ 9 & \text{if } m = 2. \end{cases}$$

To this end, consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{X_1}(-D) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_{X_2} \longrightarrow 0$$

with $D = X_1 \cap X_2$. We thus get

$$h^0\mathcal{O}_Z(m) \leq h^0\mathcal{O}_{X_2}(m) + h^0\mathcal{O}_{X_1}(m - D),$$

but

$$h^0\mathcal{O}_{X_2}(m) + h^0\mathcal{O}_{X_1}(m - D) \leq \begin{cases} 4 & \text{if } m = 1,\ s \geq 2, \\ 9 & \text{if } m = 2,\ s \geq 3, \end{cases}$$

hence the result. □

3.3. Proposition. *Fix an integer $m \in \{1, 2\}$. Let X be a reduced curve such that $h^0\mathscr{I}_X(m) = 0$, and let L be a k-secant line, $k \geq m + 1$. If X is m-normal, then $X \cup L$ is also m-normal.*

Proof. Analogous to the proof of Proposition 3.2. □

Consider the following particular case:

3.4. Corollary. *Let X be a plane curve and let L be a line meeting X at one point. Then $X \cup L$ is linearly normal.*

Proof. Arguing as in Proposition 3.2, we get $h^0\mathcal{O}_{X\cup L} \leq h^0\mathcal{O}_X(1) + h^0\mathcal{O}_L(1) - 1 = 4$. □

3.5. Proposition. *Let $Z = X_0 \cup X_1 \cup X_2$ be the nodal union of three non-singular plane curves X_0, X_1, X_2 of degrees d_0, d_1 and d_2 ($1 \leq d_0 \leq d_1 \leq d_2$), such that no two of them are contained in the same plane and*

(1) *X_1 and X_2 meet quasi-transversally at s points, $3 \leq s \leq d_1$,*
(2) *X_0 meets the curve $X_1 \cup X_2$ quasi-transversally at t points, where*

$$t \geq \begin{cases} 2 & \text{if } d_0 = 1, \\ 4 & \text{if } d_0 = 2, \\ 5 & \text{if } d_0 \geq 3. \end{cases}$$

Then Z is quadratically normal.

Proof. As before, consider the exact sequence

$$0 \longrightarrow H^0\mathcal{I}_Z(2) \longrightarrow H^0\mathcal{O}_{P^3}(2) \longrightarrow H^0\mathcal{O}_Z(2) \longrightarrow H^1\mathcal{I}_Z(2) \longrightarrow 0. \tag{13}$$

Since Z is not contained in any quadric $h^0\mathcal{I}_Z(2) = 0$ and so it is enough to show that $h^0\mathcal{O}_Z(2) \leq 10$. To this end, consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{X_0}(-D) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_{X_1 \cup X_2} \longrightarrow 0$$

with $D = X_0 \cap (X_1 \cup X_2)$. We thus get

$$h^0\mathcal{O}_Z(2) \leq h^0\mathcal{O}_{X_1 \cup X_2}(2) + h^0\mathcal{O}_{X_0}(2 - D),$$

but by hypothesis $h^0\mathcal{O}_{X_1 \cup X_2}(2) \leq 9$ (Corollary 3.2) and

$$h^0\mathcal{O}_{X_0}(2) = \begin{cases} 3 & \text{if } d_0 = 1, \\ 5 & \text{if } d_0 = 2, \\ 6 & \text{if } d_0 \geq 3. \end{cases}$$

hence $h^0\mathcal{O}_Z(2) \leq 10$. □

Let C be a smooth curve of type (a, b), $a \leq b$, on a smooth quadric surface Q. It is well known that $h^1\mathcal{I}_C(m) \neq 0$ if and only if $a \leq m \leq b - 2$. Thus, the curve C is linearly normal if and only if either $a \geq 2$ or $(a, b) \in \{(1, 1), (1, 2), (0, 1), (0, 2)\}$. In a similar way, the curve C will be quadratically normal if and only if either $a \geq 3$ or $(a, b) \in \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 3)\}$.

3.6. Corollary. *Fix an integer $m \in \{1, 2\}$. Let X be a smooth plane curve of degree d and let C be a smooth curve of type (a, b), $m + 1 \leq a \leq b$, lying on a smooth quadric surface and meeting X in a set of points S. If $\sharp S \geq 2m + 1$ then $Z = C \cup X$ is m-normal.*

Proof. Since the proof is analogous to that of Proposition 3.2, it is left to the reader. □

4. Non-obstructed linearly normal nodal curves

We will now apply the results of the two preceding sections in order to construct nodal curves which are both non-obstructed and linearly normal.

4.1. Theorem. *Fix an integer $d \geq 7$ and let r be any integer such that $2 \leq r \leq \frac{1}{2}(d - 3)$. Let Y be the nodal curve defined as the union*

$$Y = D \cup C \cup L_1 \cup \cdots \cup L_t,$$

where

(1) *C and D are non-singular plane curves of degrees $r - t$ and $d - r$, respectively, meeting transversally in s points, $2 \leq s \leq r - t$.*

(2) $L_1, \dots, L_t$ *are t skew lines, with* $t = 0$ *if* $r = 2, 4$ *and*

$$0 \le t \le \begin{cases} 1 & \text{if } r = 6, \\ \frac{1}{2}r - 1 & \text{if } r \text{ is even, } r \ge 8, \\ \frac{1}{2}(r-1) - 1 & \text{if } r \text{ is odd,} \end{cases}$$

and such that each line meets C at one point and D is one (different) point.

Assume that the points $\{q_i\}_{1 \le i \le t} = D \cap [\bigcup_{i=1}^t L_i]$ *(resp.* $\{p_i\}_{1 \le i \le t} = C \cap [\bigcup_{i=1}^t L_i]$*) are in general position in the plane containing D (resp. C). Then Y is a non-obstructed, linearly normal curve of degree d and arithmetic genus*

$$g = g(t,s) := \tfrac{1}{2}(d^2 - (2r+3)d + 2r^2 + t^2 + 5t - 2rt + 2s + 2).$$

Proof. Since $2 \le s \le r - t$, then by Proposition 3.2 we know that $C \cup D$ is linearly normal (reduced and non-degenerated). Here, applying Proposition 3.3 we have Y linearly normal. On the other hand, since $r \le \frac{1}{2}(d-3)$ and $r \ge s$, we then have $s \le \deg(D) - 3$, while from the hypothesis on t we deduce that $t \le \frac{1}{2}(r - t - 2)(r - t - 3)$. Thus, by Proposition 2.11, Y is non-obstructed. Moreover, $g(Y) = g(C \cup D) + g(\bigcup_{i=1}^t L_i) + 2t - 1$ (see for example [12, Proposition 4]), which yields the result. $\square$

4.1.1. Remark. For each fixed pair of integers (d, r) such that $d \ge 7$, $2 \le r \le \frac{1}{2}(d-3)$, we easily see that for every t, the function $g(t,s)$ satisfies

$$g(t-1, 2) = g(t, r-t),$$

hence, for $r \notin \{2,4,6\}$, when t ranges from 0 to $E[r/2] - 1$ and s ranges from 2 to $r - t$, the function $g(t,s)$ yields all the values of $g(Y)$ such that

$$g(0,r) \ge g(Y) \ge \begin{cases} g(\frac{1}{2}r - 1, 2) & \text{if } r \text{ is even, } r \ge 8, \\ g(\frac{1}{2}(r-1) - 1, 2) & \text{if } r \text{ is odd.} \end{cases}$$

More precisely, we have

$$\tfrac{1}{2}(d^2 - (2r+3)d + 2r^2 + 2r + 2)$$

$$\ge g(Y) \ge \begin{cases} \frac{1}{2}(d^2 - (2r+3)d + r^2 + \frac{1}{4}r^2 + \frac{7}{2}r + 2) & \text{if } r \text{ even, } r \ge 8, \\ \frac{1}{2}(d^2 - (2r+3)d + r^2 + \frac{1}{4}r^2 + 4r + \frac{3}{4}) & \text{if } r \text{ odd.} \end{cases}$$

On the other hand, $g(Y) = g(0,2) = \frac{1}{2}(d^2 - 7d + 14)$ if $r = 2$, while

$$\tfrac{1}{2}(d^2 - 11d + 42) = g(0,4) \ge g(Y) \ge g(0,2) = \tfrac{1}{2}(d^2 - 11d + 38) \quad \text{if } r = 4,$$

$$\tfrac{1}{2}(d^2 - 15d + 86) = g(0,6) \ge g(Y) \ge g(1,2) = \tfrac{1}{2}(d^2 - 15d + 72) \quad \text{if } r = 6,$$

4.2. Proposition. *Fix an integer* $d \ge 10$ *and let r be an integer such that* $4 \le r \le \frac{1}{2}(d-2)$. *Let D be a smooth plane curve of degree* $d - r$ *and let C be a* *non-singular curve of type*

$$\left(\frac{r}{2}-t,\frac{r}{2}+t\right) \quad \text{if } r \text{ is even,}$$
$$\left(\frac{r-1}{2}-t,\frac{r+1}{2}+t\right) \quad \text{if } r \text{ is odd}$$

($0 \le t \le E[r/2]-2$), on a non-singular quadric Q. Assume that C and D meet quasi-transversally at $3 \le s \le r$ points and that the plane containing D is transversal to Q. Then the nodal curve $Y = C \cup D$ is a non-obstructed, linearly normal curve of degree d and arithmetic genus

$$g = h(t,s) := \begin{cases} \frac{1}{2}(d^2-(2r+3)d+\frac{3}{2}r^2+r-2t^2+2+2s) & \text{if } r \text{ is even,}\\ \frac{1}{2}(d^2-(2r+3)d+\frac{3}{2}r^2+r-2t^2-2t+\frac{3}{2}+2s) & \text{if } r \text{ is odd.}\end{cases}$$

Proof. Since $t \le \frac{1}{2}r-2$, then $\frac{1}{2}r-t \ge 2$ and thus, by Corollary 3.6, $C \cup D$ is linearly normal ($s \ge 3$). On the other hand, from the inequalities $r \le \frac{1}{2}(d-2)$ and $s \le r$ it follows that $s \le 2\deg(D)-7 = 2(d-r)-7$. Hence Y is non-obstructed by Proposition 2.14. The arithmetic genus of g is obtained as usual. $\square$

4.2.1. Remark. For each pair of integers (d,r) such that $d \ge 10$, $\frac{1}{2}(d-2) \ge r \ge 4$, we easily see that for every t, the function $h(t,s)$ satisfies, for r even,

$$h(t-1,3) = \tfrac{1}{2}(d^2-(2r+3)d+\tfrac{3}{2}r^2+r-2t^2+4t+6),$$
$$h(t,r) = \tfrac{1}{2}(d^2-(2r+3)d+\tfrac{3}{2}r^2+3r-2t^2+2).$$

But $t \le \frac{1}{2}r-1$ implies that $2+2r \ge 4t+6$ and thus $h(t,r) \ge h(t-1,3)$. Similarly, for r odd, we get

$$h(t-1,3) = \tfrac{1}{2}(d^2-(2r+3)d+\tfrac{3}{2}r^2+r-2t^2+2t+\tfrac{15}{2}),$$
$$h(t,r) = \tfrac{1}{2}(d^2-(2r+3)d+\tfrac{3}{2}r^2+r-2t^2-2t+\tfrac{3}{2}+2r),$$

but $t \le \frac{1}{2}(r-3)$ implies that $h(t,r) \ge h(t-1,3)$. Hence, in both cases, when t ranges from 0 to $E[r/2]-2$ and s ranges from 3 to r, the function $h(t,s)$ yields all values of $g(Y)$ such that

$$h(0,r) \ge g(Y) \ge h(E[r/2]-2,3),$$

i.e.,

$$\frac{1}{2}\left(d^2-(2r+3)d+\frac{3}{2}r^2+3r+2\right)$$
$$\ge g(Y) \ge \frac{1}{2}(d^2-(2r+3)d+r^2+5r) \quad \text{if } r \text{ is even,}$$

$$\frac{1}{2}\left(d^2-(2r+3)d+\frac{3}{2}r^2+3r+\frac{3}{2}\right)$$
$$\ge g(Y) \ge \frac{1}{2}(d^2-(2r+3)d+r^2+5r) \quad \text{if } r \text{ is odd.}$$

In particular, for the cases $r = 4$ and $r = 6$, we obtain the following maximal values:

$$h(0,4) = g(Y) = \tfrac{1}{2}(d^2 - 11d + 38) \quad \text{and} \quad h(0,6) = g(Y) = \tfrac{1}{2}(d^2 - 15d + 74).$$

4.2.2 Remark. Observe that for each pair (d, r) satisfying the conditions of Theorem 4.1 and Proposition 4.2, we have

$$h(0,r) \geq g\left(\frac{r-1}{2} - 1, 2\right),$$

which means that, for each fixed pair (d, r), there is no gap between the values $g(Y)$ obtained in Theorem 4.1 and those of Proposition 4.2.

5. Non-obstructed, quadratically normal, nodal curves

We now turn to the case of nodal curves which are 2-normal. Although some of the following constructions resemble those of the preceding section, we treat them separately for the sake of clarity. There are also some subtleties which need a different approach. For example, the construction given in Proposition 4.1 cannot be generalized to obtain 2-normal curves (see Proposition 3.2), except for the following two easy cases:

5.1. Proposition. *Fix an integer $d \geq 9$ and let r be any integer such that $3 \leq r \leq \frac{1}{2}(d-3)$. Let $Y = D \cup C$ be the nodal union of two non-singular plane curves D and C of degrees $d - r$ and r, respectively, meeting transversally in s points, $3 \leq s \leq r$. Then Y is a non-obstructed, quadratically normal curve of degree d and arithmetic genus*

$$g = \tfrac{1}{2}(d^2 - (2r+3)d + 2r^2 + 2s + 2).$$

Proof. It is an immediate consequence of Proposition 3.2 and Corollary 2.9. $\square$

5.1.1. Remark. The result still holds if we replace the condition $r \leq \frac{1}{2}(d-3)$ by the weaker $r \leq \frac{1}{2}d$, as long as we assume that $s \leq d - r - 3$.

On the other hand, s ranges from 3 to r, the above construction yields, for each fixed pair $(d, r) \in \mathbb{Z}^2$ such that $d \geq 9$, $3 \leq r \leq \frac{1}{2}(d-3)$, all possible values of $g(Y)$ such that

$$\tfrac{1}{2}(d^2 - (2r+3)d + 2r^2 + 2r + 2) \geq g(Y) \geq \tfrac{1}{2}(d^2 - (2r+3)d + 2r^2 + 8).$$

5.2. Proposition. *Fix an integer $d \geq 12$ and let r be any integer such that $5 \leq r \leq \frac{1}{2}(d-2)$. Let Y be the nodal curve defined as the union*

$$Y = D \cup C \cup L,$$

where D and C are non-singular plane curves of degrees $d-r$ and $r-1$, respectively, meeting transversally in s points, $3 \leq s \leq r-1$, and L is a 2-secant meeting each of the curves C and D in one point. Then Y is a non-obstructed, quadratically normal curve of degree d and arithmetic genus

$$g = \tfrac{1}{2}(d^2 - (2r+3)d + 2r^2 - 2r + 2s + 8).$$

Proof. It follows from Proposition 3.5 (with $X_0 = L$, $X_1 = C$ and $X_2 = D$), together with Proposition 2.11 and the usual genus formula. □

5.2.1. Remark. As s ranges from 3 to $r-1$, the above construction yields, for each fixed pair $(d,r) \in \mathbb{Z}^2$ such that $d \geq 12$, $5 \leq r \leq \frac{1}{2}(d-2)$, all possible values of $g(Y)$ such that

$$\tfrac{1}{2}(d^2 - (2r+3)d + 2r^2 + 6) \geq g(Y) \geq \tfrac{1}{2}(d^2 - (2r+3)d + 2r^2 - 2r + 14).$$

Since we cannot extend to the quadratically normal case the construction given in Proposition 4.1 (we should require to adjoin 3-secants), we have the following alternative constructions:

5.3. Proposition. *Fix an integer $d \geq 15$ and let r be any integer such that $7 \leq r \leq \frac{1}{2}(d-1)$. Let Y be the nodal curve defined as the union.*

$$Y = D \cup C \cup K,$$

where

(1) D and C are non-singular plane curves of degrees $d-r$ and $r-2$, respectively, meeting transversally in s points, $3 \leq s \leq r-2$, and

(2) K is a conic meeting C (resp. D) transversally in two points.

Then Y is a non-obstructed, quadratically normal curve of degree d and arithmetic genus

$$g = \tfrac{1}{2}(d^2 - (2r+3)d + 2r^2 - 4r + 2s + 18).$$

Proof. Since $r-2 \leq d-r$, $3 \leq s \leq r-2$ and K meets $C \cup D$ at four points, then, by Proposition 3.5, we know that Y is 2-normal. On the other hand, since $7 \leq r \leq \frac{1}{2}(d-1)$ we have $r-2 \geq 5$ and $s \leq d-r-3$ (because $s \leq r-2$), so we can apply Proposition 2.12, and we deduce that Y is non-obstructed and non-smoothable. □

5.3.1. Remark. As s ranges from 3 to $r-2$, the above construction yields, for each fixed pair $(d,r) \in \mathbb{Z}^2$ such that $d \geq 15$, $7 \leq r \leq \frac{1}{2}(d-1)$, all possible values of $g(Y)$

such that

$$\tfrac{1}{2}(d^2 - (2r + 3)d + 2r^2 - 2r + 14) \geq g(Y) \geq \tfrac{1}{2}(d^2 - (2r + 3)d + 2r^2 - 4r + 24).$$

5.4. Proposition. *Fix an integer $d \geq 16$ and let r be any integer such that $8 \leq r \leq \frac{1}{2}d$. Let Y be the nodal union.*

$$Y = D \cup C \cup X$$

of three non-singular plane curves D, C and X of degrees $d - r$, $r - n$ and n, respectively, such that

(1) *We have*

$$3 \leq n \leq \begin{cases} 3 & \text{if } r = 8, \\ \frac{1}{2}r & \text{if } r \geq 9, \end{cases}$$

(2) *C and D meet transversally in s points, $3 \leq s \leq r - n$, and*
(3) *X meets C (resp. D) transversally in 2 (resp. 3) points.*

Then Y is a non-obstructed, quadratically normal curve of degree d and arithmetic genus

$$g(Y) = k(n, s) := \tfrac{1}{2}(d^2 - (2r + 3)d + 2r^2 - 2rn + 2n^2 + 2s + 12).$$

Proof. Since $r - n \leq d - r$, $3 \leq s \leq r - n$ and X meets $C \cup D$ at five points, then, by Proposition 3.5, we know that Y is 2-normal. On the other hand, the hypothesis on d, r and n implies that $r - n \geq 5$, $d - r \geq 6$ and $s \leq d - r - 3$ (since $s \leq r - n$), so we can apply Proposition 2.12, and we deduce that Y is non-obstructed and non-smoothable. $\square$

5.4.1. Remark. For each pair of integers (d, r) satisfying the hypothesis of Proposition 4.4, and for every $n \geq 4$, the function $k(n, s)$ satisfies

$$k(n - 1, 3) \leq k(n, r - n).$$

Hence, when n ranges from 3 to $\frac{1}{2}r$ and s ranges from 3 to $r - n$, the function $k(n, s)$ yields all the values of $g(Y)$ such that

$$k(3, r - 3) \geq g(Y) \geq \begin{cases} k(\frac{1}{2}r, 3) & \text{if } r \text{ is even}, \\ k(\frac{1}{2}(r - 1), 3) & \text{if } r \text{ is odd}, \end{cases}$$

i.e.,

$$\begin{aligned} &\tfrac{1}{2}(d^2 - (2r + 3)d + 2r^2 - 4r + 24) \\ &\quad \geq g(Y) \geq \begin{cases} \frac{1}{2}(d^2 - (2r + 3)d + \frac{3}{2}r^2 + 18) & \text{if } r \text{ even}, \quad r > 8, \\ \frac{1}{2}(d^2 - (2r + 3)d + \frac{3}{2}r^2 + 18 + \frac{1}{2}) & \text{if } r \text{ is odd}, \end{cases} \end{aligned}$$

while, for $r = 8$ we get

$$\tfrac{1}{2}(d^2 - 19d + 120) \geq g(Y) \geq \tfrac{1}{2}(d^2 - 19d + 116).$$

We now turn to the analogue of Proposition 4.2.

5.5. Proposition. *Fix an integer $d \geq 13$ and let r be an integer such that $6 \leq r \leq \frac{1}{2}(d-1)$. Let D be a smooth plane curve of degree $d - r$ and let C be a non-singular curve of type*

$$\left(\frac{r}{2} - t, \frac{r}{2} + t\right) \quad \text{if } r \text{ is even,}$$
$$\left(\frac{r-1}{2} - t, \frac{r+1}{2} + t\right) \quad \text{if } r \text{ is odd}$$

($0 \leq t \leq E[r/2] - 3$), on a non-singular quadric Q. Assume that C and D meet quasi-transversally in $5 \leq s \leq r$ points and that the plane containing D is transversal to Q. Then the nodal curve $Y = C \cup D$ is a non-obstructed, quadratically normal curve of degree d and arithmetic genus

$$g = h(t,s) := \begin{cases} \frac{1}{2}(d^2 - (2r+3)d + \frac{3}{2}r^2 + r - 2t^2 + 2 + 2s) & \text{if } r \text{ is even,} \\ \frac{1}{2}(d^2 - (2r+3)d + \frac{3}{2}r^2 + r - 2t^2 - 2t + \frac{3}{2} + 2s) & \text{if } r \text{ is odd.} \end{cases}$$

Proof. It is left to the reader. □

5.5.1. Remark. For each pair of integers (d,r) such that $d \geq 13$, $\frac{1}{2}(d-1) \geq r \geq 6$, we easily see that for every t, the function $h(t,s)$ satisfies, for r even,

$$h(t-1,5) = \tfrac{1}{2}(d^2 - (2r+3)d + \tfrac{3}{2}r^2 + r - 2t^2 + 4t + 10),$$

$$h(t,r) = \tfrac{1}{2}(d^2 - (2r+3)d + \tfrac{3}{2}r^2 + r - 2t^2 + 2 + 2r).$$

But $t \leq \frac{1}{2}r - 3$ implies that $2 + 2r \geq 4t + 10$ and thus $h(t,r) \geq h(t-1,5)$. Similarly, for r odd, we get

$$h(t-1,5) = \tfrac{1}{2}(d^2 - (2r+3)d + \tfrac{3}{2}r^2 + r - 2t^2 + 2t + \tfrac{23}{2}),$$

$$h(t,r) = \tfrac{1}{2}(d^2 - (2r+3)d + \tfrac{3}{2}r^2 + r - 2t^2 - 2t + \tfrac{3}{2} + 2r),$$

but $t \leq E[r/2] - 2$ implies that $h(t,r) \geq h(t-1,5)$. Hence, in both cases, when t ranges from 0 to $E[r/2] - 2$ and s ranges from 5 to r, the function $h(t,s)$ yields all values of $g(Y)$ such that

$$h(0,r) \geq g(Y) \geq h(E[r/2] - 2, 5),$$

i.e.,

$$\frac{1}{2}\left(d^2-(2r+3)d+\frac{3}{2}r^2+3r+2\right)$$
$$\geq g(Y) \geq \frac{1}{2}(d^2-(2r+3)d+r^2+5r+4) \quad \text{if } r \text{ is even},$$
$$\frac{1}{2}\left(d^2-(2r+3)d+\frac{3}{2}r^2+3r+\frac{3}{2}\right)$$
$$\geq g(Y) \geq \frac{1}{2}(d^2-(2r+3)d+r^2+5r+4) \quad \text{if } r \text{ is odd}.$$

5.5.2. Remark. We easily verify that, for each fixed pair (d, r), there is no gap between the values of g obtained from Proposition 5.4 and those obtained from Proposition 5.5.

6. Smooth components of $M = M(2; c_1, c_2, c_3)$

In this section we will show the existence of generically smooth components of the moduli scheme $M = M(2; c_1, c_2, c_3)$ which parametrizes isomorphism classes of rank 2 stable reflexive sheaves on $\boldsymbol{P}^3$ with Chern classes c_1, c_2 and c_3.

On one hand, the non-obstructed, m-normal curves constructed in Sections 4 and 5 will yield, via Serre's correspondence, rank 2 stable reflexive sheaves $\mathscr{F}$ on $\boldsymbol{P}^3$ which, by Corollary 1.7, will also be non-obstructed. On the other hand, we will apply Proposition 1.4 to obtain non-obstructed sheaves even in the case when the corresponding curve is not linearly (or quadratically normal).

First of all, recall that Theorem 0.1 determines precisely for which triples (c_1, c_2, c_3) the set

$$A_{c_1}(c_2) = \{c_3 \in \mathbb{Z} \mid M(2; c_1, c_2, c_3) \neq \emptyset\}$$

is non-empty. On the other hand, for each fixed $c_2 \geq 1$, let $B_{c_1}(c_2) \subset A_{c_1}(c_2)$ be the subset defined by

$$B_{c_1}(c_2) = \{c_3 \in A_{c_1}(c_2) \mid M(2; c_1, c_2, c_3) \text{ has a generically smooth component}\}.$$

Since every rank 2 reflexive sheaf can be normalized it is enough to consider the cases $c_1 = 0$ and $c_1 = -1$. Since both cases are analogous, we will first consider in detail the case $c_1 = -1$ and we will then sketch the proofs for the case $c_1 = 0$.

Applying Proposition 1.4, we obtain the following result:

6.1. Proposition. *For every pair of integers (c_2, c_3) such that $c_3 \in A_{-1}(c_2)$, $c_2 \geq 1$ and $0 \leq c_3 \leq 3c_2 - 2$, the moduli scheme $M(2; -1, c_2, c_3)$ has a generically smooth component of dimension $8c_2 - 5$.*

Proof. Given $c_2 \geq 1$, we will construct a non-obstructed rank 2, stable, reflexive sheaf $\mathscr{E}$ on $\boldsymbol{P}^3$ with Chern classes $c_1\mathscr{E} = -1$, $c_2\mathscr{E} = c_2$ and $c_3\mathscr{E} = c_3$, as an extension

$$\xi : 0 \longrightarrow \mathscr{O} \longrightarrow \mathscr{E}(1) \longrightarrow \mathscr{I}_Y(1) \longrightarrow 0, \quad 0 \neq \xi \in H^0\omega_Y(3),$$

where Y is a curve of degree c_2 which is the union of m mutually disjoint rational curves, $1 \leq m \leq c_2$.

By construction, $\mathscr{E}$ is a rank 2 stable reflexive sheaf on $\boldsymbol{P}^3$ with Chern classes $c_1\mathscr{E} = -1$, $c_2\mathscr{E} = c_2$ and $c_3 = 2g(Y) - 2 + 3c_2$. Since the curve Y satisfies the conditions of Proposition 1.4, we know that $\mathscr{E}$ is non-obstructed. On the other hand, $g(Y) = -(m-1)$ and thus $c_3\mathscr{E} = 3c_2 - 2m$, so for each fixed c_2, when m varies from 1 to c_2, we obtain all $c_3 \in A_{-1}(c_2)$ such that

$$c_2 \leq c_3 \leq 3c_2 - 2.$$

To obtain the values of c_3 in the range $0 \leq c_3 \leq c_2$ ($c_2 \geq 2$), we consider again the curve Y but we construct $\mathscr{E}$ as an extension

$$\xi : 0 \longrightarrow \mathscr{O} \longrightarrow \mathscr{E}(2) \longrightarrow \mathscr{I}_Y(3) \longrightarrow 0, \quad 0 \neq \xi \in H^0\omega_Y(3).$$

In this case we have $\deg(Y) = c_2 + 2$ and we apply the same arguments as before to show that $\mathscr{E}$ is a rank 2 stable reflexive sheaf which is non-obstructed. But in this case $c_2\mathscr{E} = c_2 + 2$ and $c_3\mathscr{E} = c_2 - 2(m-1)$. Thus, we obtain all possible values of $c_3 \in A_{-1}(c_2)$ such that $0 \leq c_3 < c_2$, hence the result. $\square$

The next two results deal with the upper range of values of $c_3 \in A_{-1}(c_2)$.

6.2. Theorem. *For every pair of integers* (c_2, c_3) *such that* $c_2 \geq 5$, $c_3 \in A_{-1}(c_2)$ *and*

$$c_2^2 - 2rc_2 + 2r(r+1) \geq c_3 \geq c_2^2 - 2rc_2 + r^2 + 5r - 2$$

for some integer r, $1 \leq r \leq \frac{1}{2}(c_2 - 3)$, *there exists a non-obstructed rank* 2, *stable, reflexive sheaf* $\mathscr{E}$ *on* $\boldsymbol{P}^3$ *with Chern classes* $c_1E = -1$, $c_2E = c_2$ *and* $c_3E = c_3$.

Proof. The cases $r = 1, 2$ follow from [13, Theorem 3.2]. Assume $r \geq 3$. We will consider two different constructions:

Construction 1: We take a curve $Y = C \cup D \cup L_1 \cup \cdots \cup L_t$ where C and D are non-singular plane curves of degrees $r - t$ and $c_2 - r$, respectively, meeting transversally in a set of s points, $2 \leq s \leq r - t$, and $L_1, L_2, \ldots, L_t$ are t skew lines with $t = 0$ if $r = 4$ and

$$0 \leq t \leq \begin{cases} 1, & r = 6, \\ (r/2) - 1, & r \equiv 0 \,(\mathrm{mod}\, 2),\ r \geq 8, \\ ((r-1)/2) - 1, & r \equiv 1 \,(\mathrm{mod}\, 2), \end{cases}$$

each line meeting C in one point and D in another point. Assume also that the points $D \cap [\bigcup_{i=1}^t L_i]$ are in general position in the plane containing D, and that the points

$C \cap [\bigcup_{i=1}^{t} L_i]$ are in general position in the plane containing C. By Theorem 4.1, we know that Y is a non-obstructed, linearly normal curve of degree $d = c_2$ and arithmetic genus

$$g(Y) = \tfrac{1}{2}(c_2^2 - (2r+3)c_2 + 2r^2 + t^2 + 5t - 2rt + 2s + 2).$$

It is immediate to see that, for $c_2 - r \geq 0$, $r - t \geq 0$, there exists a non-zero global section $\xi \in H^0\omega_Y(3)$ generating the sheaf $\omega_Y(3)$ except at a finite number of points. We thus have an extension

$$\xi : 0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}(1) \longrightarrow \mathcal{I}_Y(1) \longrightarrow 0,$$

where $\mathcal{E}$ is a rank 2 reflexive sheaf with Chern classes $c_1\mathcal{E} = -1$, $c_2\mathcal{E} = c_2(\mathcal{E}(1)) = d = c_2$ and $c_3\mathcal{E} = 2g(Y) - 2 + 3d$ (see 1.3). We thus have

$$c_3\mathcal{E} = c_2^2 - 2rc_2 + 2r^2 + t^2 + 5t - 2rt + 2s$$

so for each fixed pair (c_2, r), when t ranges from 0 to $E[r/2] - 1$, and s ranges from 2 to $r - t$, we obtain all the possible values of c_3 within the range

$$c_2^2 - 2rc_2 + 2r(r+1)$$
$$\geq c_3 \geq \begin{cases} c_2^2 - 2rc_2 + r^2 + \frac{1}{4}r^2 + \frac{7}{2}r, & r \equiv 0 \pmod 2,\ r \geq 8, \\ c_2^2 - 2rc_2 + r^2 + \frac{1}{4}r^2 + 4r - \frac{5}{4}, & r \equiv 1 \pmod 2, \end{cases}$$

while for the cases $r = 4$ and $r = 6$ we have

$$c_2^2 - 8c_2 + 40 \geq c_3 \geq c_2^2 - 8c_2 + 36 \quad \text{if } r = 4,$$

$$c_2^2 - 12c_2 + 84 \geq c_3 \geq c_2^2 - 12c_2 + 70 \quad \text{if } r = 6$$

(see Remark 4.1.1). Moreover, Y is not contained in any plane, hence $\mathcal{E}$ is stable. Finally, since Y is non-obstructed and linearly normal then $\mathcal{E}$ is non-obstructed by Corollary 1.7.

Construction 2: Assume $r \geq 4$ (and thus $c_2 \geq 11$). We take a curve $Y = C \cup D$ where D is a smooth plane curve of degree $c_2 - r$ and C is a non-singular curve of type

$$\left(\frac{r}{2} - t, \frac{r}{2} + t\right) \qquad \text{if } r \text{ is even},$$

$$\left(\frac{r-1}{2} - t, \frac{r+1}{2} + t\right) \quad \text{if } r \text{ is odd}$$

($0 \leq t \leq E[r/2] - 2$), on a non-singular quadric Q, such that C and D meet quasi-transversally at s points, $3 \leq s \leq r$, and the plane containing D is transversal to Q. Then, by Proposition 4.2, Y is a non-obstructed linearly normal curve of degree $d = c_2$ and genus

$$g = \begin{cases} \frac{1}{2}(c_2^2 - (2r+3)c_2 + \frac{3}{2}r^2 + r - 2t^2 + 2 + 2s) & \text{if } r \equiv 0 \pmod 2, \\ \frac{1}{2}(c_2^2 - (2r+3)c_2 + \frac{3}{2}r^2 + r - 2t^2 + \frac{3}{2} + 2s) & \text{if } r \equiv 1 \pmod 2. \end{cases}$$

For $c_2 - r \geq 0$ and $r - t \geq 0$ we know there is a non-zero global section $\xi \in H^0\omega_Y(3)$ generating the sheaf $\omega_Y(3)$ except at a finite number of points. We thus have an extension

$$\xi: 0 \longrightarrow \mathcal{O} \longrightarrow \mathscr{E}(1) \longrightarrow \mathscr{I}_Y(1) \longrightarrow 0,$$

where $\mathscr{E}$ is a rank 2 reflexive sheaf with Chern classes $c_1\mathscr{E} = -1$, $c_2\mathscr{E} = c_2(\mathscr{E}(1)) = d = c_2$ and $c_3\mathscr{E} = 2g(Y) - 2 + 3d$ (see 1.3). We thus have

$$c_3\mathscr{E} = \begin{cases} c_2^2 - 2rc_2 + \frac{3}{2}r^2 + r - 2t^2 + 2s & \text{if } r \equiv 0 \pmod 2, \\ c_2^2 - 2rc_2 + \frac{3}{2}r^2 + r - 2t^2 - 2t + 2s - \frac{1}{2} & \text{if } r \equiv 1 \pmod 2) \end{cases}$$

so for each fixed pair (c_2, r), when t ranges from 0 to $E[r/2] - 2$, and s ranges from 3 to r, we obtain all the possible values of c_3 within the range

$$c_2^2 - 2rc_2 + \tfrac{3}{2}r^2 + 3r \geq c_3 \geq c_2^2 - 2rc_2 + r^2 + 5r - 2, \quad r \equiv 0 \pmod 2),$$

$$c_2^2 - 2rc_2 + \tfrac{3}{2}r^2 + 3r - \tfrac{1}{2} \geq c_3 \geq c_2^2 - 2rc_2 + r^2 + 5r - 2, \quad r \equiv 1 \pmod 2)$$

(see Remark 4.2.1). Moreover, since Y is not contained in any plane, $\mathscr{E}$ is stable and the result follows from Corollary 1.7. □

In [9], Kleppe has shown that, for every pair of integers (d, g) such that the Hilbert scheme $H(d, g)$ of smooth connected curves in $\boldsymbol{P}^3$ is non-empty, there exists a generically smooth component of $H(d, g)$. As a consequence, we obtain the following result:

6.3. Proposition. *For every pair of integers (c_2, c_3) such that $c_2 \geq 6$, $c_3 \in A_{-1}(c_2)$ and*

$$\tfrac{1}{3}c_2^2 + 2c_2 \geq c_3 \geq 5c_2 - 6,$$

there exists a non-obstructed rank 2, stable, reflexive sheaf $\mathscr{E}$ on $\boldsymbol{P}^3$ with Chern classes $c_1E = -1$, $c_2E = c_2$ and $c_3E = c_3$.

Proof. Consider a linearly normal, smooth, non-obstructed curve Y of degree $d \geq 6$ and arithmetic genus g,

$$d - 2 \leq g \leq 1 + \frac{d(d-3)}{6}$$

(for the existence of such curve see [9]). We have an extension

$$\xi: 0 \longrightarrow \mathcal{O} \longrightarrow \mathscr{E}(1) \longrightarrow \mathscr{I}_Y(1) \longrightarrow 0,$$

where $\mathscr{E}$ is a rank 2, stable, reflexive sheaf on $\boldsymbol{P}^3$ which is non-obstructed (Corollary 1.7) and has Chern classes $c_1\mathscr{E} = -1$, $c_2\mathscr{E} = c_2$ and $c_3\mathscr{E} = 2g - 2 + 3c_2$, hence the result. □

We now turn to the case $c_1 = 0$.

6.4. Proposition. *For every pair of integers (c_2, c_3) such that $c_3 \in A_0(c_2)$, $c_2 \geq 1$ and*

$$0 \leq c_3 \leq 2c_2,$$

the moduli scheme $M(2; 0, c_2, c_3)$ has a generically smooth component of dimension $8c_2 - 3$.

Proof. Given $c_2 \geq 1$, we construct a non-obstructed rank 2, stable, reflexive sheaf $\mathscr{E}$ on $\boldsymbol{P}^3$ with Chern classes $c_1E = 0$, $c_2E = c_2$ and $c_3E = c_3$, as an extension

$$\xi : 0 \longrightarrow \mathcal{O} \longrightarrow \mathscr{E}(1) \longrightarrow \mathscr{I}_Y(2) \longrightarrow 0, \quad 0 \neq \xi \in H^0\omega_Y(2),$$

where Y is a curve of degree $c_2 + 1$ which is the union of m mutually disjoint smooth rational curves, $1 \leq m \leq c_2 + 1$. By construction, $\mathscr{E}$ is a rank 2 stable reflexive sheaf on $\boldsymbol{P}^3$ with Chern classes $c_1\mathscr{E} = 0$, $c_2\mathscr{E} = c_2$ and $c_3 = 2c_2 - 2(m-1)$. The result then follows from Proposition 1.4. □

6.5. Proposition. *For every pair of integers (c_2, c_3) such that $c_2 \geq 8$, $c_3 \in A_0(c_2)$ and*

$$c_2^2 - (2r-1)c_2 + 2r^2 \geq c_3 \geq c_2^2 - (2r-1)c_2 + 2r^2 - 2r + 6$$

for some integer r, $3 \leq r \leq \frac{1}{2}(c_2 - 2)$, there exists a non-obstructed rank 2, stable, reflexive sheaf $\mathscr{E}$ on $\boldsymbol{P}^3$ with Chern classes $c_1E = 0$, $c_2E = c_2$ and $c_3E = c_3$.

Proof. It is left to the reader. □

6.6. Proposition. *For every pair of integers (c_2, c_3) such that $c_2 \geq 11$, $c_3 \in A_0(c_2)$ and*

$$c_2^2 - (2r-1)c_2 + 2r^2 - 2r + 4 \geq c_3 \geq c_2^2 - (2r-1)c_2 + 2r^2 - 4r + 12$$

for some integer r, $5 \leq r \leq \frac{1}{2}(c_2 - 1)$, there exists a non-obstructed rank 2, stable, reflexive sheaf $\mathscr{E}$ on $\boldsymbol{P}^3$ with Chern classes $c_1E = 0$, $c_2E = c_2$ and $c_3E = c_3$.

Proof. We take a curve $Y = D \cup C \cup L$ which is the nodal union of two non-singular plane curves D and C of degrees $c_2 + 1 - r$ and $r - 1$, respectively, meeting transversally at s points, $3 \leq s \leq r - 1$, and a 2-secant line L meeting each of the curves D and C in one point. We consider an extension

$$\xi : 0 \longrightarrow \mathcal{O} \longrightarrow \mathscr{E}(1) \longrightarrow \mathscr{I}_Y(2) \longrightarrow 0,$$

where $\mathscr{E}$ is a rank 2, stable, reflexive sheaf on $\boldsymbol{P}^3$ with Chern classes $c_1\mathscr{E} = 0$, $c_2\mathscr{E} = c_2(\mathscr{E}(1)) - 1 = d - 1 = c_2$ and $c_3\mathscr{E} = 2g(Y) - 2 + 2d$. The result then follows from Proposition 5.2, Remark 5.2.1 and Corollary 1.7. □

6.7. Proposition. *For every pair of integers (c_2, c_3) such that $c_2 \geq 14$, $c_3 \in A_0(c_2)$ and*

$$c_2^2 - (2r-1)c_2 + 2r^2 - 4r + 12 \geq c_3 \geq c_2^2 - (2r-1)c_2 + 2r^2 - 6r + 22$$

for some integer r, $7 \le r \le \frac{1}{2}c_2$, there exists a non-obstructed rank 2, stable, reflexive sheaf $\mathscr{E}$ on $\boldsymbol{P}^3$ with Chern classes $c_1E = 0$, $c_2E = c_2$ and $c_3E = c_3$.

Proof. We take a nodal curve $Y = D \cup C \cup K$ where D and C are two non-singular plane curves of degrees $c_2 + 1 - r$ and $r - 2$, respectively, meeting transversally at s points, $3 \le s \le r - 2$, and K is a conic meeting the curves D (resp. C) transversally in two points. We consider an extension

$$\xi : 0 \longrightarrow \mathcal{O} \longrightarrow \mathscr{E}(1) \longrightarrow \mathscr{I}_Y(2) \longrightarrow 0,$$

where $\mathscr{E}$ is a rank 2, stable, reflexive sheaf on $\boldsymbol{P}^3$ with Chern classes $c_1\mathscr{E} = 0$, $c_2\mathscr{E} = c_2(\mathscr{E}(1)) - 1 = d - 1 = c_2$ and $c_3\mathscr{E} = 2g(Y) - 2 + 2d$. The result then follows from Proposition 5.3, Remark 5.3.1 and Corollary 1.7. $\square$

6.8. Theorem. *For every pair of integers (c_2, c_3) such that $c_2 \ge 15$, $c_3 \in A_0(c_2)$ and*

$$c_2^2 - (2r - 1)c_2 + 2r^2 - 6r + 22 \ge c_3 \ge c_2^2 - (2r - 1)c_2 + r^2 + 3r + 2$$

for some integer r, $8 \le r \le \frac{1}{2}(c_2 + 1)$, there exists a non-obstructed rank 2, stable, reflexive sheaf $\mathscr{E}$ on $\boldsymbol{P}^3$ with Chern classes $c_1E = 0$, $c_2E = c_2$ and $c_3E = c_3$.

Proof. We will use two different constructions:

Construction 1: We take a curve $Y = D \cup C \cup X$ which is the nodal union of three non-singular plane curves D, C and X of degrees $c_2 + 1 - r$, $r - n$ and n, respectively, and such that

(1) We have

$$3 \le n \le \begin{cases} 3, & r = 8, \\ \frac{1}{2}r, & r \ge 9, \end{cases}$$

(2) C and D meet transversally in s points, $3 \le s \le r - n$,

(3) X meets C (resp. D) transversally in 2 (resp. 3) points.

We consider an extension

$$\xi : 0 \longrightarrow \mathcal{O} \longrightarrow \mathscr{E}(1) \longrightarrow \mathscr{I}_Y(2) \longrightarrow 0,$$

where $\mathscr{E}$ is a rank 2, stable, reflexive sheaf on $\boldsymbol{P}^3$ with Chern classes $c_1\mathscr{E} = 0$, $c_2\mathscr{E} = c_2(\mathscr{E}(1)) - 1 = d - 1 = c_2$ and $c_3\mathscr{E} = 2g(Y) - 2 + 2d$. Applying Proposition 5.4 and Remark 5.4.1 we obtain all values of $c_3 \in A_0(c_2)$ such that

$$c_2^2 - 15c_2 + 102 \ge c_3 \ge c_2^2 - 15c_2 + 98 \quad \text{if } r = 8$$

and

$$c_2^2 - (2r - 1)c_2 + 2r^2 - 6r + 22$$
$$\ge c_3 \ge \begin{cases} c_2^2 - (2r - 1)c_2 + \frac{3}{2}r^2 - 2r + 16 & \text{if } r \text{ is even, } r \ge 8, \\ c_2^2 - (2r - 1)c_2 + \frac{3}{2}r^2 + 16 + \frac{1}{2} & \text{if } r \text{ is odd.} \end{cases}$$

Construction 2: We take a curve $Y = C \cup D$ where D is a smooth plane curve of degree $c_2 - r$ and C is a non-singular curve of type

$$\left(\frac{r}{2} - t, \frac{r}{2} + t\right) \quad \text{if } r \text{ is even,}$$

$$\left(\frac{r-1}{2} - t, \frac{r+1}{2} + t\right) \quad \text{if } r \text{ is odd}$$

($0 \le t \le E[r/2] - 3$), on a non-singular quadric Q, such that C and D meet quasi-transversally at s points, $5 \le s \le r$, and the plane containing D is transversal to Q. We thus have an extension

$$\xi : 0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}(1) \longrightarrow \mathcal{I}_Y(1) \longrightarrow 0,$$

where $\mathcal{E}$ is a rank 2 reflexive sheaf on $\boldsymbol{P}^3$ with Chern classes $c_1\mathcal{E} = 0$, $c_2(\mathcal{E}(1)) = d = c_2 + 1$ and $c_3\mathcal{E} = 2g(Y) - 2 + 2d$. The result then follows from Propositions 5.4 and 5.5, together with Remark 5.5.1 and Corollary 1.7. □

6.9. Proposition. *For every pair of integers* (c_2, c_3) *such that* $c_3 \in A_0(c_2)$, $c_2 \ge 12$ *and*

$$6c_2 - 4 \le c_3 \le \tfrac{1}{3}(c_2^2 + 5c_2 + 4),$$

there exists a non-obstructed rank 2, stable, reflexive sheaf $\mathcal{E}$ *on* $\boldsymbol{P}^3$ *with Chern classes* $c_1E = 0$, $c_2E = c_2$ *and* $c_3E = c_3$.

Proof. Analogous to that of Proposition 6.3. □

7. Conclusions and examples

The results of the preceding section show that for "almost all" the values of c_3, the moduli scheme $M(2; c_1, c_2, c_3)$ has a generically smooth component. The purpose of the next theorem is to summarize the results obtained in order to make precise what we mean by "almost all".

7.1. Theorem. *Fix an integer* c_1 *and consider the sets*

$$A_{c_1}(c_2) = \{c_3 \in \mathbb{Z} \mid M(2; c_1, c_2, c_3) \ne \emptyset\},$$

$$B_{c_1}(c_2) = \{c_3 \in A_{c_1}(c_2) \mid M(2; c_1, c_2, c_3) \textit{ has a generically smooth component}\}.$$

We then have

$$\lim_{c_2 \to \infty} \frac{\# B_{c_1}(c_2)}{\# A_{c_1}(c_2)} = 1.$$

Proof. Since every rank 2 stable reflexive sheaf $\mathscr{F}$ in $M(2; c_1, c_2, c_3)$ can be normalized, it is enough to consider the cases $c_1 = 0$ and $c_1 = -1$. Assume that $c_1 = -1$. We must show that

$$\lim_{c_2 \to \infty} \frac{\# A_{-1}(c_2) - \# B_{-1}(c_2)}{\# A_{-1}(c_2)} = 0.$$

From Theorem 0.1 it follows that

$$\# A_{-1}(c_2) = \begin{cases} \frac{1}{2} c_2^2 + 1 - \sum_{r=1}^{b(-1,c_2)} (c_2 - r(r+1) - 1) & \text{if } c_2 \text{ is even}, \\ \frac{1}{2}(c_2^2 - 1) + 1 - \sum_{r=1}^{b(-1,c_2)} (c_2 - r(r+1) - 1) & \text{if } c_2 \text{ is odd}, \end{cases} \tag{14}$$

where $b = b(-1, c_2) = E[\frac{1}{2}(-1 + \sqrt{4c_2 - 7})]$. A simple calculation yields

$$\sum_{r=1}^{b(-1,c_2)} (c_2 - r(r+1) - 1) = -\frac{1}{3} b^3 - b^2 + \left(c_2 - \frac{5}{3}\right) b.$$

We now look at the size of the set $A_{-1}(c_2) \backslash B_{-1}(c_2)$. Since, for $c_2 \geq 18$, the constructions given in Theorem 6.2 and Proposition 6.3 overlap, then for these values we have

$$\# A_{-1}(c_2) - \# B_{-1}(c_2) \leq \sum_{r=1}^{d(-1,c_2)} \frac{1}{2}(r^2 + 5r - 2) + 2c_2 - 2,$$

where $d = d(-1, c_2) = E[\frac{1}{2}(-1 + \sqrt{8c_2 - 31})]$ is the maximum value of r for which the inequality

$$c_2^2 - 2(r+1)c_2 + 2(r+1)(r+2) + 2 \geq c_2^2 - 2rc_2 + r^2 + 5r - 2$$

holds. A simple calculation yields

$$\sum_{r=1}^{d(-1,c_2)} \frac{1}{2}(r^2 + 5r - 2) = \frac{1}{6} d(d^2 + 9d + 2).$$

Hence,

$$\# A_{-1}(c_2) - \# B_{-1}(c_2) \leq \tfrac{1}{6} d(d^2 + 9d + 2) + 2c_2 - 2. \tag{15}$$

A comparison of (14) and (15) yields the result. The proof for the case $c_1 = 0$ is completely analogous (it follows from Propositions 6.4–6.7, 6.9 and Theorem 6.8), so it is left to the reader. □

7.2. Remark. In general, for each fixed pair (c_1, c_2), there are values of c_3 for which we still do not know if there exists a generically smooth component of the moduli scheme.

However, if we let $C_{c_1}(c_2) := A_{c_1}(c_2) \backslash B_{c_1}(c_2)$ be the set of integers $c_3 \in A_{c_1}(c_2)$ for which the moduli scheme $M(2; c_1, c_2, c_3)$ has no generically smooth component, then from the above theorem it follows that, while the sets $A_{c_1}(c_2)$ and $B_{c_1}(c_2)$ are of order $\mathrm{o}(\frac{1}{2}c_2^2)$, the set $C_{c_1}(c_2)$ is at most of order $\mathrm{o}(\frac{2}{3}c_2^{3/2})$.

As an example, in the following corollary we compare $\sharp A_{c_1}(c_2)$ with the maximal size of the sets $C_{c_1}(c_2)$, for $c_1 = -1$ and $c_2 \leq 10$.

7.3. Corollary. (a) $C_{-1}(c_2) = \emptyset$ *for all* $c_2 \leq 4$.

(b) *We have*:

$$\sharp A_{-1}(5) = 11, \qquad C_{-1}(5) \subseteq \{15, 17\},$$

$$\sharp A_{-1}(6) = 16, \qquad C_{-1}(6) \subseteq \{18, 20, 22, 26\},$$

$$\sharp A_{-1}(7) = 21, \qquad C_{-1}(7) \subseteq \{21, 23, 25, 27, 31, 35, 37\},$$

$$\sharp A_{-1}(8) = 27, \qquad C_{-1}(8) \subseteq \{24, 26, 28, 30, 32, 38, 40, 42, 48, 50\},$$

$$\sharp A_{-1}(9) = 33, \qquad C_{-1}(9) \subseteq \{27, 29, 31, 33, 35, 37, 47, 53, 55, 63, 65\},$$

$$\sharp A_{-1}(10) = 41, \qquad C_{-1}(10) \subseteq \{30, 32, 34, 36, 38, 40, 42, 54, 56, 58, 60, 66, 68, 70, 80, 82\}.$$

Proof. Part (a) follows from [2, Theorem 2.5; 6, Theorem 9.2; 13, Theorem 3.2], together with Proposition 6.1, while (b) follows from Propositions 6.1, 6.3 and Theorem 6.2. □

We wonder if $C_0(c_2) = C_{-1}(c_2) = \emptyset$ for all $c_2 \geq 1$. To be more precise, we suggest the following conjecture:

7.4. Conjecture. Fix an integer c_1. Then $A_{c_1}(c_2) = B_{c_1}(c_2)$ for every $c_2 \in \mathbb{Z}$.

7.5. Remark. An affirmative answer to Conjecture 7.4 will not imply that $M(2; c_1, c_2, c_3)$ is generically smooth. In fact, there exist integers $(c_1, c_2, c_3) \in \mathbb{Z}^3$ for which $M(2; c_1, c_2, c_3)$ has a generically smooth component and a non-reduced component. For example, by Proposition 1.4, $M(2; -1, 14, 88)$ has a generically smooth component while, by [8; Example 3.2], it also has a non-reduced component.

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