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A Theorem on Three-Coloring the Edges of a Trivalent Graph*

GEORGE J. MINTY

Indiana University
and
Università di Pisa

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1. INTRODUCTION

By "graph" we mean a *finite undirected* graph; its elements are called "edges" and "vertices." We permit "multiple edges" (two or more edges joining the same pair of vertices) but forbid loops (edges associated with only one vertex). When we say "delete an edge" we do *not* delete the associated vertices. A *circuit* is a set of edges forming a simple closed curve; a *cocircuit* is a minimal separating set of edges. A graph is *trivalent* if each vertex has precisely three edges incident on it. The edges are *n-colored* if they are partioned into n sets so that any two edges incident on the same vertex are in different sets.

In a digraph, the edges of each circuit are partitioned by the orientation into two sets; letting m and n be the cardinalities of these sets, the *flow ratio* of the circuit is the ratio m/n (with $m \geq n$); it may be $+\infty$. It is shown in [5] that the vertices of a graph are n-colorable if and only if there exists an orientation of the graph such that the flow ratio of each circuit does not exceed $(n - 1)$. The flow ratio of a cocircuit is defined in an obvious analogous way, the partitioning into two subsets being equivalent to a notion of "similarly directed" and "oppositely directed" as discussed in [3].

Now, it was conjectured by Tait (see [2]) that the edges of any trivalent graph are 3-colorable; the Petersen graph gives a counterexample

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(see [2] again). It is not known whether Tait's conjecture is true in the special case of planar graphs, and in fact it is then equivalent to the Four-Color Conjecture. However, for planar trivalent graphs, one can assert that *the edges are 3-colorable if and only if there exists an orientation such that the flow ratio of each cocircuit does not exceed* 3. The proof is via the dual-graph, using the theorem of [5].

The object of this paper is to extend the above result to non-planar graphs.

The symbols J, C_4, D_2 stand for the (Abelian) groups: the integers, the cyclic group of 4 elements, the Klein four-group, respectively.

2. The Theorem

The following conditions on a trivalent (finite, undirected, not necessarily planar) graph are all equivalent:

(a) *There exists a* 3*-coloring of the edges.*

(b) *There exists a non-vanishing* 1*-cycle with values in* D_2.

(c) *There exists an orientation and a non-vanishing* 1*-cycle with values in* C_4 *on the diagraph.*

(d) *There exists an orientation and a* 1*-cycle with values in* J *on the digraph, such that only the values* $1, 2, 3 \in J$ *are assumed by the* 1*-cycle.*

(e) *There exists an orientation such that the flow ratio of each cocircuit does not exceed* 3.

Remarks: *The main point of this theorem is the equivalence of* (a) *and* (e). *The equivalence of* (a), (b), *and* (c) *is shown by Tutte in* [7], *corollary to Theorem XI, and the equivalence of* (c) *and* (d) *is shown by the same author in* [8], *Theorem* 6.3. *The settings are considerably more abstract and general, and the proofs somewhat less constructive than given here.*

Proof: The equivalence of (a) and (b) is easily seen by identifying the three "colors" with the three non-zero elements of D_2.

Let us show (a) implies (c). Assume a 3-coloring of the edges: say with red, green, and blue. Assign to each green line the element 2 of C_4 and orient the line arbitrarily. Deletion of the green lines leaves a system of circuits; for each such circuit, orient the lines "all in the same direction" relative to the circuit. Assign to the red lines and blue lines the elements 1 and 3 of C_4, respectively; the remainder of the argument is trivial verification.

Let us show (c) implies (d). Assume an orientation and a non vanishing 1-cycle with coefficients in C_4; let g be the obvious associated 1-chain with coefficients in J. The boundary ∂g of g is a 0-chain with coefficients in the set

$$\{... -8, -4, 0, +4, +8, ...\} \subset J;$$

it is easy to show that the sum of these coefficients over all the vertices is zero. If g is a 1-cycle, no further reduction is necessary. If not, choose a vertex v such that ∂g is negative on v, and let S be the set of vertices (including v) "accessible" from v, with the edges thought of as "one-way streets," the direction of permitted travel being given by the "arrow" of the orientation, and let $\bar{S}$ be the remaining vertices. Let E be the set of edges connecting a vertex of $\bar{S}$ with a vertex of S. It is not hard to show that the sum of the coecients associated with E by g is equal to the sum of the coefficients of ∂g associated with the vertices of S. Since the first sum is non-negative, so is the second, and there is a vertex v' of S with positive coefficient, and an oriented path from v to v'. Reverse the orientations of all edges of the path, replace coefficients 1 by 3 and 3 by 1 on all edges of the path (leaving the 2's unchanged) and we now have a non-vanishing 1-cycle with coefficients in C_4, which is "closer to being" a 1-cycle with coefficients in J, in the sense that: regarding it as a 1-chain with coefficients in J, the sum of the absolute values of the coefficients of its boundary is smaller than before the reduction. Finitely many iterations of the above reduction process eventually produce the desired orientation and non-vanishing 1-cycle with coefficients in J, taking on only the values 1, 2, 3.

Now let us show (d) implies (a). Given a 1-cycle h with coefficients in J and taking on only the values 1, 2, 3: let E_1 be the set of edges with coefficient 2. By listing all possibilities, one shows easily that each vertex has precisely one edge of E_1 incident on it; thus deletion of E_1 leaves a system of circuits. For each such circuit C, let $E_1(C)$ be the set of lines of E_1 incident on it. Now, the sum of the entries of h over $E_1(C)$ is zero (mod 4); thus C has an even number of vertices and hence an even number of edges. Color the edges of each C alternately red and blue, and color the edges of E_1 green.

We turn now to the equivalence of (d) and (e). For an *interval* $I \subset J$, define $(-I)$ as $\{x: -x \in I\}$, and for two intervals $I_1, I_2 \subset J$, define $I_1 + I_2$ as $\{x + y: x \in I_1, y \in I_2\}$. Recall the theorem ([3], Theorem 4.1): *Let there be given a function which assigns to each edge of a digraph,* *an interval in J. If, for each cocircuit, the sum of the intervals assigned to the edges of the cocircuits (signed + and − according to the directions of the lines) contains zero, then there exists a 1-cycle with coefficients in J such that the integer assigned to each edge lies in the corresponding interval. (This theorem is really "if and only if," the "only if" part being essentially trivial.)*

To apply this theorem, think of the intervals in J as being all $\{1, 2, 3\}$, and note that the hypothesis of the above theorem is equivalent to the condition that the flow ratio of each cocircuit does not exceed 3.

REMARK: The tool theorem adduced above is a variant on the "integer form" of the Max-Flow-Min-Cut Theorem of Ford and Fulkerson. An explicit algorithm for the construction is given in [4]. There is a connection with the method of Grundy functions [1], [6] which will not be explored here.

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