A Priori Bounds for Solutions of the Dirichlet Problem for
\[ [\Delta + \lambda^2 n(x)] u = f(x, \lambda) \text{ on an Exterior Domain} \]

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1. INTRODUCTION

In this paper we obtain a priori estimates for the solution \( u(x, \lambda) \) of the radiation problem
\[
Lu = f(x, \lambda) \quad (x \in V \subseteq \mathbb{R}^m, m = 2, 3; \lambda > 0),
\]
\[
u |_{\partial V} = u_0(x) \quad (x \in \partial V),
\]
\[
\lim_{R \to \infty} \int_{r=R}^{r=\infty} r \left| \partial_1 u \right|^2 = 0 \quad (r = (x \cdot x)^{1/2}),
\]
where
\[
Lu = \Delta u + \lambda^2 n(x)u,
\]
and
\[
\partial_1 u = u_r - i\lambda u + [(m - 1)/2r]u \quad (m = 2, 3).
\]

If \( m = 2(3) \), then \( V \) is the exterior of a smooth closed curve (surface) \( \partial V \). We assume that \( \partial V \) can be illuminated from the exterior by a convex curve (surface) contained in \( V \) (see Definition 2.1 of Section 2).

We require:

H(i) \( u_0(x) \in C^1(\partial V) \),
H(ii) \( n(x) \in C^2(V) \),
H(iii) \( f(x, \lambda) \in C(V) \) for every \( \lambda > 0 \),
H(iv) \( \int r^2 |f|^2 < \infty \), and
H(v) \( n(x) \geq n_0 > 0 \) for all \( x \in \bar{V} \).

Here \( \bar{V} = V \cup \partial V \). In addition we require that as \( r \to \infty \):

H(vi) \( \left[ \frac{1}{n(x)} \right] - 1 = O(r^{-p}) \) for some \( p > 2 \),
H(vii) \( \nabla n(x) = O(r^{-2}) \),
H(viii) \( \partial^{i+j}n(x)/\partial x^i \partial x^j = O(r^{-3}) \) \( (i + j = 2, i, j \geq 1) \).
Most of this paper is devoted to obtaining estimates for the "energy norms"

\[ \| u_r \|_{\partial V} = \left\{ \int_{\partial V} |\nu^* \cdot \nabla u|^2 \right\}^{1/2}, \]

where \( \nu^* \) is the exterior unit normal to \( \partial V \),

\[ \| \nabla u/r \|_{V} = \left[ \int_V (|\nabla u|^2/r^2) \right]^{1/2}, \]

and

\[ \| u/r \|_{V} = \left[ \int_V (|u|^2/r^2) \right]^{1/2} \]

in terms of the given boundary data \( u_0 \) and the source term \( f \). We show how to compute the constants involved in these a priori estimates explicitly in terms of \( \partial V \), \( n \), \( f \), and \( u_0 \). The estimates hold as \( \lambda \to \infty \). We use these estimates to derive an a priori estimate for the scalar field strength \( |u(x, \lambda)| \) in terms of \( f \) and \( u_0 \) that holds uniformly on \( \overline{V} \) as \( \lambda \to \infty \).

The function \( u e^{-i\lambda t} \) is the time harmonic solution of the wave equation

\[ \Delta u - n(x) \frac{\partial u}{\partial t} = f(x)e^{-i\lambda t} \quad ((x, t) \in V \times [0, \infty)), \]

that satisfies the boundary condition

\[ u \big|_{\partial V} = g(x)e^{-i\lambda t} \quad ((x, t) \in \partial V \times (0, \infty)), \]

and the radiation condition (1.3) for \( t > 0 \).

The estimates we obtain for the \( L^2 \) norms of \( u/r \) and \( \nabla u/r \) also imply an upper bound on the local energy \( E_{R_0}(ue^{-i\lambda t}) \) of the function \( u e^{-i\lambda t} \) that is contained in the region between the boundary \( \partial V \) of the scattering obstacle and a sphere of radius \( R_0 \). This local energy is shown to be bounded from above by a constant multiple of the sum of the total energy \( \lambda^2(\|u_0\|_{\partial V})^2 + (\|u_{r*}\|_{\partial V})^2 \) of the boundary data \( u_0 e^{-i\lambda t} \) and the square of the \( L^2 \) norm of \( e^{-i\lambda t}f \) (see Corollary 7.2). Here

\[ \| u_{r*} \|_{\partial V}^2 = \int_{\partial V} | \nabla u_0 - (\nu^* \cdot \nabla u_0) \nu^* |^2. \]

In a sequel to this paper we apply our estimate for \( |u(x, \lambda)| \) to the Ursell radiating body problem [15] (Problem U):

Let \( u(x, \lambda) \) be the solution of Eq. (1.1) subject to the radiation condition (1.3), and the boundary condition

\[ \alpha(x)(\nu^* \cdot \nabla u) + \beta(x)u = g(x) \quad (x \in \partial V). \]

(1.4)

Construct an asymptotic approximation \( u_M(x, \lambda) \) of \( u(x, \lambda) \) such that as \( \lambda \to \infty \)

\[ u(x, \lambda) - u_M(x, \lambda) = O(\lambda^{-M-\frac{1}{2}(1+m)}r^{-(m-1)/2}) \quad (M > \frac{1}{2}(1 + m)) \]

(1.5)
uniformly in $x$ ($x \in V$). We shall solve Problem U for a general class of scattering obstacles in the case $\alpha = 0$, $\beta = 1$, $g = u_0$ under physically reasonable hypotheses on (i) the smoothness of $\partial V$, $u_0$, $n$, and $f$, and (ii) the behavior as $r \to \infty$ of $f$, $n$, and derivatives of these functions. The approximations we construct satisfy (1.5) for positive integer values of $M$.

In the case $n(x) \equiv 1$ we shall require that the subset of $\partial V$ contained in the support of the radiating sources $f$ and $g$ consist of a finite number of smooth locally convex patches $S_i$ ($i = 1, 2, \ldots, K$) joined together so that $\partial V$ is smooth and can be illuminated from the exterior. In addition we shall impose the condition that (i) each straight line (ray) emanating orthogonally from the patch $S_i$ extends to infinity without intersecting $\partial V$. The convexity of each $S_i$ ensures that every distinct pair of straight lines (rays) emanating orthogonally from the patch $S_i$ extend to infinity without intersecting. If all of $\partial V$ is contained in the support of $f$ or $g$, then the above requirements can be satisfied only if $\partial V$ is convex.

In the case $n(x) + 1$ we shall impose analogous restrictions on $\partial V$. The portion of $\partial V$ contained in the support of $f$ or $u_0$ should consist of a finite number of patches $S_i$ that are "locally convex relative to $n(x)$," and joined smoothly together to form a curve (surface) that can be illuminated from the exterior. Every geodesic of the Riemannian metric $ds = n^{1/2} |dx|$ emanating orthogonally from $S_i$ must extend to infinity without intersecting $\partial V$. A patch $S_i$ is locally convex relative to $n$ if every pair of geodesics (rays) emanating from $S_i$ extend to infinity without intersecting.

More specifically we shall use the results of this paper to prove that under the above hypotheses, a function $u_M$ of the form

$$U_M(x, \lambda) = \sum_{n=0}^{M+1} T_n(x, \lambda) \quad (M = -1, 0, 1, 2, 3, \ldots),$$

where

$$T_n(x, \lambda) = \left[ \sum_{k=1}^{K} e^{i\lambda y_k(x)} a_{p,n}(x) \right] n^{-n} = O(\lambda^{-n} r^{1/(1-m)})$$

is a rigorous asymptotic expansion of the solution $u$ of Problem U if $\alpha = 0$ and $\beta = 1$ as $\lambda \to \infty$, uniformly in $x$ ($x \in V$).

We shall construct $u_M$ such that as $\lambda \to \infty$

$$(\Delta + \lambda^2 n(x))u_M = f(x) + O(\lambda^{-M} r^{-(m+3)/2}) \quad (x \in V),$$

and $u_M$ satisfies the boundary condition (1.2) and the radiation condition (1.3).

The a priori estimates derived in this paper will be applied to the difference $u - u_M$ to obtain the error estimate (1.5). Roughly speaking, we demonstrate the proposition that if $u_M$ satisfies Problem P approximately as $\lambda \to \infty$, then $u_M$ approximates uniformly on $V$ as $\lambda \to \infty$. 
The leading term $T_0$ in this asymptotic expansion is the approximation to the scalar field $u(x, \lambda)$ that would be constructed on the basis of Fermat's principle, and the principle of conservation of energy; i.e., on the basis of the principles of geometrical optics for an inhomogeneous medium. Examination of the structure of $T_0$ leads to the conclusions that (i) the conditions we impose on $\partial V$ and $n$ are sufficient to preclude foci and caustics in the "geometrical optics field," (ii) the scalar field $u(x, \lambda)$ at a typical point $x$ of $\overline{V}$ is determined by the values of $u_0$ at the points on each patch visible from $x$ that are closest (in the sense of the Riemannian metric $ds = n^{1/2} |dx|$) to $x$ and by $f(x)$. This is consistent with classical geometrical optics.

We remark that our a priori estimates hold under conditions where focusing of the energy of the geometrical optics field does occur and where caustics are formed. Our a priori estimate for $|u(x, \lambda)|$ provides an upper bound on the strength of the field at foci and caustics for large values of the wave number $\lambda$. We are able to predict that the strength of the field due to the sources radiating on the boundary and in the exterior is at most of the order of magnitude of $\lambda^{(1+m)/2}$ as $\lambda \to \infty$. If $u(x, \lambda)$ vanishes on the boundary we find that $|u(x, \lambda)| = \mathcal{O}(\lambda^{(1+m)/2})$ at every point of the exterior.

A more precise statement about the behavior of the field in cases where foci or caustics are present, or where multiple reflection occurs would, of course, require the construction of an appropriate approximate solution of Problem P.

The interest in solving Problem U arises on the one hand from the fact that Eqs. (1.1), (1.3), and (1.4) provide a mathematical model for the propagation of time harmonic waves in an inhomogeneous medium. The solution $u(x, \lambda)$ of Problem U can be interpreted as the amplitude of the steady-state solution of the following initial boundary value problem:

$$\begin{align*}
\Delta W - n(x) W_{tt} &= e^{-i\lambda t} f(x) \quad (x \in V \times [0, \infty)), \\
\alpha(x)(\nu^x \cdot \nabla W) + \beta(x)W &= e^{-i\lambda t} g(x) \quad (x \in \partial V \times [0, \infty)), \\
W(x, 0) &= h_1(x), \\
W_t(x, 0) &= h_2(x) \quad (x \in V).
\end{align*}$$

Bloom has proved in [4] that $W(x, t)$ tends to $u(x, \lambda)e^{-i\lambda t}$ at an algebraic rate with respect to $t$ as $t \to \infty$, if (i) $\alpha \equiv 0, \beta \equiv 1$; (ii) $h_1(x)$ and $h_2(x)$ have compact support; and (iii) $\partial V$ is star-shaped. For general obstacles, if $W(x, t)$ is the solution of Problem U', Eidus has proved [6] that $W(x, t)$ tends to $u(x, \lambda)e^{-i\lambda t}$ as $t \to \infty$. No rate of approach to the steady state can be established for the class of scattering obstacles treated by Eidus (see [13]).

Ursell [15] has solved Problem U in two dimensions for the case $n(x) \equiv 1, \beta \equiv 0, f(x) \equiv 0$, and $\partial V$ convex. In Ursell's work $u(x, \lambda)$ is interpreted as the amplitude of the time harmonic component of the velocity potential of the acoustic field produced by a double-layer distribution of radiating sources on the boundary of a slightly compressible homogeneous medium. He constructed a Fredholm integral equation for the boundary values of $u(x, \lambda)$ with
a kernel that tends to zero as \( \lambda \to \infty \). He proved that the leading term of the Neumann series for this equation is an asymptotic approximation of the boundary values of the exact solution, and he then used this result to derive the leading term of an asymptotic expansion of the velocity potential in the exterior of the radiating obstacle. Unfortunately, there are formidable mathematical obstacles that prevent his approach from being generalized to higher dimensions, nonconvex boundaries, or to the case of variable \( n(x) \). The method of Ursell can probably be adapted to the more general boundary condition (1.4).

Ursell's fundamental paper [15] has had a strong influence on much of the recent work in scattering theory. Grimshaw [8] has successfully used a method similar to Ursell's in his treatment of Problem U in two dimensions in the case of a convex scattering obstacle with \( n(x) = 1, \alpha = 1, \beta = 0, g = 0, \) and \( f(x) = \delta(x, x_0) \). He solved Problem U for integer values of \( M = \frac{1}{2}(1 + m) \) if \( x \) is in the region of \( V \) illuminated by point source at \( x_0 \). Other applications of Ursell's method can be found in [1, 2, 11]. Interest in the radiating body problem is heightened because it is a scalar diffraction problem with a "weak" shadow boundary, none if \( f(x) = 0 \) in (1.1).

It is an open problem to separate and describe the contributions to the total field of diffraction and other lower-order effects if \( g \neq 0 \) and \( f(x) \) is not a point source. Because of the absence of a clearly defined shadow boundary, the asymptotic solution we construct for Problem U has a relatively simple form. Diffraction effects are absorbed into the error terms.

It also remains to achieve one of Ursell's original goals, to solve Problem U in three dimensions if \( u_\ast \) is prescribed on \( \partial V \), at least in the case where \( n = 1, f = 0, \) and \( \partial V \) is convex.

Ludwig and Morawetz [9] have considered Problem U for \( m = 2, 3 \) in the case \( n = 1, \partial V \) convex, \( \alpha = 0, \beta = 1, g = 0, f(x) = \delta(x, x_0) \). Their approach is to first construct a function \( u_A(x, \lambda) \) that (i) vanishes on \( \partial V \), (ii) satisfies the radiation condition (1.3), and (iii) is an approximate solution of the reduced wave equation on \( \partial V \cup V \). They derived a priori bounds that hold for solutions of Problem P in the case that \( \partial V \) is star-shaped and \( n(x) = 1 \), and they applied them to obtain an estimate for the difference between the exact solution and \( u_A(x, \lambda) \) as an asymptotic approximation of \( u(x, \lambda) \) in the "illuminated" portion of \( V \).

The a priori estimates obtained by Morawetz and Ludwig in [9] do not include an estimate for \( (\| \nabla u/\gamma \|_V)^2 \). Such an estimate is obtained in a more recent paper of Morawetz [10] on energy decay for the wave equation, but it is obtained by an argument that is different from ours; see Strauss [14] for a related estimate.

Bloom [3] has derived a priori estimates for the solution of Problem P in the case of a general second-order elliptic operator, but only for star-shaped \( \partial V \). Our pointwise estimate for \( u(x, \lambda) \) improves upon Bloom's because it is uniform over \( V \), and because we allow a much wider class of scattering obstacles,
e.g., "snake-shaped" bodies. For real λ our local energy estimate generalizes a recent estimate of Morawetz [10] which holds for complex λ, and a general class of scattering obstacles.

Our derivation of the a priori estimates in Theorem 7.1 and Theorem 8.1 is similar in structure to Bloom's proof in [3]. There are significant differences between the choice of multipliers made here and in [3], and in the various delicate calculations and estimates. There are also several simplifications in our work resulting from the fact that we treat the differential operator \( A + \lambda \delta(x) \) rather than a general self-adjoint operator with variable coefficients. In particular we are able to avoid the patching argument used in [3, Appendix III], and the complicated choice of \( \rho(x) \) made in [3, Appendix II].

The multipliers we use are related to those defined in [5]. There we established decay rates for the local energy of solutions of the wave equation defined in exterior regions with boundaries that can be illuminated from the interior.

This paper is organized as follows. Section 2 deals with geometry. We describe there the coordinate system in terms of which the multipliers, and other auxiliary functions are defined.

In Section 3 we derive a differential inequality for functions \( u \in C^4(V) \cap C^2(\mathbb{V}) \). This inequality is obtained from a basic divergence identity proved in [3]. We integrate this inequality over the region \( V(R) \) bounded by \( \partial V \), and the sphere \( S(R) = \{ x : |x| = R \} \). The divergence terms give rise to an integral over \( \partial V \), and an integral over \( S(R) \) of quadratic functions of \( U \) and first derivatives of \( U \). The remaining terms obtained are integrals over \( V(R) \). Our a priori estimates are derived from this inequality.

Sections 4 and 5 are devoted to carefully estimating the integrands of the integrals over \( V(R) \). In Section 6 we prove that the integral over \( S(R) \) is bounded from below by a function of \( R \) that tends to zero as \( R \to \infty \), if \( u \) satisfies the radiation condition (1.3).

In Section 7 we apply these results to the integral inequality derived in Section 3. We assume that \( u \) is the solution of (P) and we let \( R \to \infty \). We obtain a preliminary estimate for \( \| r^{-1} \nabla u \|_V \) and \( \| \nu^* \cdot \nabla u \|_{\partial V} \) in terms of the boundary data \( u_0 \), the source term \( u_0 \), and \( \| r^{-1} u \|_V \).

At this point it appears that little has been accomplished since \( \| r^{-3} u \|_V \) is as yet an unknown quantity. We overcome this difficulty by establishing a "small-multiples" estimate. This is an upper bound for \( \| r^{-1} u \|_V \) in terms of norms of \( f \), \( u_0 \), and small multiples of the unknown quantities \( \| r^{-1} \nabla u \|_V \) and \( \| \nu^* \cdot \nabla u \|_V \), multiples that approach zero as \( \lambda \to \infty \). We use this upper bound in the preliminary estimate to bound \( \| \nu^* \cdot \nabla u \|_{\partial V} \) and \( \| r^{-1} \nabla u \|_V \) from above in terms of norms of \( f \) and \( u_0 \). We then use this result in the small multiples inequality to obtain an upper bound for \( \| r^{-3} u \|_V \) in terms of norms of \( f \) and \( u_0 \).

Finally, in Section 8 we use the estimates for \( \| r^{-1} \nabla u \|_V \), \( \| r^{-1} u \|_V \), and
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\[\| \nu^* \cdot \nabla u \|_{\partial V} \]

obtained in Section 7 in an integral representation for \(u\) to derive an a priori estimate for \(|u(x, \lambda)|\) that holds uniformly on \(\overline{V}\) as \(\lambda \to \infty\).

We remark that the estimates of this paper imply a uniqueness theorem for the solution of Problem P if \(\lambda\) is sufficiently large. For if \(f = u_0 = 0\), our estimate for \(|u(x, \lambda)|\) reduces to \(|u(x, \lambda)| = 0\) for every \(x \in V\).

2. GEOMETRIC PRELIMINARIES

Let \(C\) be a convex body in \(\mathbb{R}^m (m = 2 \text{ or } 3)\) with smooth boundary \(\partial C\). Let \(\nu\) and \(\nu^*\) be the unit exterior normals to \(\partial C\) and to \(\partial V\), respectively.

**Definition 2.1.** A scattering obstacle \(\partial V\) can be illuminated from the exterior if and only if there exist a \(c_0 > 0\) and a convex body \(C\), with \(\partial V \subset C\), such that (i) if \(X_0 \in \partial C\) and \(x \in \partial V\) lie on the same interior normal to \(\partial C\), then \(\nu(x) \cdot \nu^*(x) \geq c_0\), and (ii) any two interior normals to \(\partial C\) intersect only after passing through \(\partial V\).

Definition 2.1 means that each point of \(\partial V\) can be seen along one and only one interior normal to \(\partial C\). An example of a scattering obstacle \(\partial V\) that can be illuminated from the exterior, but which is neither star-shaped nor illuminable from the interior is a "snake" (Fig. 1). Henceforth \(\partial V\) will be an obstacle that can be illuminated from the exterior.

\[x = X^0(\tau) \quad (\tau_1 \leq \tau \leq \tau_0)\]

be a representation of \(C\), with \(X^0(\tau_1) = X^0(\tau_2)\). The normals to \(\partial C\) are described by

\[x = \nu(\tau) \sigma + X^0(\sigma),\]

(2.1)
where \( \nu(\tau) \) is the unit exterior normal to \( \partial C \) at the point \( X^0(\tau) \), and \( | \sigma | \) measures distance from \( \partial C \) along this normal: \( \sigma > 0 \) in \( \text{ext} \ C \), \( \sigma = 0 \) on \( \partial C \), \( \sigma < 0 \) in \( \text{int} \ C \). Equation (2.1) defines a coordinate system in \( V \). For each \( x \) in \( V \) there is a unique ordered pair \( (\sigma(x), \tau(x)) \) such that (2.1) holds. Further there exists a "half-strip" in \( (\sigma, \tau) \) space, call it \( \mathcal{S} \), bounded by the curves \( \tau = \tau_1 \), \( \tau = \tau_2 \), \( \sigma = \sigma_2(\tau) \) such that (2.1) defines a smooth \( 1 \)-\( 1 \) mapping \( x = X(\sigma, \tau) \) from \( \mathcal{S} \) onto \( V \) and \( \{ x \mid x = x(\sigma(\tau), \tau), \tau_1 \leq \tau \leq \tau_2 \} = \partial V \) (Fig. 2).

In \( \mathbb{R}^3 \) the situation is more complicated due to the existence of umbilic points on \( \partial C \). (At an umbilic point \( x \in \partial C \) the curvature is the same in all directions; the two principal curvatures are equal.) However, by a theorem of Feldman [7], we can assume without loss of generality that there are only a finite number of umbilic points on \( \partial C \) and that \( \partial C \) can be subdivided into a finite number of regions with boundaries that contain all the umbilics. In each such region \( R_i \) the arcs of constant principal curvature \( \tau^1 = \text{const} \), \( \tau^2 = \text{const} \) define a local coordinate system. Using the local coordinates \( (\sigma, \tau) \) of \( R_i \) (suppressing subscripts on \( (\sigma, \tau) \)), we again write

\[
x = X_i^0(\tau) \quad (\tau = (\tau^1, \tau^2))
\]

for \( x \in \partial C \cap R_i \). Corresponding to each \( R_i \) is a local coordinate zone \( V_i \) in \( V \). This zone is defined by the ray equation

\[
x = \nu(\tau)\sigma + X_i^0(\tau),
\]  

where \( \nu(\tau) \) and \( \sigma \) have meanings similar to those in the case of \( \mathbb{R}^2 \). The zones \( V_i \) cover all of \( V \) except for the points that lie on normals emanating from the umbilics on \( \partial C \). Again we associate with each \( x \) in a zone \( V_i \) the ray coordinates \( (\sigma(x), \tau(x)) \) (suppressing the subscript \( i \) on \( \tau \)) that correspond to it through (2.2). Finally, we denote the \( 1 \)-\( 1 \) mapping determined by (2.2), from a "half-cylinder" \( \mathcal{S}_i \) in \( (\sigma, \tau) \) space onto \( V_i \) (minus the rays emanating from umbilics) by \( X_i \); that is, for each \( x \in \partial V_i \) (minus the rays emanating from umbilics)

\[
x = X_i(\sigma(x), \tau(x)).
\]
In view of its geometrical interpretation, the local coordinate function $\sigma(x)$ can be extended to a globally continuous function of $x$ on all of $V$, while the local coordinate function $\tau$ cannot be so extended.

In $\mathbb{R}^2$ let $\rho_1(x)$ be the radius of curvature of $\partial C$ at the point $x'$ where the normal to $\partial C$ passes through $x$. In $\mathbb{R}^3$ let $\rho_i(x)$ ($i = 1, 2$) be the principal radii of curvature of $\partial C$ at the point $x'$ where the normal to $\partial C$ passes through $x$.

We make the following observations for use in the sequel.

**Lemma 2.1.** If the obstacle $\partial V$ is illuminated from the exterior by $\partial C$, then

$$\min_{\partial V} [\sigma(x) + \rho_1(x), \sigma(x) + \rho_2(x)] > 0 \quad (s = 1, m - 1; m = 2, 3).$$

**Proof.** Since $V$ is illuminated from the exterior by $\partial C$, every $x$ in $V$ lies on some coordinate surface $\Sigma = \{x' \mid \sigma(x') = \sigma(x)\} \cap \overline{V}$.

Suppose $m = 3$. The Gaussian curvature $K(x)$ of $\Sigma$ at $x$ is

$$K(x) = \frac{[(\sigma(x) + \rho_1(x))(\sigma(x) + \rho_2(x))]^{-1}}{\sigma(x)}.$$

Since $\Sigma$ is convex, $K(x)$ must be positive at each $x \in V$, which implies the desired conclusion. In $\mathbb{R}^2$ the convexity of $\Sigma$ (defined analogously) implies that the curvature $K(x) = [\sigma(x) + \rho_1(x)]^{-1} > 0$, and the desired conclusion follows.

**Lemma 2.2.** The function $\sigma$, defined by (2.1) in two dimensions and by (2.2) in three dimensions, is a smooth function on $V$. Furthermore,

$$\nu = \nu(\sigma(x)) = \nabla \sigma(x)$$

so that $\nu$ is a smooth function of $x$ on $\overline{V}$. Also $|\nabla \sigma(x)|^2 = 1$ on $\overline{V}$.

**Lemma 2.3.** The function $\Delta \sigma$ is continuous on $V$.

The proof of Lemma 2.2 is easy and is left to the reader. Lemma 2.3 follows immediately from Lemma 2.1 and the identity

$$\Delta \sigma(x) = \sum_{i=1}^{m-1} [\sigma(x) + \rho_i(x)]^{-1} \quad (m = 2, 3).$$

**3. The Basic Inequality**

The starting point in obtaining our a priori estimates is the following divergence identity [3, Appendix I]

$$-\nabla \cdot \text{Re} \mathcal{L} = \mathcal{E} + \mathcal{G} + \mathcal{H} - \mathcal{P} + \mathcal{I},$$

(3.1a)
where
\[ \mathcal{F} = -\text{Re}[b \cdot \nabla u + (\imath \rho + \gamma)u][((\Delta u + \lambda^2 nu)/n], \]  
(3.1b)
\[ \mathcal{B} = -\text{Re}[\bar{u} \nabla (\gamma/n) \cdot \nabla u], \]  
(3.1c)
\[ \mathcal{H} = |\nabla (\rho/n) \cdot \nabla u|^2/4\omega, \]  
(3.1d)
\[ \mathcal{P} = \omega |[\nabla (\rho/n) \cdot \nabla u]/2\omega| - \imath \lambda u|^2, \]  
(3.1e)
\[ \mathcal{T} = \{\nabla u[\nabla(b/n) - \frac{1}{2}(b \cdot \nabla(1/n))I - (\omega/n)I] \cdot \nabla u, \]  
(3.1f)
\[ \lambda = \frac{|(\nabla \cdot b)/2| - \gamma, \]  
(3.1g)
and
\[ \mathcal{L} = \left[(\nabla \cdot b/n) \nabla u - \frac{h}{2n} |\nabla u|^2 + \frac{(\imath \lambda n + \gamma)}{\omega} \bar{u} \nabla u + \frac{\lambda^2 h}{2} |u|^2 \right]. \]  
(3.1h)

In the above definitions \(\nabla'(b/n)\) is the matrix \((\partial(b'/n)/\partial x^2)\) and \(I\) is the identity matrix. The identity (3.1) holds if \(u\) is a twice continuously differentiable complex valued function, \(n, \gamma, \rho\) are real valued, continuously differentiable functions, and \(b\) is a vector with continuously differentiable, real valued components. Both \(b(x)\) and \(\gamma(x)\) must be chosen so that \(\omega > 0\).

We want to bound the right-hand side of (3.1a) from above by
\[ -(\nabla u \mathcal{P}) \cdot \nabla \bar{u} - \frac{1}{2} \mathcal{P} + A |Lu|^2 + B |u|^2, \]
where \(A\) and \(B\) are positive functions with \(A = \mathcal{O}(r^2)\) and \(B = \mathcal{O}(r^{-2})\) on \(\bar{V}\) and \(\mathcal{P}\) is a positive definite matrix such that \(\mathcal{P} = \mathcal{O}(r^{-2})\) on \(\bar{V}\). To this end we write
\[ \mathcal{F} = \text{Re} \sum_{i=1}^{4} \mathcal{F}_i Lu, \]  
(3.2a)
where
\[ \mathcal{F}_1 = (\rho/n)[(c \cdot \nabla u)|/\omega] - \imath \lambda u], \]  
(3.2b)
\[ \mathcal{F}_2 = \{(b \cdot c)/|c| \} - (\rho |c/|\omega))c \cdot \nabla u)/n |c|, \]  
(3.2c)
\[ \mathcal{F}_3 = \{- (b \cdot c)/|c|/|c| \} + (b/n) \} \cdot \nabla u, \]  
(3.2d)
\[ \mathcal{F}_4 = (\gamma/n)u, \]  
(3.2e)
and
\[ Lu = \Delta u + \lambda^2 nu. \]  
(3.3)

Next we set
\[ c = \frac{1}{2} \nabla (\rho/n) \quad \text{and} \quad \omega = 2 |c|^2. \]  
(3.4)

Since, by definition (3.1g), \(\gamma = \frac{1}{2}(\nabla \cdot b) - \gamma\), the multiplier \(\gamma\) is now defined in terms of \(\rho\) and the vector \(b\). It is also convenient to define two auxiliary functions \(\mu\) and \(\eta\) on \(V\) by the equations
\[ \mu(x) = \frac{1}{2} + \eta(x), \quad \eta(x) = (e/3) h^2(x), \quad h(x) = [\sigma(x) + \sigma_0]^{-1}, \]  
(3.5)
where $\sigma$ is defined in Section 1 and $\sigma_0$ is some positive number such that $\sigma + \sigma_0 > 0$ on $V$. It follows from Lemma 2.2 that $\mu$ and $\eta$ are continuous functions on $V$. We choose $\epsilon$ to be any positive number such that
\[ 1 - \epsilon h^2(x) > \frac{1}{2} \quad \text{if} \quad \sigma \geq \sigma_0. \] (3.6)

With these definitions in mind it is easy to derive the following inequalities for the terms on the right-hand side of (3.2a):
\[
-\text{Re}(\nabla \cdot \mathbf{u}) \leq \frac{1}{2} \mathcal{P} + A_1 |Lu|^2,
-\text{Re}(\nabla \cdot \nabla \mathbf{u}) \leq 2\eta \mathcal{P} + A_2 |Lu|^2,
-\text{Re}(\mathcal{F} \nabla \cdot \mathbf{u}) \leq -2\mu \mathcal{P} + \mu |\nabla u|^2 + A_3 |Lu|^2,
-\text{Re}(\nabla \cdot \mathbf{u}) \leq (h\gamma |u|/2n)^2 + A_4 |Lu|^2.
\] (3.7)

Here
\[
A_1 = \left( -\frac{\rho}{2n} \frac{|c|}{|c|} \right), \quad A_2 = \frac{(\rho |c|)^2}{16\eta n^2 |c|^4} \left( \frac{(b \cdot c)}{\rho |c|} - 1 \right)^2,
A_3 = \frac{|b|^2}{4\mu n^2}, \quad A_4 = h^{-2}.
\]

Moreover, if we set
\[ 2d = \nabla (\gamma/n) \quad \text{and} \quad \theta = \frac{1}{2} \eta > 0, \] (3.8)
then
\[ \mathcal{G} \leq (|d|^2 |u|^2/\theta) + \theta |\nabla u|^2. \] (3.9)

Using inequalities (3.7) and (3.9), we obtain an upper bound for $\mathcal{F} + \mathcal{G}$. Using this estimate, the definition of $\omega$ given by (3.4) and the condition (3.6), we obtain the following basic inequality from (3.1a):
\[ -\text{Re} \nabla \cdot \mathbf{L} \leq -(\nabla \mathbf{u} \cdot \nabla \omega) \cdot \nabla u - \frac{1}{2} \mathcal{P} + A |Lu|^2 + B |u|^2, \] (3.10)
where
\[ A = \sum_{i=1}^{4} A_i, \quad B = [\gamma^2/(4n^2 A_4)] + (|d|^2/\theta), \] (3.11)
and
\[ \omega = \nabla' (b/n) - \frac{1}{2} b \cdot \nabla (n^{-1}) I - \frac{1}{2} (1 + 3\eta) I. \] (3.12)

Upon integrating both sides of (3.10) over the region exterior to the obstacle
\[\partial V \text{ and interior to a large sphere of radius } R \text{ and using the divergence theorem to evaluate the integral of the left-hand side, we obtain the result}\]

\[\int_{\partial V} ((\nabla u \mathcal{B}) \cdot \nabla u + \frac{1}{2} \mathcal{P}) - \int_{r=R} (x/r) \cdot \text{Re } \mathcal{L} + \int_{\partial V} v^* \cdot \text{Re } \mathcal{L} \lesssim \int_{\partial V} [A \mid L u \mid^2 + B \mid u \mid^2], \tag{3.13}\]

Here \(v^*\) is the unit exterior normal to \(\partial V\) and \(V(R) = V \cap \{x \mid r \leq R\}\).

In the remainder of this paper we carry out the analysis that leads from (3.13) to estimates of the form (7.1).

The success of our further argument hinges on choosing the multipliers \(b\) and \(\rho\) so that

\[\int_{r=R} (x/r) \cdot \mathcal{L} \geq I_2(R) = o(1) \quad (R \to \infty), \tag{3.14}\]

and so that

\[I_1 \equiv \int_{\partial V} v^* \cdot \text{Re } \mathcal{L} \geq p_1 \|u_{*r}\|^2_{\partial V} + (\text{terms involving } u \text{ and } u_{r*} \text{ on } \partial V), \tag{3.15}\]

\[I_3 \equiv \int_{V} (\nabla u \mathcal{B}) \cdot \nabla u \geq p_2 \|\nabla u/r\|^2_V, \tag{3.16}\]

and

\[I_4 \equiv \int_{V} B \mid u \mid^2 \leq p_3 \|u/r\|^2_V, \tag{3.17}\]

where the \(p_i\) are positive constants independent of \(\lambda\).

An inequality of the form (3.15) does hold if \(b\) is chosen so that \(v^* \cdot b\) is strictly positive on \(\partial V\). To see this, note first that

\[I_1 = \int_{\partial V} \left[\left(\frac{b \cdot v^*}{2n}\right) \mid u_{*r}\mid^2 + \text{Re}\{u_{r+}u_{r}/n\} + \text{Re}\{(i\lambda \rho + \gamma) u_{r+}/n\} \right.\]

\[+ \frac{1}{2} \lambda^2 (v^* \cdot b) \mid u \mid^2 - \frac{1}{2} (v^* \cdot b) \mid u_{r^*}\mid^2 \right]. \tag{3.18}\]

An application of the elementary inequality \(ab \leq \frac{1}{2}(a^2 + b^2)\) to the second and third terms in the integrand of (3.18) leads to the estimate

\[I_1 \geq (1 - 2 \epsilon_3) \int_{\partial V} \left(\frac{(v^* \cdot b)}{2n}\right) \mid u_{*r}\mid^2 - \int_{\partial V} [F_1 \mid u_{r+}\mid^2 + F_2 \mid u \mid^2], \tag{3.19}\]

where

\[F_1 = \frac{(v^* \cdot b)}{n} + \frac{|b|^2}{2 \epsilon_3 n (v^* \cdot b)}.\]
and
\[
F_2 = \left[ -\frac{\lambda^2}{2} (\nu^* \cdot b) + \frac{\lambda^2 p^2 + \gamma^2}{2 \varepsilon_m (\nu^* \cdot b)} \right].
\]

The desired lower bound for $I_1$ follows immediately from (3.19) if we choose $\varepsilon_1 > 0$ and sufficiently small, provided that both $b \cdot \nu^*$ and $\nu$ are strictly positive on $\partial V$. The positive definiteness of $\nu(x)$ is one of our basic hypotheses made in the Introduction.

In Section 4 of this paper we shall choose $\rho$ and $b$ so that $b \cdot \nu^* > 0$ on $\partial V$, which implies (3.15) holds. We derive a lower bound for $(\nabla u, \partial) \cdot \nabla \tilde{u}$ that implies (3.16) under physically reasonable hypotheses on $\nu(x)$. In Section 5 we derive bounds on $A$ and $B$ (defined in (3.11)) that imply (3.17) and
\[
\int_{\partial V} A |Lu|^2 \leq p_4 \|rLu\|^2, \tag{3.20}
\]
for some positive number $p_4$ independent of $\lambda$. In Section 6 we prove that with our choice of $b$ and $\rho$ made in Section 4, (3.14) holds.

We make use of (3.14) in (3.13) and take limits of both sides of the resulting inequality to obtain
\[
I_1 + \int_{\partial V} [(\nabla u, \partial) \cdot \nabla \tilde{u} + \frac{\gamma}{2}] \leq \int_{\partial V} [A |Lu|^2 + B |u|^2]. \tag{3.21}
\]
This inequality implies the preliminary estimate (7.2) by virtue of (3.19) and the results of Sections 4–6. It follows immediately from (7.2) that a priori estimates of the form
\[
\|u_r\|_{\partial V} \leq p \|u/r\|_{\partial V} + \|rLu\|_{\partial V} + \lambda \|u\|_{\partial V} + \|u_{\nu^*}\|_{\partial V} \tag{3.22}
\]
hold, when $p$ is a positive constant independent of $u$ and $\lambda$. In Section 8 we use the fundamental solution $H^{(1)}_0(\lambda \mid x \cdot x')$ of $Lu = \delta(x, x')$ and Green's identity to derive our a priori pointwise estimate for $|u(x, \lambda)|$ from Theorem 7.1.

### 4. A LOWER BOUND FOR $(\nabla u, \partial) \cdot \nabla \tilde{u}$

In this section we choose $b$ and $\rho$ and make hypotheses on the smoothness and far field behavior of the index of refraction $n^{1/2}$ sufficient to guarantee that the quadratic form $(\nabla u, \partial) \cdot \nabla \tilde{u}$ is bounded from below by a strictly positive multiple of $|\nabla u|^2$ on $\tilde{V}$. Let $\sigma_0$, $\sigma = \sigma(x)$, and $\nu = \nabla \sigma(x)$ be defined as in Section 2, and recall the definition (3.5) of $h$. We further recall from Section 2 that $\sigma(x)$, $\tau(x)$ are the coordinates of $x \in \tilde{V}$ under the transformation (2.1) in $\mathbb{R}^2$ and (2.2) in $\mathbb{R}^3$. 


HYPOTHESES. There exist constants $p > 2$, $n_0 > 0$, $C_i > 0$ ($i = 1, 2$), and $\alpha_1 \geq \alpha_0$ such that in both the case of $\mathbb{R}^2$ and the case of $\mathbb{R}^3$

\begin{equation}
\n(x) \geq n_0 > 0 \quad (x \in V);
\end{equation}

\begin{equation}
| 1 - n^{-1}(x) | \leq C_2h^2(x) \quad (x \in \overline{V}, \theta(x) \geq \theta_2),
\end{equation}

\begin{equation}
| n_n | = | \nu \cdot \nabla n | \leq C_2h^2(x) \quad (x \in \overline{V}, \theta(x) \geq \theta_2).
\end{equation}

Further, there exists a positive number $c_0$ such that for all $x \in \overline{V}$

\begin{equation}
n + \frac{1}{2}h^{-1}(\nu \cdot \nabla n) \geq c_0.
\end{equation}

Note that hypothesis (4.3) restricts the magnitude of $n_n = \nu \cdot \nabla n$ only on subsets of $\overline{V}$, where $n < 0$. The hypotheses (4.2) and (4.3) are physically reasonable. Radical changes in $n$ relative to its magnitude can produce concentration of energy. The hypotheses we make ensure that energy flows out to infinity at a reasonable rate. This is consistent with our local energy norm interpretation of our a priori $L_2$ estimates, which we mentioned in the Introduction.

We next introduce an auxiliary function $\Gamma(x)$ to be used in the definition of the multiplier $b(x)$.

**Definition 4.1.** $\Gamma$ is the $C^4(\overline{V})$ function with piecewise continuous second derivatives given by

\begin{equation}
\Gamma(x) = \begin{cases}
1 - \varepsilon^{-1} & \text{if } \theta(x) \leq \theta_2, \\
\varepsilon^{-1}[1 + \varepsilon' \cdot (k(x) - 1)^2] & \text{if } \theta(x) > \theta_2,
\end{cases}
\end{equation}

where $k(x) = (\theta_2 + \theta_1)k(x)$. In this definition $\varepsilon'$ is a positive number, which we shall later assume to be suitably small. We now define our multipliers $b$ and $\rho$.

**Definition 4.2.** The multiplier $b$ is the $C^4(\overline{V})$ vector function with piecewise continuous second derivatives defined by

\begin{equation}
b(x) = n(x)\Gamma(x)h^{-1}(x)\nabla \theta(x),
\end{equation}

and $\rho$ is the $C^4(\overline{V})$ function defined by

\begin{equation}
\rho(x) = n(x)[h^{-1}(x) + 2[\varepsilon'(\theta_2 + \theta_1)]^{-1} + \varepsilon h(x)].
\end{equation}

In (4.6) $\varepsilon$ is the positive number chosen in (3.5) and $\varepsilon'$ is the positive number occurring in (4.4).

The following lemma plays a key role in the proof of Proposition 4.1 below.
**Lemma 4.1.** If we define \( \hat{b}(x) = \nabla \sigma(x)/h(x) \), then the matrix \( \nabla' \hat{b} = (b_j^i) \) has the property

\[
\nabla' \hat{b} \geq I \quad (x \in \bar{V});
\]

that is, for each complex \( m \) vector \( \psi (\psi \nabla' \hat{b}) \cdot \psi \geq \psi \cdot \psi \) on \( \bar{V} \).

**Proof.** We give the proof only in the case of \( \mathbb{R}^3 \). It is easy to specialize the argument to \( \mathbb{R}^2 \). We first observe that

\[
\hat{b} = \alpha + \nu \sigma_0,
\]

where \( \alpha = \nu \sigma = (\nabla \sigma) \sigma \) is the vector multiplier used in [5]. In a typical coordinate zone (see Section 2)

\[
b_j^i = \alpha_j^i + \sigma_0 \sigma_{ij} \quad (i, j = 1, 2, 3),
\]

where the subscripts denote differentiation with respect to \( x^i \) and, as computed in [5, Eq. (8.7)],

\[
\alpha_j^i = \delta_{ij} - \sum_{l=1}^{2} \left[ \rho_l X_{\sigma_i}^{0l} X_{\sigma_j}^{0l}/(\sigma + \rho_l) \left| X_{\sigma_i}^{0l} \right|^2 \right].
\]

On the other hand, since \( \sigma = X_{\sigma}^i \), in a typical coordinate zone

\[
\sigma_{ij} = (X_{\sigma}^i)_j = \sum_{l=1}^{2} \left[ X_{\sigma_i}^{0l} X_{\sigma_j}^{0l}/(\sigma + \rho_l) \left| X_{\sigma_i}^{0l} \right|^2 \right].
\]

Substituting for \( \alpha_j^i \) and \( \sigma_{ij} \) from (4.9) and (4.10) in (4.8), we find that in a typical coordinate zone

\[
\nabla' \hat{b} = I + \sum_{l=1}^{2} \left[ (\sigma_0 - \rho_l)(X_{\sigma_i}^0)^{tr} X_{\sigma_j}^0/(\sigma + \rho_l) \left| X_{\sigma_i}^0 \right|^2 \right],
\]

where \( (X_{\sigma_i}^0)^{tr} \) is the transpose of \( X_{\sigma_i}^0 \), and

\[
(X_{\sigma_i}^0)^{tr} X_{\sigma_i}^0 = (X_{\sigma_i}^{0l} X_{\sigma_i}^{0l}).
\]

By Lemma 2.1 both \( \sigma + \rho_1 \) and \( \sigma + \rho_2 \) are positive on \( \bar{V} \). We now further restrict our choice of \( \sigma_0 \) so that both

\[
\sigma + \sigma_0 > 0 \quad \text{and} \quad \sigma_0 - \rho_l > 0 \quad (l = 1, 2) \text{ in } \bar{V}.
\]

Having thus chosen \( \sigma_0 \), we see that since

\[
(\Psi[(X_{\sigma_i}^0)^{tr} X_{\sigma_i}^0]) \cdot \Psi = \left| \sum_{l=1}^{2} \Psi^{0l} X_{\sigma_i}^{0l} \right|^2 \geq 0
\]

on \( \bar{V} \), the desired inequality (4.7) holds on \( \bar{V} \). This completes the proof of Lemma 4.1.
We now state and prove the main result of this section, which holds in $\mathbb{R}^m$ ($m = 2, 3$).

**Proposition 4.1.** Suppose that $\sigma_0$ satisfies (4.12), $b$ and $\rho$ are chosen according to Definition 4.2, $\epsilon$ and $\epsilon'$ are sufficiently small, and hypotheses (4.1)-(4.3) are satisfied. Then the matrix $\mathcal{Q}$, which is defined by (3.12), (3.5), and the choice of $b$ and $\sigma_0$, is strictly positive definite on $\mathcal{V}$. In particular, there exists a positive constant $C$ independent of $x$ such that on $\mathcal{V}$

$$(\nabla u \mathcal{Q}) \cdot \nabla u \geq C h^2(x) |\nabla u|^2.$$  \hspace{1cm} (4.13)

Our proof of Proposition 1 is considerably simpler than the proof of the analogous result in [3, Appendices II and III]. Our choice of $\rho$ is simpler, and we avoid the patching argument used by Bloom.

**Proof.** Using the definition of $b$, we rewrite $\mathcal{Q}$ as

$$\mathcal{Q} = \sum_{i=1}^4 \mathcal{Q}_i,$$  \hspace{1cm} (4.14)

where

$$\mathcal{Q}_1 = \Gamma(\nabla b + [(\hat{b} \cdot \nabla n)/2n]I), \hspace{1cm} \mathcal{Q}_2 = (\nabla \Gamma)' \hat{b},$$

$$\mathcal{Q}_3 = -(1/2n) |\nabla (\rho/n)|^2 I, \hspace{1cm} \mathcal{Q}_4 = -\frac{1}{2}(1 + 3\eta) I.$$

We consider two cases.

**Case I.** $x \in S_1$, where $S_1 = \{x \mid \sigma(x) \leq \sigma_2\} \cap \mathcal{V}$.

In this case $\Gamma$ is constant so that $\mathcal{Q}_2 = 0$. Furthermore, the constant value of $\Gamma$ can be made arbitrarily large by taking $\epsilon'$ sufficiently small. Therefore, if

$$\nabla' b + [(\hat{b} \cdot \nabla n)/2n]I > 0,$$  \hspace{1cm} (4.15)

then $\mathcal{Q}_1$ can be made as large as we please by taking $\epsilon'$ small enough. On the other hand, both $\mathcal{Q}_3$ and $\mathcal{Q}_4$ are independent of $\epsilon'$ and bounded on $S_1$ since $S_1$ is compact. Thus if (4.15) holds, then $\mathcal{Q}$ is positive definite on $S_1$. But by Lemma 4.1

$$\nabla' b + [(\hat{b} \cdot \nabla n)/2n]I \geq (1 + [(\nu \cdot \nabla n)/2n h(x)])I.$$  \hspace{1cm} (4.16)

By our hypothesis (4.3) the right-hand side of (4.16) is positive definite; hence (4.13) holds, and $\mathcal{Q}$ is positive definite on $S_1$. (Our hypothesis (4.3) should be compared to [3, conditions II-2].)

Finally, since $S_1$ is compact, it follows that on $S_1$

$$(\nabla u \mathcal{Q}) \cdot \nabla u \geq C_1 h^2(x) |\nabla u|^2,$$  \hspace{1cm} (4.17)

where

$$C_1 = \min_{S_1} [h^{-2}(x)(\min_{|\psi| = 1} (\psi \mathcal{Q} \cdot \psi))].$$
Case II. \( x \in S_\Pi \), where \( S_\Pi = \{ x \mid \alpha(x) \geq \alpha_0 \} \cap \bar{V} \).

Our approach is direct: we compute and estimate the terms in (4.14). The far field behavior of \( \pi \) now comes into play. Hypotheses (4.2) imply that

\[
\mathcal{L}_1 \geq \Gamma[1 - C_1 h^p(x)(1 + C_2 h^p(x))] I. \tag{4.18}
\]

The choice of \( \sigma_2 \) does not affect our proof. We choose it so large that the function in square brackets in (4.18) is positive.

Estimates of \( \mathcal{L}_2 \) and \( \mathcal{L}_3 \) are easy to obtain. We find by straightforward computation that

\[
\mathcal{L}_2 = -\frac{2}{\varepsilon'} k(x)[1 - k(x)](\nabla \sigma) \cdot \nabla \sigma,
\mathcal{L}_3 = -\frac{1}{2}[1 - \varepsilon h^2(x)]^2 + \mathcal{O}(h^p(x)). \tag{4.19}
\]

In estimating \( \mathcal{L}_3 \) we have again used hypotheses (4.2).

Recalling definition (3.5) of \( \eta \), we see that

\[
\mathcal{L}_4 = -\frac{1}{2}[1 + \varepsilon h^2(x)]. \tag{4.20}
\]

We now use the estimates (4.18)–(4.20) to conclude that for all \( x \in S_\Pi \)

\[
(\nabla \mathcal{L}_2) \cdot \nabla u \geq \Gamma[1 + \mathcal{O}(h^p(x))] | \nabla u |^2 - (2/\varepsilon') k(x)[1 - k(x)] | \nabla \sigma |^2
+ \left[ -1 + \frac{1}{2} \varepsilon h^2(x)[1 - \varepsilon h^2(x)] + \mathcal{O}(h^p(x)) \right] | \nabla u |^2. \tag{4.21}
\]

But \( \Gamma = 1 + (\Gamma - 1) \), and

\[
\Gamma - 1 = [k(x)/\varepsilon'][2 - k(x)](1/\varepsilon') k(x)[2 - k(x)]. \tag{4.22}
\]

Making use of (4.22) in (4.21), we find that

\[
(\nabla \mathcal{L}_2) \cdot \nabla \varepsilon \geq \left\{ [1 + \mathcal{O}(h^p(x))] + (1/\varepsilon') k(x)[2 - k(x)][1 + \mathcal{O}(h^p(x))]^2
+ \left[ \varepsilon/2 h^2(x)[1 - \varepsilon h^2(x)] + \mathcal{O}(h^p(x)) \right] | \nabla u |^2
- (2/\varepsilon') k(x)[1 - k(x)] | \nabla \sigma |^2 \right\}. \tag{4.23}
\]

Since \( -| \nabla \sigma | > -| \nabla u | \), we deduce from (4.23) that on \( S_\Pi \)

\[
(\nabla \mathcal{L}_2) \cdot \nabla \varepsilon \geq \{(\sigma_2 + \sigma_0)^2 + \frac{1}{2} \varepsilon[1 - \varepsilon h^2(x)] + \mathcal{O}(h^{p-2}(x)) \} h^2(x) | \nabla u |^2.
\tag{4.24}
\]

We now choose \( \sigma_2 \) so large that it satisfies our previous conditions and

\[
\inf_{\sigma_0 < \sigma_2} q(x) > 0.
\]

With this choice of \( \sigma_2 \) it follows from (4.24) that

\[
(\nabla \mathcal{L}_2) \cdot \nabla \varepsilon \geq C_{11} h^2(x) | \nabla u |^2, \tag{4.25}
\]
where
\[ C_{11} = \min\{H^{-2}(x)[\min(\psi, \bar{\psi})] \cdot \psi] \}. \]

Finally, on the basis of (4.17) and (4.25) we conclude that Proposition 4.1 holds with \( C = \min(C_1, C_{11}) \).

5. Far Field Behavior of Coefficients of the \( |Lu|^2 \) and \( |u|^2 \) Terms

In this section we establish the behavior as \( \sigma \to \infty \) of the coefficients of \( |Lu|^2 \) and \( |u|^2 \) in our basic inequality (3.10). We use the choices of \( b \) and \( \rho \) made in the last section.

Several straightforward calculations using hypotheses (4.1) and (4.2) lead to the conclusion that, as \( D \to \infty \),
\[ A_2 \cdot A_3 = \mathcal{O}(\sigma^2), \]
\[ A_1 + A_4 = \mathcal{O}(\sigma^2), \]
where the \( A_i \) are defined after (3.7). In view of (5.1) the coefficient \( A \) of \( |Lu|^2 \) in (3.10) grows no faster than a constant multiple of \( \sigma^2 \) as \( \sigma \to \infty \), and
\[ K_3 = \sup_{\nu \in R^m}|h^2(x)A| < \infty. \]

Next, we examine the behavior of the coefficient \( B \) of \( |u|^2 \) in (3.10) as \( \sigma \to \infty \). We use assumptions (4.1) and (4.2) and we also assume:

**Hypotheses.** For all \( x \in \overline{V} \),
\[ |\nabla n| \leq C_3 h^2(x), \]
\[ |n_{\sigma\sigma}| \leq C_3 h^2(x), \]
\[ |n_{\sigma i}| \leq C_3 h^2(x) \quad (i = 1, m - 1) \]
in \( R^m(m = 2, 3) \).

These conditions are implied by hypotheses H(vii) and (viii) in the Introduction.

The calculations necessary to obtain the estimates to follow are tedious and straightforward for the most part. Therefore we omit most of them. All the big-oh estimates in the remainder of this Section hold for \( x \in \overline{V} \).

First, it follows from the definitions of \( \omega, \rho, \) and \( \Gamma \) that
\[ \rho/n = \mathcal{O}(h^{-1}(x)), \]
\[ 2\omega = |\nabla(\rho/n)|^2 = (\rho/n)^2 = \mathcal{O}(1), \]
\[ \nabla \omega = (\rho/n) \cdot (\rho/n)_{\sigma\sigma} \nu = \mathcal{O}(h^2(x)), \]
DIRICHLET PROBLEM FOR \([A + h^2n(x)]u = f(x, \lambda)\)

and

\[ \Gamma = \mathcal{O}(1), \quad \Gamma_\sigma = \mathcal{O}(h^2(x)), \quad \Gamma_{\sigma\sigma} = \mathcal{O}(h^2(x)). \quad (5.4d) \]

Furthermore,

\[ \nabla \sigma = \sum_{i=1}^{m-1} \left[ \frac{1}{(\nu + \mu_i)} \right] - \mathcal{O}(\sigma^{-1}) \quad (m = 2, 3); \quad (5.5) \]

and since \(|\text{grad } \rho_1| = \mathcal{O}(1),\)

\[ \text{grad}(\nabla \sigma) = - \sum_{i=1}^{m-1} \left[ (\nu + \text{grad } \rho_1) / (\sigma + \rho_1)^2 \right] = \mathcal{O}(\sigma^{-2}). \quad (5.6) \]

A straightforward calculation using (5.4d) and (5.5) gives the result

\[ \nabla \cdot b = n_o h^{-1} + n [h^{-1} \Gamma + \Gamma'] + n h^{-1} \Delta \sigma \]
\[ = n_o \mathcal{O}(\sigma) + n \mathcal{O}(1). \]

Thus by hypotheses (4.2)

\[ \nabla \cdot b = \mathcal{O}(1). \quad (5.7) \]

This result together with (5.4b) imply that

\[ \gamma = \frac{1}{2} \nabla \cdot b - \omega = \mathcal{O}(1). \quad (5.8) \]

An immediate consequence of the last estimate and our hypotheses on \(n\) is that

\[ \gamma h^2 / 4n = \mathcal{O}(\sigma^{-2}). \quad (5.9) \]

We next estimate \(d^2/\theta\), recalling that \(d = \text{grad}(\gamma/n)\). This calculation is rather tedious since second derivatives of \(b\), and hence \(n\), are involved through \(\text{grad} \gamma\). In estimating \(\text{grad}(\text{div} b)\) we use the results (5.4d), (5.5), and (5.6) to estimate \(\Gamma\) and the derivatives of \(\Gamma\) and \(\sigma\), while we use the hypotheses (4.1), (4.2), and (5.3) to estimate \(n\) and its derivatives. We find that

\[ \nabla(\nabla \cdot b) = \mathcal{O}(\sigma^{-2}). \quad (5.10) \]

This result together with (5.4c) implies that

\[ \nabla \gamma = \mathcal{O}(\sigma^{-2}). \quad (5.11) \]

Finally, using results (5.9) and (5.8) in the definition of \(d\), we obtain the estimate

\[ \frac{d^2}{\theta} = \mathcal{O}(\sigma^{-2}) \quad (5.12) \]

since \(\theta = \mathcal{O}(\sigma^{-2})\) by (3.5).

In view of estimates (5.9) and (5.12) we conclude that \(B\) grows no faster than a constant multiple of \(\sigma^{-2}\) as \(\sigma \to \infty\); that is

\[ K_4 = \sup_{\nu} \frac{h^2(x)B}{\nu} < \infty. \quad (5.13) \]
6. THE RADIATION INTEGRAL

One cannot expect that Problem P, stated in the Introduction, will have a unique solution unless an outgoing radiation condition such as (1.3) is imposed. It follows that somewhere in a proof of a priori estimates for solutions of Problem P (estimates that imply uniqueness) the radiation condition must be used. It plays its role in this section. Our goal is to derive the result (3.14), namely, to prove that

\[-\int_{r=R} \frac{x}{r} \cdot \text{Re} \mathcal{L} \geq o(1) \quad (R \to \infty). \tag{6.1}\]

This result is used to deduce (3.21) from (3.13). We first rewrite the integrand in (6.1) as a quadratic form in \( u, u_r, \) and \( \mathcal{D}_1 u, \) where

\[\mathcal{D}_1 u = u_r - i\lambda u + [(m - 1)/2r]u \tag{6.2}\]

and \( u_r \) is the component of \( \text{grad} u \) lying in the plane perpendicular to the position vector \( x \) on the sphere \( |x| = r. \) To do this we write

\[\nabla u = u_r(x/r) + u_r T, \quad u_r = \mathcal{D}_1 u + i\lambda u - [(m - 1)/2r]u\]

and

\[b^r = (x \cdot h)/r, \quad h^T = T \cdot h.\]

The result is:

\[-\int_{r=R} \frac{x}{r} \cdot \text{Re} \mathcal{L} = \int_{r=R} \left\{ \frac{b^r}{2n} \left[ |u_r|^2 - \mathcal{D}_1 u|^2 \right] - \text{Re} \left\{ \frac{b^T}{n} \mathcal{D}_1 u \right\} \right.\]

\[+ \frac{|u|^2}{n} \left[ \lambda^4 (\rho - b^r) + \frac{1}{2} b^r (1 - n) \right] + \frac{\gamma (m - 1)}{2r} \]

\[- \frac{1}{2} b^r \left( \frac{m - 1}{2r} \right)^2 \left[ i \lambda (\rho - b^r) + \left( \gamma - \frac{m - 1}{2r} b^r \right) \frac{\bar{u}}{n} \mathcal{D}_1 u \right] \]

\[- \text{Re} \left\{ \left( i \lambda - \frac{m - 1}{2r} b^r \frac{\bar{u}}{n} u \right) \right\} \tag{6.3}\]

We use the inequality \(|ab| \leq \frac{1}{2} |c| |a|^2 + c^{-1} |b|^2|\) to estimate the cross product terms involving \( u_r \) in (6.3). We choose \( c = \frac{1}{2} \) so that the resultant \(|u_r|^2\) terms exactly cancel the term \( b^r |u_r|/2n \) in (6.3). Note that \( b^r \) is positive for sufficiently large \( r \) since

\[b^r = r + O(1) \quad (r \to \infty)\]
DIRICHLET PROBLEM FOR \([A + \lambda^2 n(x)]u = f(x, \lambda)\)

(see (6.7)). We next use the inequality \(|ab| \leq \frac{1}{2}|a|^2 + r^{-1}|b|^2\), to estimate the \(uD_1u\) terms in (6.3). We thus obtain

\[-\int_{r=R} (x/r) \cdot \Re \mathcal{L} \geq \mathcal{I}_a(R) + J(R),\]  

(6.4)

where

\[\mathcal{I}_a(R) = \int_{r=R} \frac{r}{n} \left\{ \frac{1}{b^r} - \frac{|b^T|^2}{rb^r} - \frac{1}{2} \right\},\]  

(6.5)

and

\[J(R) = \int_{r=R} \left\{ \frac{1}{n} \left[ (\rho - b^r) - \frac{|b^T|^2}{b^r} + \frac{1}{2} b^r (1 - n) - \frac{(\rho - b^r)^2}{2r} \right] \right\} \]

\[+ \gamma (m - 1) - \frac{1}{2} b^r \left( \frac{m - 1}{2r} \right)^2 - \left( \frac{m - 1}{2r} \right)^2 \frac{|b^T|^2}{b^r} \]

\[- \left( \gamma - \frac{(m - 1) b^r}{2r} \right)^2 \frac{1}{2r} \} \].

(6.6)

We use the local coordinate transformation (2.1) or (2.2) to estimate the terms within curly brackets in these last two integrals in each coordinate zone. Of course, we employ our various hypotheses on \(n\). The results are that as \(r \to \infty\),

\[\nu_r = 1 + \frac{1}{2} r^{-2} (\nu - X^0 - |X^0|^2) + O(r^{-4}),\]  

(6.7a)

\[\sigma = r - \nu \cdot X^0 - (2r)^{-1} (\nu \cdot X^0 - |X^0|^2) + O(r^{-3}),\]  

(6.7b)

\[b^r = r + \left[ \frac{2(\sigma_2 + \sigma_0)}{\epsilon} + \sigma_0 - \nu \cdot X^0 \right] + O(r^{-1}),\]  

(6.7c)

\[\rho - b^r = \left[ \frac{(\sigma_2 + \sigma_0)^2}{\epsilon} + \epsilon - \frac{1}{2} (\nu \cdot X^0 - |X^0|^2) \right] r^{-1} \]

\[+ O(r^{1-\gamma}) \quad (\rho > 2),\]  

(6.7d)

\[1 - n = O(r^{-\nu}),\]  

(6.7e)

\[\frac{|b^T|^2}{b^r} = -r^{-1} (\nu \cdot X^0 - |X^0|^2) + O(r^{-2}),\]  

(6.7f)

\[\gamma - \frac{m - 1}{2r} b^r = -\sum_{t=1}^{m-1} \rho tr^{-1} + O(r^{-2}) = O(r^{-1}).\]  

(6.7g)

It follows that the term multiplying \(r |D_1u|^2/n\) in (6.5) is \(O(1)\) as \(r \to \infty\). Therefore, by the radiation condition (1.3),

\[\mathcal{I}_a(R) = O(1) \quad (R \to \infty).\]  

(6.8)
Next we observe that the coefficient of $\lambda^2$ in the integrand of (6.6) can be made positive. We accomplish this by first choosing $\epsilon'$ small enough to make the coefficient of the $O(r^{-1})$ part of

$$\rho - b^r - (b^r |^2 / b^r)$$

positive and then choosing $R$ so large that this $O(r^{-1})$ term dominates; say $R > R_0$. Then we can choose $\lambda$ so large, say $\lambda \geq \lambda_0$, that the integrand in (6.6) is nonnegative if $R \geq R_0$, since the remaining terms not involving $\lambda$ in the curly brackets are $\epsilon'^{-1}O(r^{-1})$. Thus

$$J(R) \geq 0 \quad \text{for} \quad R \geq R_0 \quad \text{and} \quad \lambda \geq \lambda_0. \quad (6.9)$$

Results (6.8) and (6.9) imply (6.1) if $\lambda \geq \lambda_0$.

We close this section with a lemma concerning radiation integrals that will be useful a little later on.

**Lemma 6.1.** If $u \in C^4(V)$ and $u$ satisfies the radiation condition

$$\lim_{R \to \infty} \int_{r=R} r | \partial u |^2 = 0,$$

then for each $\delta > 0$

$$\lim_{R \to \infty} \int_{r=R} r^{-1-\delta} | u |^2 = 0. \quad (6.10)$$

**Proof.** It is a direct consequence of the divergence theorem that in $R^m$

$$\int_{r=R} | u |^2 r = \int_{\partial V} \frac{\nu \cdot x}{r^2} | u |^2 \equiv \int_{V(R)} \text{div} \left( \frac{x | u |^2}{r^2} \right) = \int_{V(R)} \frac{2 \text{Re} \bar{u} \partial u}{r} + (m-2) \int_{V(R)} \frac{|u|^2}{r^2},$$

where $V(R)$ is the intersection of $V$ with a large ball of radius $R$. Recall that

$$u_r = \partial u + i\lambda u - [(m-1)/2r]u.$$

Therefore,

$$\int_{r=R} \frac{|u|^2}{r} \equiv \int_{\partial V} \frac{\nu \cdot x}{r^2} | u |^2 + \int_{V(R)} \frac{2 \text{Re} \bar{u} \partial u}{r} - \int_{V(R)} \frac{|u|^2}{r^2}.$$

It follows that

$$\int_{r=R} (| u |^2 / r) \leq \int_{\partial V} (\nu \cdot x | r^2 | | u |^2 + \int_{V(R)} | \partial u |^2.$$
The integral over $\partial V$ is bounded by our assumptions on $u$. Thus we may rewrite the preceding inequality as

$$\int_{r=R} (|u|^2/r) \leq \text{const} + \int_{V(R)} |D_1u|^2.$$  

By virtue of the radiation condition,

$$\int_{V(R)} |D_1u|^2 = \Theta(\ln R) \quad (R \to \infty).$$

Therefore

$$\int_{r=R} (|u|^2/r) \leq \Theta(\ln R) \quad (R \to \infty).$$

We now divide both sides of the last relation by $R^8$ to obtain the desired conclusion.

7. A PRIORI ESTIMATES IN WEIGHTED $L_2$ NORMS

The results obtained in Sections 3–6 together with an auxiliary estimate for $\|u/r\|_V$, to be derived in this section, imply the following theorem.

**Theorem 7.1.** Suppose $u$ is a solution of Problem P, with $f$ replaced by $g(x, \lambda)$, that lies in $C^2(V) \cap C^1(V^c)$. Assume that $V$ can be illuminated from the exterior. Finally assume that hypotheses (4.1)–(4.3) and (5.3) each hold. Then if $\lambda$ is sufficiently large, there exist positive constants $\Gamma_1$ and $\Gamma_2$, independent of $\lambda$ and $u$, such that

$$\|u/r\|_V \leq \Gamma_1[\|u_0\|_{\partial V} + \|u_T\|_{\partial V} + \|rg\|_V], \quad (7.1a)$$

and

$$\|u/r\|_V \leq \Gamma_2[\lambda \|u_0\|_{\partial V} + \|u_T\|_{\partial V} + \|rg\|_V]. \quad (7.1b)$$

Here $\|\cdot\|_S$ is the $L_2$ norm over the set $S$. The inequalities of Theorem 7.1 imply the following corollaries.

**Corollary 7.2.** Under the hypotheses of Theorem 7.1, if $\lambda$ is sufficiently large, there exists a positive constant $\Gamma_3$, independent of $\lambda$ and $u$ such that

$$E_{R_1}(e^{-\lambda u}) \leq \frac{\lambda^2}{2} \|u\|_{V(R_1)} + \|\text{grad } u\|_{V(R_1)} \leq \Gamma_3[\lambda^2 \|u_0\|_{\partial V} + \|u_T\|_{\partial V} + \|rg\|_V]$$
where \( E_{R_1} \) is the portion of the energy of \( u e^{-i\lambda t} \) contained in \( V(R_1) = V \cap \{ x | | x | \leq R_1 \} \) and

\[
\Gamma_3 = \left[ \frac{1}{V(R_1)} \right]^{r \rightarrow 2} \left[ \Gamma_1^2 + \Gamma_2^2 \right].
\]

**Corollary 7.3.** Under the hypotheses of Theorem 7.1, if \( \lambda \) is sufficiently large, then solutions of Problem \( P \) are unique in the class \( C^0(V) \cap C^1(\overline{V}) \).

The remainder of this section is devoted to proving Theorem 7.1. From the result obtained in Section 6 that (3.20) holds for functions which satisfy the radiation condition (1.3) and from (3.13), we conclude that (3.21) holds for the solution \( u \) of Problem \( P \). We now observe that the vector multiplier \( \mathbf{b} \) has been chosen so that on \( \partial V \), \( \nu^* \cdot \mathbf{b} > 0 \) (the auxiliary function \( \mathbf{I} \) is a positive constant on \( \partial V \)). The multiplier \( \rho \) is strictly positive on \( \partial V \). Therefore it follows from (3.19) that

\[
I_1 \geq K_2 \| \mathbf{u}^* \|_{\partial V}^2 - K_6 \| \mathbf{u}^* \|_{\partial V}^2 - \lambda^2 K_6 \| u \|_{\partial V}^2,
\]

where

\[
K_2 = (1 - 2\epsilon_1)(\inf_{\partial V} \nu^* \cdot \mathbf{b}) > 0 \quad (0 < \epsilon_1 < \frac{1}{2}),
\]

\[
K_6 = \sup_{\partial V} F_1 < \infty, \quad \text{and} \quad K_8 = \lambda^{-2} \sup_{\partial V} F_2 < \infty.
\]

Making use of (7.2), Proposition 4.1, and results (5.2) and (5.13) in (3.21), we conclude that if \( u \) is the solution of Problem \( P \), then for \( \lambda \) sufficiently large,

\[
K_1 \| h \mathbf{u} \|_{\partial V}^2 + \frac{1}{2} \int_V \mathcal{P} + K_3 \| \mathbf{u}^* \|_{\partial V}^2
\]

\[
\leq K_2 \| h \mathbf{u} \|_{\partial V}^2 + K_4 \| h^{-1}g \|_{\partial V}^2 + K_5 \| \mathbf{u}^* \|_{\partial V}^2 + \lambda^2 K_6 \| u \|_{\partial V}^2,
\]

where

\[
K_1 = \min(C_1, C_{11}) > 0
\]

and \( K_3 \) and \( K_4 \) are defined by (5.2) and (5.13). Note that the \( K_i \) (\( i = 1, \ldots, 5 \)) are independent of \( \lambda \) and \( K_6 = \mathcal{O}(1) \) as \( \lambda \rightarrow \infty \).

It follows from the ray equation \( x = \nu \sigma + X^0(\tau) \) that in each coordinate zone

\[
r = [\sigma^2 + 2(\nu \cdot X^0(\tau))\sigma + |X^0(\tau)|^2]^{1/2}.
\]

Since \( \nu \cdot X^0 > 0 \), \( r \geq \sigma \) on \( \overline{V} \) whenever \( \sigma(x) > 0 \). Thus there exist constants \( \omega_i \) (\( i = 1, 2 \)) such that

\[
\| h^{-1}g \|_{\partial V} \leq \omega_1 \| rg \|_{\partial V} \quad \text{and} \quad \| hw \|_{\partial V} \geq \omega_2 \| \omega/r \|_{\partial V},
\]

(7.5)
where $w$ is either $u$ or $\nabla u$. The inequalities (7.5) and (7.3) imply (3.22) for $\lambda$ sufficiently large, say $\lambda \geq \lambda_0$, with

$$\rho = \frac{\max(\omega_2 K_3, \omega_1 K_4, K_5, K_6 |_{\lambda = \lambda_0})}{\min(\omega_2 K_1, K_2)}.$$

(7.6)

It still appears that we are far from our goal to obtain a priori estimates for $\|u/r\|_V$, $\|\nabla u/r\|_V$, and $\|u_r\|_V$. For while the integrals on the left-hand side of (3.22) or (7.3) are the unknown quantities we wish to estimate a priori, they are bounded from above in (3.22) or (7.3) by a linear combination of given quantities and the unknown quantity $\|u/r\|_V$ or $\|hu\|_V$. However, we shall now demonstrate that this latter quantity can be bounded from above by the sum of small multiples of the quantities we desire to estimate in (7.1a) and small multiples of known quantities (see (7.9) below). These multiples can be made as small as we please by choosing $\lambda$ sufficiently large. By using this bound on $\|hu\|_V^2$ in (7.3) we obtain an inequality that immediately yields (7.1a). We then use the estimates for the quantities on the left-hand side of (7.1a) in the “small multiples” estimate (7.9) for $\|hu\|_V^2$ to obtain an inequality that immediately yields (7.1b).

We begin to carry out this program with the identity

$$\nabla \cdot (h^2 \bar{u} \nabla u) = h^2 |\nabla u|^2 - 2h^2 \bar{u}(\nabla \sigma \cdot \nabla u) - \lambda^2 h^2 n |u|^2 + (\bar{u} Lu) h^2.$$

(7.7)

This identity holds for solutions of Problem $P$. We integrate it over $V(R)$ and use the divergence theorem. The result after letting $R \to \infty$ is

$$-\int_{\partial V} h^2 \bar{u} u_{r,*} = \int_V [h^2 \{\bar{u}g + |\nabla u|^2\} - 2h^2 \bar{u}(\nabla \sigma \cdot \nabla u)]$$

$$- \lambda^2 \int_V h^2 n |u|^2 - \lim_{R \to \infty} \int_{r=R} h^2 \bar{u} u_r.$$

(7.8)

We next show that

$$\lim_{R \to \infty} \int_{r=R} h^2 \bar{u} u_r = 0.$$

(7.9)

To prove (7.9) we observe that

$$\left| \int_{r=R} h^2 \bar{u} u_r \right| = \left| \int_{r=R} h^2 \bar{u} D_1 u + \int_{r=R} \left( i\lambda - \frac{(m - 1)}{2r} \right) h^2 |u|^2 \right|$$

$$\leq \text{const} \int_{r=R} |D_1 u|^2 + \text{const} \left( \frac{\lambda}{r} + |\lambda| + \frac{m - 1}{2r} \right) \int_{r=R} \frac{|u|^2}{r^2}.$$

(7.10)
But by Lemma 6.1 and the radiation condition (1.3), the right-hand side of
the last inequality converges to 0 as $R \to \infty$. This implies (7.9).

We next estimate the various terms that remain in (7.8) with the help of the
inequality $ab \leq \frac{1}{2} [c \cdot |a|^2 + c^{-1} |b|^2]$. The result is

\[
\left(1 - \frac{2}{\lambda^2}\right) \int h^2n \cdot |u|^2 \leq \frac{\lambda^2}{2} \int_{\partial V} \left[ \epsilon_2 \cdot |u_v|^2 + h^4 \cdot |u|^2 \right] + \lambda^2 \int h^2 \left[ \frac{|\xi|^2}{4n} + \left(1 + \frac{h^2}{n}\right) \cdot |\nabla u|^2 \right].
\]  

(7.11)

If $\lambda$ is so large that $[1 - 2\lambda^{-2}] > 0$ and $n(x) \geq n_0$ on $V$ (see (4.1ii)), then (7.11)
implies the “small multiples” estimate

\[
\|hu\|^2 \leq \lambda^{-2}[1 - (2/\lambda^2)]^{-1} [D_4 \|u_v\|^2 + D_5 \|h\|^2],
\]  

(7.12)

where

\[
D_4 = \frac{1}{h} n_0^{-1}, \quad D_5 = n_0^{-1} [1 + \text{Sup}_V (h^2 n^{-1})];
\]

\[
D_3 = \frac{1}{h} n_0^{-1} \text{Sup}_V h^4, \quad D_4 = 1/4 n_0^2.
\]

Using (7.12) to estimate the unknown term on the right-hand side of (7.3),
we obtain the result that for all $\lambda \geq \lambda_0$

\[
\|u_v\|^2 + \|\nabla u\|^2 + \frac{1}{2} \left| \omega^{1/2} \left( \frac{c \cdot \nabla u}{\omega} - i\lambda u \right) \right|^2 \leq A_1 \lambda^2 \|u_v\|^2 + \|u_{T^*}\|_{\partial V}^2 + \|h^{-1}g\|_{\partial V}^2.
\]  

(7.13)

Here

\[
A_1 = \left( \text{Min} \left[ 1, K_1 - \frac{D_2 K_2}{\lambda_0^2} \left(1 - \frac{2}{\lambda_0^2}\right)^{-1}, K_2 - \frac{D_4 K_3}{\lambda_0^2} \left(1 - \frac{2}{\lambda_0^2}\right)^{-1} \right] \right)^{-1}
\]

\[
\cdot \text{Max} \left[ K_3, K_6 + \frac{K_5 D_3}{\lambda_0^4} \left(1 - \frac{2}{\lambda_0^2}\right)^{-1}, K_4 + \frac{K_8 D_4 (1 - 2\lambda_0^{-2})^{-1}}{\lambda_0^2 (1 + \lambda_0^2)} \right],
\]  

(7.14)

and the number $\lambda_0$ is so large that all the quantities involved in the definition
of $A_1$ are positive, $[1 - 2\lambda_0^2]$ is positive, and (7.3) holds for $\lambda \geq \lambda_0$.

Next we employ (7.13) and (7.5) in (7.12) to find that for $\lambda \geq \lambda_0$

\[
\|u_t\|_{\partial V}^2 \leq \lambda^{-2}[1 - (2/\lambda_0^2)]^{-1} A_2 \lambda^2 \|u_0\|_{\partial V}^2 + \|u_{T^*}\|_{\partial V}^2 + \|h^{-1}g\|_{\partial V}^2.
\]  

(7.15)
where
\[ A_2 = \frac{1}{\omega_2} \text{Max}[A_1 D_1 + A_1 D_2 + D_3, (A_1 D_1 + A_1 D_2), (A_1 D_1 + A_1 D_2 + (D_d^2 + D_2)^3)^{-1}], \]

(7.16)

where \( \lambda_0 \) is so large that (7.13) and (7.5) both hold.

The estimates of Theorem 7.1 are direct consequences of (7.13), (7.15), and (7.5). Corollary 7.2 is a simple consequence of Theorem 7.1. Corollary 7.3 follows from the pointwise a priori estimate for \( |u(x, \lambda)| \) that we establish in the next section using Theorem 7.1.

8. An a Priori Estimate for \( |u(x, \lambda)| \)

Our final objective in this paper is to obtain an a priori estimate for \( |u(x, \lambda)| \) which holds uniformly in \( x \) on \( V \) for \( \lambda \geq \lambda_0 \). We first derive an upper bound for \( |u| \) in terms of

\[ \| u \|_\nu, \max_{x \in \partial V} |u_0(x)|, \quad \| u/r \|_\nu, \quad \text{and} \quad \| u_{rr} \|_{\partial V}. \]

Then, making use of inequalities (7.1) of Theorem 7.1, to estimate the last two quantities, we obtain the desired pointwise estimate for \( |u(x, \lambda)| \).

Let \( H(x, x') \) be the (fundamental) solution of

\[ \Delta H + \lambda^2 H = \delta(x, x') \]

(8.1)

that satisfies the radiation condition (1.3); namely, let

\[ H(x, x') = e^{i|x-x'|} |x - x'|, \quad (x, x' \in \mathbb{R}^3), \quad (8.2a) \]

\[ H(x, x') = (i/4) H_0'(x - x'), \quad (x, x' \in \mathbb{R}^2), \quad (8.2b) \]

where \( H_0'(z) \) is the Hankel function of first kind of order zero.

As usual, we begin with an identity to which we shall apply the divergence theorem:

\[ \nabla \cdot (u \nabla H) - \nabla \cdot (H \nabla u) = -H(\Delta u + \lambda^2 n(x')u) + \lambda^2 u(x') H_u + u(\Delta H + \lambda^2 H) - \lambda^2 u H. \]

(8.3)

Here \( u \) is the solution of Problem P with \( f \) replaced by \( g \) and \( H \) is as just defined above. The variables of differentiation in (8.3) are the \( x' \) variables. We integrate (8.3) over the region \( V(R) \), which is the intersection of \( V \) with a large ball of radius \( R \). The result after applying the divergence theorem is

\[ u(x, \lambda) = \int_{V(R)} H(x, x')(g(x', \lambda) - \lambda^2 [n(x') - 1] u(x')) \, dx' \]

\[ + \int_{\partial V} \{H(x, x') u_\nu(x', \lambda) - u(x', \lambda) H_\nu(x, x')\} \, dS(x') \]

(8.4)

\[ + \int_{|x'|=R} (u H_\nu - H u_\nu) \, dS(x'). \]
Since
\[ \int_{|x'|=R} (uH_r - Hu_r) = \int_{|x'|=R} (u\mathcal{D}_1 H - H\mathcal{D}_1 u), \]
it is possible to conclude that
\[ \lim_{R \to \infty} \int_{|x'|=R} (uH_r - Hu_r) = 0. \quad (8.5) \]

To see this note first that for \( \delta > 0 \)
\[ \left| \int_{|x'|=R} u\mathcal{D}_1 II \right| \leq \left[ \int_{|x'|=R} \frac{|u|^2}{r^{1+\delta}} \right]^{1/2} \left[ \int_{|x'|=R} r^{1+\delta} |\mathcal{D}_1 II|^2 \right]^{1/2}. \quad (8.6) \]
The first integral on the right-hand side of (8.6) has the limit 0 as \( R \to \infty \) by Lemma 6.1. The second integral is \( O(R^{\delta/3}) \) for \( m = 2 \) or 3 by the properties of \( H \). Thus choosing \( \delta = 1 \), we conclude that the left-hand side of (8.6) has limit zero as \( R \to \infty \). Similarly we find that
\[ |H\mathcal{D}_1 u| = o(R^{-1/2}) \quad (R \to \infty). \]

These results imply (8.5).

We let \( R \to \infty \) in (8.4) and use (8.5). With a little care we conclude from the resultant identity that [9, Lemma 3]
\[ |u(x, \lambda)| \leq r^{(1-m)/2} \left\{ \max_{x \in \mathcal{P}} \|r^{(m-1)/2}(x) r^{-1} H(x, \cdot)r\|_Y \right\} \|rg]\|_Y \]
\[ + \lambda^2 \left[ \max_{x \in \mathcal{P}} r^2 \left| \frac{1}{n} \right| \right] \left[ \max_{x \in \mathcal{P}} \|r^{-1} H(x, \cdot)r\|_Y \right] \|u/r\|_Y \]
\[ + \lambda \left[ \max_{x \in \mathcal{V}} \int_{\partial \mathcal{V}} \lambda^{-1} r^{(m-1)/2}(x) \left| H_{\mu}(x, \cdot) \right| \max_{x \in \mathcal{P}} |u_0| \right] \]
\[ + \left[ \max_{x \in \mathcal{P}} \|r^{(m-1)/2}(x) H(x, \cdot)r\|_Y \right] \|u\|_{\partial \mathcal{V}}. \quad (8.7) \]

It can be shown that the factors involving \( H \) in (8.7) are bounded by \( C\lambda^{-3+m}/2 \) \( (m = 2, 3) \), where \( C \) is a constant independent of \( \lambda \) (see [3], for example). Consequently, it follows from (8.7) that if \( \lambda \) is sufficiently large,
\[ |u(x, \lambda)| \leq C'\lambda^{-(3-m)/2} r^{(1-m)/2} \left\{ \|rg\|_Y + \lambda^2 \|u/r\|_Y + \lambda \max_{x \in \mathcal{P}} |u_0| + \|u\|_{\partial \mathcal{V}} \right\}, \quad (8.8) \]
where \( C' \) is some positive number independent of \( x \) and \( \lambda \).

Finally, using Theorem 1 to estimate the terms that are a priori unknown on the right-hand side of (8.8), we obtain:
TEOREM 8.1. If the hypotheses of Theorem 7.1 hold and \( \lambda \) is sufficiently large \( (\lambda \geq \lambda_0) \), then there exists a positive constant \( \Gamma_3 \), independent of \( \lambda \) and \( x \), such that for \( \lambda \geq \lambda_0 \) and all \( x \) in \( V \)

\[
|u(x, \lambda)| \leq \Gamma_3^4(1+m)^2 \tau(1-m)^2[\|rg\|_{\nu} + \|u\|_{\nu} + \lambda \max_{x \in \partial V} |u_x|],
\]

\( (8.9) \)

where \( u \) is the solution of Problem \( P \).

REFERENCES

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