Asymptotic distribution of the even and odd spectra of real symmetric Toeplitz matrices

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Dedicated to Hans Schneider on the occasion of his 70th birthday

Abstract

If \( T_n = (t_{r-s})_{r,s=0}^n \) is a real symmetric Toeplitz (RST) matrix then \( \mathbb{R}^n \) has a basis consisting of \( \lfloor n/2 \rfloor \) eigenvectors \( x \) satisfying (A) \( Jx = x \) and \( \lceil n/2 \rceil \) eigenvectors \( y \) satisfying (B) \( Jy = -y \), where \( J \) is the flip matrix. We say that an eigenvalue \( \lambda \) of \( T_n \) is even if a \( \lambda \)-eigenvector of \( T_n \) satisfies (A), or odd if a \( \lambda \)-eigenvector of \( T_n \) satisfies (B). We call the collection of even (odd) eigenvalues of \( T_n \) the even (odd) spectrum of \( T_n \). In the case where \( t_r = 1/\pi \int_0^\pi f(x) \cos rx \, dx \) a great deal is known about the asymptotic distribution of the eigenvalues of \( T_n \) as \( n \to \infty \), under suitable assumptions on \( f \). However, the question of the separate asymptotic distributions of the even and odd spectra does not seem to have been raised. This is the subject of this paper. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

There are many results on the asymptotic distribution of the eigenvalues of the family \( \{ T_n \}_{n=1}^\infty \) of Hermitian Toeplitz matrices defined by

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\[ T_n = (t_{r,s})_{r,s=1}^n, \quad n = 1, 2, 3, \ldots, \]

where

\[ t_r = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{irx} \, dx \tag{1} \]

and \( f \) is a real valued function. This paper treats a related problem for real symmetric Toeplitz (RST) matrices which, to the author’s knowledge, has not attracted previous attention.

We assume henceforth that \( f \in L^2[0, \pi] \) and \( f(-x) = f(x) \), so that (1) reduces to

\[ t_r = \frac{1}{\pi} \int_{0}^{\pi} f(x) \cos rx \, dx \tag{2} \]

and \( T_n \) is an RST matrix for every \( n \). From a result of Szegö [5, p. 64], the eigenvalues

\[ \lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \cdots \leq \lambda_n^{(n)} \]

of \( T_n \) satisfy the inequalities

\[ \alpha \leq \lambda_i^{(n)} \leq \beta, \]

where \( \alpha \) and \( \beta \) are the essential lower and upper bounds of \( f \) on \([0, \pi]\); that is, \( \alpha \) is the largest number and \( \beta \) the smallest such that \( \alpha \leq f(x) \leq \beta \) almost everywhere on \([0, \pi]\). (These definitions of \( \alpha \) and \( \beta \) apply throughout this paper.)

If \( \lambda \) is an eigenvalue of an RST matrix \( T_n \) then \( T_n \) has \( \lambda \)-eigenvector \( x \) such that \( Jx = \pm x \), where \( J \) is the flip matrix. Following Andrew [1], we say that an \( n \)-vector \( x \) is symmetric (skew-symmetric) if \( Jx = x \) (\( Jx = -x \)). We say that \( \lambda \) is an even (odd) eigenvalue of \( T_n \) if \( T_n \) has a symmetric (skew-symmetric) \( \lambda \)-eigenvector. In fact, \( T_n \) always has exactly \( \lfloor n/2 \rfloor \) even eigenvalues and \( \lceil n/2 \rceil \) odd eigenvalues. (A repeated eigenvalue is necessarily both even and odd [3].) We call the set of even (odd) eigenvalues of \( T_n \) the even (odd) spectrum of \( T_n \). In this paper we are interested in the ordering relationship between the even and odd spectra. For example, it was shown in [8] that if \( f \) is monotonic then the even and odd spectra of \( T_n \) are interlaced for every \( n \). Although it is possible to construct pathological examples where the even eigenvalues are at one end of the spectrum and the odd eigenvalues are at the other [7], extensive computational experiments performed for [6] show that it is rare to encounter long runs (more than three or four) of successive eigenvalues of the same parity.

In this paper we provide a reason for this behavior. Specifically, we show that if \( f \in L^2[0, \pi] \), then the even and odd spectra of \( T_n \) are asymptotically
distributed in a sense like the spectra of $T_{(n/2)}$ and $T_{(n/2)}$, respectively, and that if $f$ is bounded then this result holds in a stronger sense.

2. Results

The following theorems are obtained by specializing results stated explicitly in [2], but already implicit in [1].

**Theorem 1.** The even eigenvalues

\[ \mu_1^{(2n)} \leq \mu_2^{(2n)} \leq \cdots \leq \mu_n^{(2n)} \]

of $T_{2n}$ are the eigenvalues of

\[ A_n = (t_{r-s} + t_{r+s-1})_{r,s=1}^n, \tag{3} \]

and the odd eigenvalues

\[ \nu_1^{(2n)} \leq \nu_2^{(2n)} \leq \cdots \leq \nu_n^{(2n)} \]

of $T_{2n}$ are the eigenvalues of

\[ B_n = (t_{r-s} - t_{r+s-1})_{r,s=1}^n. \tag{4} \]

**Theorem 2.** The even eigenvalues

\[ \mu_1^{(2n+1)} \leq \mu_2^{(2n+1)} \leq \cdots \leq \mu_n^{(2n+1)} \]

of $T_{2n+1}$ are the eigenvalues of $C_n = (c_{r,s})_{r,s=0}^n$, where

\[ c_{rs} = \begin{cases} 
  t_0 & \text{if } r = s = 0, \\
  \sqrt{2t_r} & \text{if } s = 0, \\
  \sqrt{2t_s} & \text{if } r = 0, \\
  t_{r-s} + t_{r+s} & 1 \leq r, s \leq n,
\end{cases} \tag{5} \]

and the odd eigenvalues

\[ \nu_1^{(2n+1)} \leq \nu_2^{(2n+1)} \leq \cdots \leq \nu_n^{(2n+1)} \]

of $T_{2n+1}$ are the eigenvalues of

\[ D_n = (t_{r-s} - t_{r+s})_{r,s=1}^n. \]

If $f \in L^2[0, \pi]$ then $\sum_{r=1}^{\infty} t_r^2 < \infty$. Conversely, by the Riesz–Fischer theorem, the hypothesis of the following theorem implies that (2) holds for some $f \in L^2[0, \pi]$.

**Theorem 3.** If $\sum_{r=1}^{\infty} t_r^2 < \infty$ then

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( \mu_i^{(2n)} - \mu_i^{(n)} \right)^2 = 0, \tag{6} \]

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( \nu_i^{(2n)} - \nu_i^{(n)} \right)^2 = 0, \tag{7} \]

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( \mu_i^{(2n+1)} - \mu_i^{(n+1)} \right)^2 = 0. \tag{8} \]
and
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( v_{i}^{(2n+1)} - \lambda_{i}^{(n)} \right)^{2} = 0. \tag{9}
\]

The proof uses the following lemma.

**Lemma 1.** If \( \{a_r\} \) is a sequence of nonnegative numbers such that \( \sum_{r=1}^{\infty} a_r < \infty \) then
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{r,s=1}^{n} a_{r+s} = 0.
\]

The proof of the lemma uses the estimate
\[
\frac{1}{n} \sum_{r,s=1}^{n} a_{r+s} = \frac{1}{n} \sum_{r=1}^{n} ra_{r+1} + \frac{1}{n} \sum_{r=n+2}^{2n} (2n - r + 1)a_r
\leq \frac{1}{n} \sum_{r=1}^{n} ra_{r+1} + \sum_{r=n+2}^{\infty} a_r. \tag{10}
\]

We omit the details.

**Proof of Theorem 3.** We begin with the proof of (6). Since \( \lambda_{1}^{(n)}, \ldots, \lambda_{n}^{(n)} \) are the eigenvalues of \( T_n \) and \( \mu_{1}^{(2n)}, \ldots, \mu_{n}^{(2n)} \) are the eigenvalues of \( A_n \) in (3), the Wielandt–Hoffman theorem implies that
\[
\sum_{i=1}^{n} \left( \mu_{i}^{(2n)} - \lambda_{i}^{(n)} \right)^{2} \leq \sum_{r,s=1}^{n} t_{r+s-1}^{2},
\]
where the quantity on the right is the square of the Frobenius norm of \( T_n - A_n \). Since \( \sum_{r=1}^{\infty} t_{r-1}^{2} < \infty \) by assumption, Lemma 1 implies (6). Since \( v_{1}^{(2n)}, \ldots, v_{n}^{(2n)} \) are the eigenvalues of \( B_n \) in (4) and \( v_{1}^{(2n+1)}, \ldots, v_{n}^{(2n+1)} \) are the eigenvalues of \( D_n \), we obtain (7) and (9) similarly. As for (8), since \( v_{1}^{(2n+1)}, \ldots, v_{n}^{(2n+1)} \) are the eigenvalues of \( C_n \) in (5), the Wielandt–Hoffman theorem implies that
\[
\sum_{i=1}^{n+1} \left( \mu_{i}^{(2n+1)} - \lambda_{i}^{(n+1)} \right)^{2} \leq 2(\sqrt{2} - 1)^{2} \sum_{r=1}^{n} t_{r}^{2} + \sum_{r,s=1}^{n} t_{r+s}^{2},
\]
where the quantity on the right is the square of the Frobenius norm of \( C_n - T_{n+1} \). Therefore Lemma 1 and our assumption on \( \{t_r\} \) imply (8).
Theorem 4. If $t_r = O(r^{-\rho})$ with $\rho > 1/2$ then
\[
\frac{1}{n} \sum_{i=1}^{n} \left( \mu_i^{(2n)} - \lambda_i^{(n)} \right)^2 = O(\phi(n)),
\]
\[
\frac{1}{n} \sum_{i=1}^{n} \left( \nu_i^{(2n)} - \lambda_i^{(n)} \right)^2 = O(\phi(n)),
\]
\[
\frac{1}{n} \sum_{i=1}^{n} \left( \mu_i^{(2n+1)} - \lambda_i^{(n+1)} \right)^2 = O(\phi(n)),
\]
and
\[
\frac{1}{n} \sum_{i=1}^{n} \left( \nu_i^{(2n+1)} - \lambda_i^{(n)} \right)^2 = O(\phi(n)),
\]
with
\[
\phi(n) = \begin{cases} 
  n^{-2\rho+1} & \text{if } 1/2 < \rho < 1, \\
  n^{-1} \log n & \text{if } \rho = 1, \\
  n^{-1} & \text{if } \rho > 1.
\end{cases}
\]

The proof can be obtained by using (10) with appropriate choices of $\{a_i\}$. The assumption holds, for example, if $f$ satisfies the Lipschitz condition
\[
|f(x_1) - f(x_2)| \leq |x_1 - x_2|^\rho, \quad 0 \leq x_1, x_2 \leq \pi,
\]
with $1/2 < \rho < 1$, or with $\rho = 1$ if $f$ is of bounded variation [4]. Of course, in both cases the stated condition is sufficient, but not necessary.

If $f$ in (2) is bounded then stronger results can be obtained. We need the following definition.

**Definition 1.** Suppose that $f$ is bounded. For each $n$ let $\{a_i^{(n)}\}_{i=1}^{n}$ and $\{b_i^{(n)}\}_{i=1}^{n}$ be sets of real numbers such that
\[
\alpha \leq a_i^{(n)}, b_i^{(n)} \leq \beta, \quad i = 1, 2, \ldots, n.
\]
Then we say that the sets $\{a_i^{(n)}\}_{i=1}^{n}$ and $\{b_i^{(n)}\}_{i=1}^{n}$ are **absolutely equally distributed** if
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left| G(b_i^{(n)}) - G(a_i^{(n)}) \right| = 0
\]
whenever $G$ is continuous on $[\alpha, \beta]$.

If the absolute value signs are deleted from (11) this becomes Weyl’s definition of equal distribution of $\{a_i^{(n)}\}_{i=1}^{n}$ and $\{b_i^{(n)}\}_{i=1}^{n}$ [5].
Theorem 5. If \( f \) is bounded then the sets

\[
\{ \mu_i^{(2n)} \}_{i=1}^n, \quad \{ \nu_i^{(2n)} \}_{i=1}^n, \quad \{ \nu_i^{(2n+1)} \}_{i=1}^n, \quad \text{and} \quad \{ \lambda_i^{(n)} \}_{i=1}^n
\]

are all absolutely equally distributed, as are the sets

\[
\{ \mu_i^{(2n+1)} \}_{i=1}^{n+1} \quad \text{and} \quad \{ \lambda_i^{(n+1)} \}_{i=1}^{n+1}.
\]

Proof. We begin by showing that if \( k \) is an arbitrary positive integer then (11) holds with \( a_i = \lambda_i^{(n)}, \quad b_i = \mu_i^{(n)}, \quad \text{and} \quad G(u) = u^k \). Let

\[
W_n^{(k)} = \sum_{i=1}^n \left| \left( \mu_i^{(2n)} \right)^k - \left( \lambda_i^{(n)} \right)^k \right|.
\]

By the mean value theorem,

\[
W_n^{(k)} \leq M_k \sum_{i=1}^n \left| \mu_i^{(2n)} - \lambda_i^{(n)} \right|,
\]

with \( M_k = k \max \{|x|^{k-1}, |\beta|^{k-1}\} \). Now Schwarz’s inequality implies that

\[
W_n^{(k)} \leq M_k \sqrt{nS_n},
\]

where

\[
S_n = \sum_{i=1}^n \left( \mu_i^{(2n)} - \lambda_i^{(n)} \right)^2.
\]

If \( f \) is bounded then \( f \in L_2[0, \pi] \), so \( \sum_{i=1}^\infty \hat{t}_i^2 < \infty \). From the proof of Theorem 3, this implies that \( \lim_{n \to \infty} S_n/n = 0 \). Therefore \( \lim_{n \to \infty} W_n^{(k)}/n = 0 \).

It now follows that (11) holds if \( G \) is a polynomial. Now suppose that \( G \) is an arbitrary continuous function on \([x, \beta]\), and let

\[
W_n(G) = \sum_{i=1}^n \left| G(\mu_i^{(2n)}) - G(\lambda_i^{(n)}) \right|.
\]

Let \( \epsilon > 0 \) be given. From the Weierstrass approximation theorem, there is a polynomial \( P \) such that \( |G(u) - P(u)| < \epsilon \) for all \( u \) in \([x, \beta]\). Therefore \( W_n(G) < W_n(P) + 2n\epsilon \), and

\[
\limsup_{n \to \infty} \frac{W_n(G)}{n} \leq \lim_{n \to \infty} \frac{W_n(P)}{n} + 2\epsilon = 2\epsilon.
\]

Now let \( \epsilon \to 0 \) to conclude that \( \lim_{n \to \infty} W_n(G)/n = 0 \).
Similar proofs show that \( \{v_i^{(2n)}\}_{i=1}^{n} \) and \( \{v_i^{(2n+1)}\}_{i=1}^{n} \) are absolutely equally distributed.\( \{\lambda_i^{(n)}\}_{i=1}^{n} \) and \( \{\mu_i^{(2n+1)}\}_{i=1}^{n} \) are absolutely equally distributed.

An argument of Widom [9] yields the following lemma.

\textbf{Lemma 2.} Suppose that \( f \) is bounded and, for each \( n \), \( \{a_i^{(n)}\}_{i=1}^{n} \) is a set of real numbers in \([\alpha, \beta]\), such that
\[
\lim_{n \to \infty} \frac{G(a_1^{(n)}) + G(a_2^{(n)}) + \cdots + G(a_n^{(n)})}{n} = \int_0^\pi G(f(x)) \, dx
\]
(12)
whenever \( G \) is continuous on \([\alpha, \beta]\). Let \([a, b]\) be a subinterval of \([\alpha, \beta]\) and suppose that the set \( \{x | f(x) = a \text{ or } f(x) = b\} \) has measure zero. Let \( \Omega(a, b) \) be the measure of the set \( \{x | a \leq f(x) \leq b\} \), and let \( C(a, b, n) \) be the cardinality of the set \( \{i | a \leq a_i^{(n)} \leq b\} \). Then
\[
\lim_{n \to \infty} \frac{1}{n} C(a, b, n) = \frac{1}{\pi} \Omega(a, b).
\]
A classical result of Szegő [5, pp. 64, 65] says that if \( f \) is bounded then (12) holds with \( a_i^{(n)} = \lambda_i^{(n)} \). Therefore, Theorem 5 implies that (12) also holds with
\[
a_i^{(n)} = v_i^{(2n)}, \quad \mu_i^{(2n)}, \quad v_i^{(2n+1)}, \quad \text{or } \mu_i^{(2n+1)}.
\]
This implies the following theorem.

\textbf{Theorem 6.} Suppose that \( f \) is bounded and \([a, b]\) is a subinterval of \([\alpha, \beta]\) such that the set \( \{x | f(x) = a \text{ or } f(x) = b\} \) has measure zero. Let \( \Omega(a, b) \) be the measure of the set \( \{x | a \leq f(x) \leq b\} \). Let \( L(a, b, n), M(a, b, n), \) and \( N(a, b, n) \) be the cardinalities of the sets \( \{i | a \leq \lambda_i^{(n)} \leq b\}, \{i | a \leq \mu_i^{(n)} \leq b\}, \) and \( \{i | a \leq v_i^{(n)} \leq b\} \), respectively. Then
\[
\lim_{n \to \infty} \frac{1}{n} L(a, b, n) = \lim_{n \to \infty} \frac{1}{n} M(a, b, n) = \lim_{n \to \infty} \frac{1}{n} N(a, b, n) = \frac{1}{\pi} \Omega(a, b).
\]
(13)
The assertion in (13) concerning \( L(a, b, n) \) is well known [8]; however, we believe that the assertions concerning \( M(a, b, n) \) and \( N(a, b, n) \) have not been explicitly stated elsewhere. Nevertheless, it must be noted that since the matrices \( A_n, B_n, C_{n-1}, \) and \( D_n \) all differ from \( T_n \) by matrices with Frobenius norm \( o(\sqrt{n}) \) as \( n \to \infty \), these assertions can also be obtained directly from Szegő’s theory of canonical distributions [5]. (For a specific example dealing with \( D_n \), see [5, pp. 235, 236, item (c)].) However, the proof given here is elementary by comparison with Szegő’s theory, and easily accessible to those who – like the author – are not well versed in this theory.
References