CONVEX SUBSETS OF $2^n$ AND BOUNDED TRUTH-TABLE REDUCIBILITY

Louise HAY
University of Illinois at Chicago Circle, Chicago, IL 60680, U.S.A.

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Let $2^n$ be the set of $n$-tuples of 0's and 1's, partially ordered componentwise. A characterization is given of the possible decompositions of arbitrary subsets of $2^n$ as disjoint unions of sets which are convex in this ordering; this result is used to obtain a decomposition theorem for Boolean functions in terms of monotone functions. The second half of the paper contains applications to recursion theory; in particular, canonical forms for certain minimum-norm bounded-truth-table reductions are obtained.

1. Introduction

Let $2^n$ denote the set of all $n$-tuples of 0's and 1's, partially ordered componentwise. In this paper we characterize (in terms of their characteristic functions) the subsets of $2^n$ which are convex in this ordering, and describe the possible decompositions of arbitrary subsets as unions of convex sets. These results are then used to prove a decomposition theorem for Boolean functions in terms of monotone functions. A related theorem was proved by Gilbert [2, Theorem 6] who pointed out that it was not especially helpful for minimizing negations. It may be expected to be useful, however, in a context where monotone Boolean functions are basic in some sense. This is the case in the bounded-truth-table reducibility of recursion theory; in the second half of the paper the decomposition theorem will be applied to obtain a canonical form for “shortest” bounded-truth-table reductions.

2. Notation and terminology

If $b$ is a mapping from $\{1, \ldots, n\}$ to $\{0, 1\}$, $b(i)$ will be denoted as $b_i$ and $(b_1, \ldots, b_n)$ as $b$; $2^n$ then denotes the set of all such $b$'s. $\bar{b}$ is defined by $\bar{b_i} = 1 - b_i$, $1 \leq i \leq n$. If $a_i = 0$ and $b_i = 1$ for $0 \leq i \leq n$ we denote $a$ by 0 and $b$ by 1. $2^n$ is partially ordered in the usual way by defining $a \leq b$ iff $a_i \leq b_i$ for $0 \leq i \leq n$. A Boolean function $f$ mapping $2^n$ to $\{0, 1\}$ is called increasing if $a \leq b$ implies $f(a) \leq f(b)$, decreasing if $a \leq b$ implies $f(a) \geq f(b)$. Boolean forms in the variables $x_1, \ldots, x_n$ will in general be identified with the Boolean functions they determine.
A subset of $2^n$ is convex with respect to the partial ordering of $2^n$ if $a, b \in S$ and $a \preceq c \preceq b$ implies $c \in S$, for all $a, b, c \in 2^n$. In this section we characterize the convex subsets of $2^n$ (Theorem 3.3) and describe the possible decompositions of arbitrary subsets of $2^n$ into unions of subsets (Theorem 3.9). We shall require the following two lemmas:

**Lemma 3.1.** Suppose $f$ is a Boolean function for which there exist increasing functions $g_1, \ldots, g_{2m}$, $m \geq 1$, such that $f = \sum_{i=1}^{m} g_{2i-1} g_{2i}$; then there are increasing functions $f_1, \ldots, f_{2m}$ satisfying $f_1 \geq f_2 \geq \cdots \geq f_{2m}$ such that $f = \sum_{i=1}^{m} f_{2i-1} f_{2i}$ (and therefore also $f = f_1 + f_2 + \cdots + f_{2m}$).

**Proof.** By induction on $m$. The argument is analogous to that used in [1, Proposition 1] to prove a related theorem concerning unions of differences of sets; we include it here for completeness. For $m = 1$, suppose $f = g_1 g_2$ where $g_1, g_2$ are increasing. Let $f_1 = g_1$ and $f_2 = g_2$; then $f_1, f_2$ are increasing, $f_1 \geq f_2$ and $f = f_1 f_2$. For $m = 2$, suppose $f = g_1 g_2 + g_3 g_4$ where $g_1, g_3, g_2, g_4$ are increasing. By the case $m = 1$ we may assume $g_1 \geq g_2$ and $g_3 \geq g_4$. Let

$$f_1 = g_1 + g_3, \quad f_2 = g_2 + g_4 + g_1 g_3, \quad f_3 = f_1 f_2, \quad f_4 = g_2 g_4.$$  

Then $f_1, f_2, f_3, f_4$ are increasing, $f_1 \geq f_2 \geq f_3 \geq f_4$, and we note for later use that $f_2 \geq g_4$; it is now easily verified that $f_1 f_2 = g_1 g_2 g_3 + g_1 g_3 g_4$ and $f_3 f_4 = g_1 g_2 g_3 + g_1 g_2 g_4$ so that $f_3 f_4 = g_1 g_2 g_3 + g_1 g_2 g_4 = f$.

Now assume the lemma holds for $m$ and let $f = \sum_{i=1}^{m+1} g_{2i-1} g_{2i}$ where $g_1, \ldots, g_{2m+2}$ are increasing.

By the induction hypothesis, we may assume that $g_3 \geq g_4 \geq \cdots \geq g_{2m+2}$. As in the case $m = 2$, replace $g_1, g_2, g_3, g_4$ by increasing functions $f_1', f_2', f_3', f_4'$ satisfying $f_1' \geq f_2' \geq f_3' \geq f_4'$ and

$$f = f_1' f_2' + f_3' f_4' + \sum_{i=3}^{m+1} g_{2i-1} g_{2i}.$$
Let
\[ g = f_3 f_4 + \sum_{i=1}^{m+1} g_{2i-1} g_{2i}. \]

Then \( g \leq f_3 + g_5 \) since \( g_5 \geq g_6 \geq \cdots \geq g_{2m+2} \); but \( f_2' \geq f_3 \) and, as noted above, \( f_2' \geq g_4 \geq g_5 \) and hence \( g \leq f_2' \). By the induction hypothesis, there are increasing functions \( g_3', \ldots, g_{2m+2}' \) satisfying \( g_3' \geq g_4' \geq \cdots \geq g_{2m+2}' \) such that \( g = \sum_{i=2}^{m+1} g_{2i-1} g_{2i} \). Let \( f_i = f_i' \) for \( i = 1, 2 \) and \( f_i = g_i f_i' \) for \( 3 \leq i \leq 2m+2 \). Then \( f_1, \ldots, f_{2m+2} \) are increasing, \( f_1 \geq f_2 \geq \cdots \geq f_{2m+2} \), and \( g \leq f_2' \) implies \( g = \sum_{i=2}^{m+1} f_{2i-1} f_{2i} \); hence

\[ f = f_1 f_2' + g = \sum_{i=1}^{m+1} f_{2i-1} f_{2i} \]

which completes the induction.

**Lemma 3.2.** For each \( a \in 2^n \), let \( a^+ \) denote the product of all the variables \( x_i \) such that \( a_i = 1 \) and \( a^- \) the sum of all the variables \( x_i \) such that \( a_i = 0 \), \( 1 \leq i \leq n \). (By the usual convention, \( a^+ = 1 \) if \( a = 0 \) and \( a^- = 0 \) if \( a = 1 \).) Then for all \( a, b \in 2^n \),

(a) \( a^+ \) is an increasing function \( \neq 0 \);
(b) \( b^- \) is an increasing function \( \neq 1 \);
(c) \( a \leq b \iff a^+(b) = 1 \iff b^-(a) = 0 \).

**Proof.** It is clear that \( a^+ \neq 0 \) and \( b^- \neq 1 \); that they are increasing functions follows from the well-known characterization of increasing functions as the sums of products of the variables \( x_i \) (given, e.g., in [3, Theorem 5, p. 139]). For (c), note that

\[ a \leq b \iff (\forall i)_{1 \leq i \leq n} (a_i = 1 \rightarrow b_i = 1) \]
\[ a^- (b) = 1 \]

and

\[ a \leq b \iff (\forall i)_{1 \leq i \leq n} (b_i = 0 \rightarrow a_i = 0) \]
\[ b^- (a) = 0. \]

The convex subsets of \( 2^n \) can now be characterized as follows:

**Theorem 3.3.** Let \( S \) be a subset of \( 2^n \), and let \( a_1, \ldots, a_m \), \( m \geq 0 \), be the minimal elements of \( S \). For each \( j \) let \( b_{1j}, \ldots, b_{mj} \), \( M_j \geq 1 \), be the maximal elements \( b \) of \( S \) satisfying \( a_i \leq b \). The following conditions are then equivalent:

(a) \( S \) is convex;
(b) \( S = \bigcup_{j=1}^m \bigcup_{i=1}^{M_j} \{ a : a_i \leq a \leq b_{ij} \} \);
(c) there are increasing functions \( f, g \) such that \( C_S = f g \).
Moreover, if (c) holds, then

(d1) \( S = \emptyset \iff f \leq g \); 
(d2) \( \emptyset \in S \iff f = 1 \) and \( g \neq 1 \); 
(d3) \( 1 \in S \iff f \neq 0 \) and \( g = 0 \); 
(d4) \( S = 2^+ \iff f = 1 \) and \( g = 0 \); 
(d5) \( \emptyset \notin S \) and \( 1 \notin S \iff f \leq g \) or \( f \neq 0, 1 \) and \( g \neq 0, 1 \).

**Proof.** (a) \( \rightarrow \) (b): Assume \( S \) is convex. If \( S = \emptyset \), then \( m = 0 \) and \( \bigcup_{j=1}^{m-1} = \emptyset = S \). If \( S \neq \emptyset \) then since every element of \( S \) lies between a minimal and a maximal element,

\[
S \subseteq \bigcup_{j=1}^{m} \bigcup_{i=1}^{M_j} \{ a : a_i \leq a \leq b_i \}.
\]

The converse inclusion follows from the convexity of \( S \).

(b) \( \rightarrow \) (c): Assume

\[
S = \bigcup_{j=1}^{m} \bigcup_{i=1}^{M_j} \{ a : a_i \leq a \leq b_i \}.
\]

If \( m = 0 \), then \( S = \emptyset \) and \( C_s = f \bar{g} \) for \( f = 0 \), \( g = 1 \). If \( m > 0 \), then

\[
a \in S \iff (\exists j)_{1 \leq j \leq m} (\exists i)_{1 \leq i \leq M_j} (a_i \leq a \leq b_i) \iff [(\exists j)_{1 \leq j \leq m} (a_i \leq a)] \text{ and } [(\forall j)_{1 \leq j \leq m} (a_i \leq a) \implies (\exists i)_{1 \leq i \leq M_j} (a_i \leq b_i)].
\]

Let \( f = \sum_{j=1}^{m} a_j^+ \), \( g = \sum_{j=1}^{m} (a_j^+ - b_j^-) \). By Lemma 3.2, \( f \) and \( g \) are increasing functions, and

\[
a \in S \iff [(\exists j)_{1 \leq j \leq m} a_j^+(a) = 1] \text{ and } [(\forall j)_{1 \leq j \leq m} a_j^+(a) = 0 \implies (\exists i)_{1 \leq i \leq M_j} b_i^-(a) = 0] \iff f(a) = 1 \text{ and } g(a) = 0.
\]

Hence \( C_s = f \bar{g} \).

(c) \( \rightarrow \) (a): Assume \( C_s = f \bar{g} \) where \( f, g \) are increasing functions. Then

\[
a \in S \iff f \bar{g}(a) = 1 \iff f(a) = 1 \text{ and } g(a) = 0.
\]

Suppose \( a \leq c \leq b \) where \( a, b \in S \); then \( f(c) = 1 \) and \( g(b) = 0 \). Moreover, since \( f, g \) are increasing functions, \( f(a) = 1 \) implies \( f(c) = 1 \) and \( g(b) = 0 \) implies \( g(c) = 0 \), so that \( c \in S \); hence \( S \) is convex.

To prove (d), assume \( C_s = f \bar{g} \) where \( f, g \) are increasing functions. We note first the following easily verified facts: (i) \( S \) is convex \( \iff \bar{S} \) is convex; (ii) \( f \) is increasing \( \iff f^D \) is increasing; (iii) \( C_s = f \bar{g} \iff C_s = g^D f_D = g^D f^D \).

(d1) It is clear that \( S = \emptyset \iff f \bar{g} = 0 \iff f \leq g \).
(d₂) If \( \emptyset \in S \) then \( f\bar{g}(\emptyset) = 1 \); hence \( f(\emptyset) = 1 \) and \( \bar{g}(\emptyset) = 1 \), which implies \( f = 1 \) and \( g \neq 1 \). Conversely, suppose \( f = 1 \) and \( g \neq 1 \). Then \( C_S = \bar{g} \neq 0 \); hence for some \( a \in 2^n \), \( g(a) = 0 \), which implies \( g(\emptyset) = 0 \) and \( \emptyset \in S \).

(d₃) As noted above, \( \tilde{S} \) is convex and \( C_{\bar{S}} = \bar{g} \bar{D} \tilde{f} \bar{D} \); hence by (d₂), \( 1 \in S \iff \emptyset \in \tilde{S} \iff \bar{g} \bar{D} = 1 \) and \( f \bar{D} \neq 1 \iff g = 0 \) and \( f \neq 0 \).

(d₄) If \( S = 2^n \) then \( \emptyset \in S \) and \( 1 \in S \); hence by (d₂) and (d₃), \( f = 1 \) and \( g = 0 \). The converse is obvious.

\( (d₅) \) Suppose \( \emptyset \notin S \) and \( 1 \notin S \). If \( S = \emptyset \) then by (d₁), \( f \leq g \). If \( S \neq \emptyset \) then \( f \neq 0 \), \( g \neq 1 \) and hence by (d₂), \( \emptyset \notin S \) implies \( f \neq 1 \) and by (d₃), \( 1 \notin S \) implies \( g \neq 0 \). Conversely, if \( f \leq g \) then by (d₁). \( S = \emptyset \); if \( f \neq 0 \) or \( 1 \) and \( g \neq 0 \) or \( 1 \) then by (d₂) and (d₃), \( \emptyset \notin S \) and \( 1 \notin S \). This completes the proof of Theorem 3.3.

**Corollary 3.4.** \( S \) is a convex subset of \( 2^n \) containing \( 1 \) if and only if \( C_S \) is a non-zero increasing function.

**Proof.** Assume \( S \) is convex and \( 1 \in S \). Then by Theorem 3.3 (c) and (d₃), \( C_S = f\bar{g} \) where \( f \) is increasing, \( f \neq 0 \) and \( g = 0 \); hence \( C_S = f \neq 0 \). Conversely, if \( C_S = f \neq 0 \) is increasing then \( C_{\bar{S}} = f \cdot \bar{g} \) which by Theorem 3.3 implies \( S \) is convex and \( 1 \in S \).

**Corollary 3.5.** \( S \) is a convex subset of \( 2^n \) containing \( 0 \) if and only if \( C_S \) is a non-zero decreasing function.

**Proof.** By Corollary 3.4, since \( \emptyset \in S \iff 1 \in \tilde{S} \iff C_{\bar{S}} = g \) is increasing and \( \neq 0 \iff C_S = \bar{g} \bar{D} \) where \( g \bar{D} \) is decreasing and \( \neq 0 \).

**Corollary 3.6.** Let \( S \) be a subset of \( 2^n \) such that \( S \neq \emptyset \) and \( \tilde{S} \neq \emptyset \). Then \( S \) is a convex set containing \( 0 \) if and only if \( \tilde{S} \) is a convex set containing \( 1 \).

**Proof.** By Corollary 3.4 and 3.5, \( S \) is convex, \( \emptyset \in S \) and \( \tilde{S} \neq \emptyset \iff C_S = \bar{g} \) where \( g \) is increasing and \( \neq 0 \), \( 1 \iff C_{\bar{S}} = \bar{g} \) is increasing and \( \neq 0 \), \( 1 \iff \tilde{S} \) is convex, \( 1 \in S \) and \( S \neq \emptyset \).

**Corollary 3.7.** \( S \) is a non-empty convex subset of \( 2^n \) not containing \( 0 \) or \( 1 \) if and only if \( \tilde{S} \) is not convex but there exist disjoint convex sets \( S_0, S_1 \) such that \( \emptyset \in S_0 \), \( 1 \in S_1 \) and \( \tilde{S} = S_0 \cup S_1 \).

**Proof.** Assume \( S \) is convex, \( S \neq \emptyset \), \( 0 \notin S \) and \( 1 \notin S \). By Theorem 3.3, \( C_S = f\bar{g} \) where \( f, g \) are increasing, \( f \neq 0 \) or \( 1 \) and \( g \neq 0 \) or \( 1 \). Then \( C_{\tilde{S}} = \bar{f} + \bar{g} = \bar{f} + fg \); hence \( \tilde{S} = S_0 \cup S_1 \) if \( S_0 = \{ a : \bar{f}(a) = 1 \} \) and \( S_1 = \{ a : fg(a) = 1 \} \), and \( S_0, S_1 \) are clearly disjoint. By Corollary 3.5, \( C_{\bar{S}} = \bar{f} \) implies \( S_0 \) is convex and \( \emptyset \in S_0 \); and by Corollary 3.4, \( C_{\bar{S}} = fg \) implies \( S_1 \) is convex and \( 1 \in S_1 \). Finally, \( \emptyset \in \tilde{S} \) and \( 1 \in \tilde{S} \) implies \( \tilde{S} \) is not convex since \( S \neq \emptyset \). Conversely, assume there exist disjoint convex sets \( S_0, S_1 \) such that \( \emptyset \in S_0 \), \( 1 \in S_1 \) and \( \tilde{S} = S_0 \cup S_1 \), but \( \tilde{S} \) is not convex. Then in particular \( \tilde{S} \neq 2^n \).
and hence $S \neq \emptyset$, $r \notin S$, $1 \notin S$. Also, by Corollaries 3.4 and 3.5 there are increasing functions $f, g$ such that $C_a = \tilde{f} \neq 0$ and $C_{g} = g \neq 0$; hence $S = S_0 \cup S_1$ implies $C_a = \tilde{f} + g$. Then $C_a = f \tilde{g}$ which by Theorem 3.3 implies $S$ is convex.

**Corollary 3.8.** Let $S$ be any subset of $2^n$. If $S$ is the union of $m$ convex sets, then

(a) $\tilde{S}$ is a disjoint union of $m+1$ convex sets;

(b) if $0 \in S$ or $1 \in S$ then $\tilde{S}$ is a disjoint union of $m$ convex sets;

(c) if $0 \in S$ and $1 \in S$ then $\tilde{S}$ is a disjoint union of $m-1$ convex sets.

**Proof.** Suppose $S = \bigcup_{i=1}^{m} S_i$ where each $S_i$ is convex. Then by Theorem 3.3, for each $i, 1 \leq i \leq m$, there are increasing functions $f_i, g_i$ such that $C_n = f_i \tilde{g}_i$. Hence $C_S = \sum_{i=1}^{m} f_i \tilde{g}_i$; since by Lemma 3.1 we may assume $f_1 \geq g_1 \geq \cdots \geq f_m \geq g_m$ it follows that

$$C_S = \tilde{f}_1 + g_m + \sum_{i=1}^{m-1} g_i \tilde{f}_{i+1}.$$ 

Let

$$T_i = \{a : f_i(a) = 1\}, \quad T_{m+1} = \{a : g_m(a) = 1\},$$

$$T_{i+1} = \{a : g_i(a) = 1\} \text{ for } 1 < i < m.$$ 

Then each $T_i$ is convex by Theorem 3.3, and $f_i \geq g_i \geq \cdots \geq f_m \geq g_m$ implies $T_i \cap T_j = \emptyset$ if $i \neq j$; clearly $S = \bigcup_{i=1}^{m+1} T_i$, which proves (a). If $0 \in S$, we may assume without loss of generality that $0 \in S_1$; then by Theorem 3.3(d), $C_n = f_1 \tilde{g}_1$ implies $f_1 = 1$, so that $T_1 = \emptyset$ and hence $S = \bigcup_{i=2}^{m+1} T_i$; the second half of (b) follows by considering $\tilde{S}$, since $S = S$ is a disjoint union of $m$ convex sets; $\tilde{S}$ is a disjoint union of $m$ convex sets. For (c), suppose $0 \in S$ and $1 \in S$; we may again assume that $0 \in S_1$ and hence $f_1 = 1$ and $T_1 = \emptyset$. If $1 \notin S_1$ also then since $S_1$ is convex, $S_1 = S = 2^n$ and $\tilde{S} = \emptyset$. If $1 \in S_1$ we may assume $1 \in S_m$ where $m > 1$ so that $g_m = 0$, $T_{m+1} = \emptyset$ and $S = \bigcup_{i=1}^{m} T_i$; in either case, $S$ is a disjoint union of $m-1$ convex sets, which proves (c).

We can now completely characterize arbitrary subsets of $2^n$ as disjoint unions of convex sets, as follows.

**Theorem 3.9.** Let $S$ be any subset of $2^n$. Then

(a) if neither of $0, 1$ are in $S$ then for some $m, 0 \leq m \leq \lfloor \frac{1}{2} n \rfloor$, $S$ is a disjoint union of $m$ convex sets and $\tilde{S}$ is a disjoint union of $m + 1$ convex sets;

(b) if exactly one of $0, 1$ is in $S$ then for some $m, 0 \leq m \leq \lfloor \frac{1}{2} (n - 1) \rfloor$, $S$ and $\tilde{S}$ are each a disjoint union of $m + 1$ convex sets.

(c) if both of $0, 1$ are in $S$ then for some $m, 0 \leq m \leq \lfloor \frac{1}{2} n \rfloor$, $S$ is a disjoint union of $m + 1$ convex sets and $\tilde{S}$ is a disjoint union of $m$ convex sets.

**Proof.** For each $a \in 2^n$ let the weight $w(a)$ be defined by

$$w(a) = |\{i : c_i = 1\}|.$$
If \( \emptyset \notin S \), define
\[
S_i = \{a \in S : w(a) = 2i + 1 \text{ or } 2i + 2\}. \quad 0 \leq i \leq \lfloor \frac{1}{2} n \rfloor.
\]
Clearly \( S_i \cap S_j = \emptyset \) if \( i \neq j \); moreover, each \( S_i \) is convex since for \( a, b \in S_n \), \( a < b \) only if \( b \) is an immediate successor of \( a \) in the partial ordering of \( 2^n \) and hence \( a < c < b \) vacuously implies \( c \in S_i \). If \( 1 \notin S \), then \( S = \bigcup_{i=0}^{\lfloor \frac{1}{2} n \rfloor} S_i \); hence \( S \) is a disjoint union of \( m \) convex sets for some \( m, 0 \leq m \leq \lfloor \frac{1}{2} n \rfloor \) and, by Corollary 3.8(a), \( \bar{S} \) is a disjoint union of \( m + 1 \) sets, which proves (a). Part (c) follows by interchanging \( S \) and \( \bar{S} \). If \( 1 \in S \) then if \( n \) is even, \( S = \bigcup_{i=0}^{\lfloor \frac{1}{2} n \rfloor-1} S_i \) and if \( n \) is odd, \( S = \bigcup_{i=0}^{\lfloor \frac{1}{2} n \rfloor} S_i \); in either case \( S \) is a disjoint union of \( m + 1 \) convex sets for some \( m, 0 \leq m \leq \lfloor \frac{1}{2} (n - 1) \rfloor \) and by Corollary 3.8(b) so is \( \bar{S} \). This proves half of (b); the second half follows by interchanging \( S \) and \( \bar{S} \).

The following Proposition shows that the bounds in Theorem 3.9 are best possible.

**Proposition 3.10.** Let \( B_n = \{a \in 2^n : w(a) \text{ is odd}\} \). Then

(a) if \( n \) is even, \( B_n \) is a disjoint union of \( m \) convex sets or \( \bar{B}_n \) is a disjoint union of \( m + 1 \) convex sets if and only if \( m \geq \lfloor \frac{1}{2} n \rfloor \).

(b) if \( n \) is odd, \( B_n \) or \( \bar{B}_n \) is a disjoint union of \( m + 1 \) convex sets if and only if \( m \geq \lfloor \frac{1}{2} (n - 1) \rfloor \).

**Proof.** The "if" part follows from Theorem 3.9 (adding empty sets if necessary). Now assume \( B_n \) is a union of \( m \) convex sets \( S_1, \ldots, S_m \). Since each \( S_i \subseteq B_n \), all elements of \( S_i \) have odd weight; hence since \( S_i \) is convex, all elements of \( S_i \) must have the same weight.

It follows that there are at least as many \( S_i \)'s as there are weights of elements of \( B_n \); hence \( m \geq \lfloor \frac{1}{2} n \rfloor \). Similarly \( \bar{B}_n = \bigcup_{i=0}^{m} S_i \) where \( S_i \) is convex. \( 0 \leq i \leq m \), implies \( m + 1 \geq 1 + \lfloor \frac{1}{2} n \rfloor \). Part (b) is proved similarly.

4. Decomposition of Boolean functions

We now apply the results of the previous section to show that every Boolean function of \( n \) variables can be decomposed as a monotone function of at most \( n \) monotone functions. Subsequent to proving this theorem, we found that an essentially equivalent result had been proved by Gilbert by a different method in [2, Theorem 6].

**Theorem 4.1.** Let \( f(x_1, \ldots, x_n) \) be any non-constant Boolean function. Then

(a) \( f(1) = 1 \) and \( f(0) = 0 \) if and only if there exist some \( m, 0 \leq m \leq \lfloor \frac{1}{2} (n - 1) \rfloor \), and non-constant increasing functions \( g_1, \ldots, g_{2m+1} \) such that

\[
f = g_{2m+1} \sum_{i=1}^{m} g_{2i-1} \bar{g}_{2i}.
\]
(b) $f(1) = 0$ and $f(0) = 0$ if and only if $n \geq 2$ and there exist some $m$, $1 \leq m \leq \frac{1}{2} n$ and non-constant increasing functions $g_1, \ldots, g_m$ such that

$$f = \sum_{i=1}^{m} g_{2i-1} \beta_{2i};$$

(c) $f(1) = 0$ and $f(0) = 1$ if and only if there exist some $m$, $0 \leq m \leq \frac{1}{2} (n-1)$ and non-constant increasing functions $g_1, \ldots, g_{m+1}$ such that

$$f = \beta_{2m+1} + \sum_{i=1}^{m} g_{2i-1} \beta_{2i};$$

(d) $f(1) = 1$ and $f(0) = 1$ if and only if $n \geq 2$ and there exist some $m$, $1 \leq m \leq \frac{1}{2} n$ and non-constant increasing functions $g_1, \ldots, g_m$ such that

$$f = g_{2m+1} + \sum_{i=1}^{m-1} g_{2i-1} \beta_{2i};$$

Proof. The “if” part follows trivially from the fact that if $g$ is a non-constant increasing function, $g(0) = 0$ and $g(1) = 1$. For the other direction, let $S = \{a : f(a) = 1\}$. If $f(1) = 1$ and $f(0) = 0$ then by Theorem 3.9(b), there exist $m$, $0 \leq m \leq \frac{1}{2} (n-1)$ and disjoint convex sets $S_1, \ldots, S_{m+1}$ such that $S = \bigcup_{i=1}^{m+1} S_i$. Let $m$ be such that each $S_i \neq \emptyset$; we may assume $1 \in S_{m+1}$. Then by Theorem 3.9(c) and (d) there are increasing functions $g_1, \ldots, g_{2m+1}$ such that $C_{S_i} = g_{2i-1} \beta_{2i}$ for $1 \leq i \leq m$ and $C_{S_{m+1}} = g_{2m+1}$; and since $1 \notin S_i$ for $i \leq m$ and $0 \notin S_i \neq \emptyset$ for $i \leq m+1$, the $g_i$'s are non-constant. Hence

$$f = C_S = \sum_{i=1}^{m+1} C_{S_i} = g_{2m+1} + \sum_{i=1}^{m} g_{2i-1} \beta_{2i};$$

which proves (a). If $f(0) = 0$ and $f(1) = 0$ then $f$ non-constant implies $n \geq 2$ and $S \neq \emptyset$; hence by Theorem 3.9(a) there are disjoint non-empty convex sets $S_1, \ldots, S_m$, $1 \leq m \leq \frac{1}{2} n$, such that $S = \bigcup_{i=1}^{m} S_i$. Since none of the $S_i$'s contain 0 or 1, it follows from Theorem 3.3 that there are non-constant increasing functions $g_1, \ldots, g_{2m}$ such that

$$f = C_S = \sum_{i=1}^{m} g_{2i-1} \beta_{2i};$$

which proves (b). Parts (c) and (d) now follow by applying (a) and (b) to $f$ and then invoking Lemma 3.1.

5. Bounded truth-table reducibility

Theorem 4.1 may be expected to have useful applications in any context where monotone increasing Boolean functions are "basic" in some sense; one such context is bounded truth-table reducibility in recursion theory. In this section we
obtain a canonical form for shorted bounded truth-table reductions in certain cases (Theorem 5.4).

The recursion-theoretic terminology and notation will be that of [8]; in particular \( \mathbb{N} \) denotes the natural numbers. If \( A, B \subseteq \mathbb{N} \), \( \overline{A} \) denotes the complement of \( A \), and \( A \times B \) the recursive Cartesian product with recursive inverse functions; by iteration we can recursively decompose \( x \in \mathbb{N} \) as \( x = (x_1, \ldots, x_n) \) for each \( n > 0 \). \( K \) denotes the complete recursively enumerable (r.e.) set; (i.e., \( K = \{ x : x \) is accepted by Turing machine \( x \} \). \( B \leq_m A \) denotes many-one reducibility and \( B \leq_T A \) Turing reducibility. \( B \leq_{\text{btt}} A \) denotes bounded truth-table reducibility; i.e., the existence of a recursive function \( \varphi \) and some \( m > 0 \) such that for each \( x \),

(a) \( \varphi(x) = ((y_1, \ldots, y_k), f_x) \), where \( 0 < k \leq m \), \( y_1, \ldots, y_k \in \mathbb{N} \) and \( f_x \) is a \( k \)-ary Boolean function; and

(b) \( x \in B \iff f_x(C_A(y_1), \ldots, C_A(y_k)) = 1 \). \( m \) is called the norm of the reduction.

As pointed out in [8], bounded truth-table reducibility is not affected (except in trivial cases) by requiring that the number \( k \) and the Boolean function \( f_x \) be the same for all \( x \) (although the norm of the possible reductions may be affected). We call these fixed reductions, and will assume below that all btt-reductions are fixed. If \( \varphi \) is such a reduction, we will use \( f_\varphi \) (or simply \( f \), if the context is unambiguous) to denote the corresponding Boolean function. We will further restrict the btt-reduction \( \varphi \) by requiring that \( f_\varphi \) be non-constant. It is evident that the only reductions lost by this restriction are of the form \( B \leq_{\text{btt}} A \) where \( B \) is recursive; and in that case we shall say by convention that \( B \leq_{\text{btt}} A \) with norm 0. In addition we shall always assume that \( A \) is non-trivial, i.e., \( A \neq \emptyset \) or \( \mathbb{N} \).

In order to make use of the decomposition theorem of the previous section, it is necessary to place restrictions on the set \( A \) which will allow the "collapsing" of conjunctions and disjunctions in truth-table conditions. In [6], a set \( A \) is called a \( p \)-cylinder if \( A \times A \leq_m A \), and \( \overline{A} \times \overline{A} \leq_m \overline{A} \). The name is justified by the following lemma, whose easy proof is left to the reader:

**Lemma 3.1.** If \( A \times A \leq_m A \), \( \overline{A} \times \overline{A} \leq_m \overline{A} \), and \( g(x_1, \ldots, x_n) \) is any non-constant increasing Boolean function, then there is an \( n \)-ary recursive function \( \psi \) such that

\[
g(C_A(y_1), \ldots, C_A(y_n)) = 1 \iff \psi(y_1, \ldots, y_n) \in A.
\]

In the terminology of [6] it follows that if \( A \times A \leq_m A \) and \( \overline{A} \times \overline{A} \leq_m \overline{A} \) then whenever \( B \leq_{\text{btt}} A \) with a "positive" truth table, \( B \leq_m A \); hence the name \( p \)-cylinder. As an example, note that \( K \times K \) is r.e. and \( \overline{K} \times \overline{K} \) is co-r.e., so that \( K \times K \leq_m K \) and \( \overline{K} \times \overline{K} \leq_m \overline{K} \), hence \( K \) is a \( p \)-cylinder.

We now introduce sets which will be shown to be "complete" with respect to btt-reductions. If \( x \in \mathbb{N} \) and \( x = (x_1, \ldots, x_n) \), \( n > 0 \), let

\[\sigma^A_n(x) = \#\{ i : x_i \in A \},\]

\[A_n = \{ x : \sigma^A_n(x) \text{ is odd} \},\]

\[A_0 = \text{any infinite, co-infinite recursive set}.\]

To simplify notation, \( \overline{A}_n \) will be used to denote \( \mathbb{N} - A_n \) (and not \( \mathbb{N} - A \)).
Lemma 5.2. For all $n$,
(a) $A_n \subseteq A_{n+1}$ and $\overline{A}_n \subseteq_m A_{n+1}$;
(b) if $m \vdash A_n$ then $B \vdash A_m$ with norm $n$.

Proof. Let $a, b$ be fixed elements of $A$ and $\bar{A}$ respectively. Define $\alpha, \beta$ as follows: if $x = (x_1, \ldots, x_m)$ then
$$\alpha(x) = (a_1, \ldots, a_m, a), \quad \beta(x) = (a_1, \ldots, a_m, b).$$
Then $\alpha, \beta$ are recursive and it is evident that
$$x \in A_n \iff \beta(x) \in A_{n+1}, \quad x \in \overline{A}_n \iff \alpha(x) \in A_{n+1}$$
which proves (a). For (b), suppose that there is a recursive function $\gamma$ such that
$$y \in B \iff \gamma(y) \in A_n \iff \sigma_n^{\alpha}(\gamma(y)) \text{ is odd.}$$
Then $B \subseteq \overline{A}$ by the reduction
$$\varphi = ((\gamma(y))_{i_1}, (\gamma(y))_{i_2}, \ldots, (\gamma(y))_{i_n}, f),$$
where $f(x_1, \ldots, x_n)$ is the Boolean function which has value 1 exactly when the weight of $(x_1, \ldots, x_n)$ is odd.

Lemma 5.3. Suppose $A$ is a p-cylinder and $B \subseteq A$ by a reduction $\varphi = ((y_1, \ldots, y_n), f)$ of norm $n > 0$. Then
(a) $f(1) = 0$ and $f(0) = 0$ implies $B \subseteq A_n$ for some $k, 1 \leq k \leq [\frac{1}{2}n]$;
(b) $f(1) = 1$ and $f(0) = 0$ implies $B \subseteq A_{k+1}$ for some $k, 0 \leq k \leq [\frac{1}{2}(n-1)]$.

Proof. (a) If $f(1) = 0$ and $f(0) = 0$ then by Theorem 4.1(b) there exist $k, 1 \leq k \leq [\frac{1}{2}n]$ and non-constant increasing functions $g_1, \ldots, g_{2k}$ such that
$$f = \sum_{i=1}^{k} g_{2i-1}g_{2i}.$$  
By Lemma 3.1 we may assume $g_1 \geq g_2 \geq \cdots \geq g_{2k}$, and hence that for $a \in 2^n$,
$$|\{i: g_i(a) = 1\}| \text{ is odd } \iff (\exists i)_{1 \leq i \leq k} (g_{2i-1}(a) = 1 \text{ and } g_{2i}(a) = 0).$$
Then
$$x \in B \iff f(C_A(y_1), \ldots, C_A(y_n)) = 1$$
$$\iff \sum_{i=1}^{k} g_{2i-1}(C_A(y_1), \ldots, C_A(y_n))g_{2i}(C_A(y_1), \ldots, C_A(y_n)) = 1$$
$$\iff (\exists i)_{1 \leq i \leq k} (g_{2i-1}(C_A(y_1), \ldots, C_A(y_n)) = 1 \text{ and }\quad g_{2i}(C_A(y_1), \ldots, C_A(y_n)) = 0)$$
$$\iff |\{j: g_j(C_A(y_1), \ldots, C_A(y_n)) = 1\}| \text{ is odd}$$
$$\iff |\{j: \psi_j(y_1, \ldots, y_n) \in A\}| \text{ is odd.}$$
where \( \psi_j \) is related to \( g_j \) as in Lemma 5.1; hence if \( \psi(x) = (\psi_1(y_1, \ldots, y_n), \ldots, \psi_{2k}(y_1, \ldots, y_n)) \), \( x \in B \rightarrow \psi(x) \in A_{2k} \) and \( B \leq_m A_{2k} \). Part (b) is proved similarly, using Theorem 4.1(a).

For convenience of notation, we will say that \( B \) has \( A \)-order \( n \) if \( B \leq_m A_n \) or \( B \leq_m \tilde{A}_n \) but \( B \leq_m A_{n-1} \) and \( B \leq_m \tilde{A}_{n-1} \).

**Theorem 5.4.** Assume \( A \) is a \( p \)-cylinder. Then \( B \) has \( A \)-order \( n \) if and only if \( B \leq_{btt} A \) with minimum norm \( n \); and in that case, all reductions \( \varphi \) of minimum norm satisfy \( f_\varphi(0) \neq f_\varphi(1) \) if \( n \) is odd and \( f_\varphi(0) = f_\varphi(1) \) if \( n \) is even (and \( n > 0 \)).

**Proof.** Assume the hypothesis. If \( B \) has \( A \)-order \( n \), then \( B \leq_m A_n \) or \( B \leq_m \tilde{A}_n \) and hence by Lemma 5.2, \( B \leq_{btt} A \), with norm \( n \). If \( B \leq_{btt} A \) with norm \( n' < n \) then by Lemma 5.3 either \( B \) or \( \tilde{B} \leq_m A_{n'} \), where \( n'' < n' < n \); hence by Lemma 5.2(a) \( B \leq_m A_{n-1} \) or \( B \leq_m \tilde{A}_{n-1} \), contrary to hypothesis. Hence \( n \) is the minimum norm. Conversely, suppose \( B \leq_{btt} A \) with minimum norm \( n \). If \( n = 0 \) then \( B \leq_m A_0 \) and hence has order 0. If \( n > 0 \), then by Lemma 5.3 and 5.2(a), \( B \) or \( \tilde{B} \leq_m A_{n'} \) for some \( n' < n \) and hence \( B \leq_m A_n \) or \( \tilde{A}_n \); while by Lemma 5.2(b) if \( B \leq_{btt} A \) or \( B \leq_{btt} \tilde{A}_n \), then \( B \leq_{btt} A \) with norm \( n - 1 \), which contradicts the minimality of \( n \). Hence \( B \) has \( A \)-order \( n \).

Now let \( \varphi \) be a reduction of minimum norm \( n > 0 \), and let \( f \) denote \( f_\varphi \). Suppose \( n \) is odd; if \( f(1) = f(0) \) then either \( f(1) = f(0) = 0 \) or \( f(1) = f(0) = 0 \). It then follows from Lemma 5.3(a) that \( B \) or \( \tilde{B} \leq_m A_{2k} \) for some \( k \leq \lfloor \frac{1}{2} n \rfloor \) and hence \( B \) has \( A \)-order \( < n \), contradicting the minimality of \( n \). Hence \( f(1) \neq f(0) \) if \( n \) is odd and it is shown similarly that \( f(1) = f(0) \) if \( n \) is even, \( n > 0 \).

One may ask whether it is possible to strengthen Theorem 5.4 by specifying the actual values of \( f_\varphi(0) \) and \( f_\varphi(1) \) for reductions \( \varphi \) of minimum norm. That this is not possible is seen by the following example.

**Lemma 5.5.** Let \( K \oplus \bar{K} \) denote the recursive join of \( K \) and \( \bar{K} \); i.e., \( K \oplus \bar{K} = \{ x : (x = 2y \text{ and } y \in K) \text{ or } (x = 2y + 1 \text{ and } y \in \bar{K}) \} \). Then

(a) \( K \oplus \bar{K} \leq_m K \oplus \bar{K} \);

(b) \( K \oplus \bar{K} \leq_{btt} K \) with minimum fixed norm 2.

**Proof.** For (a), let \( \alpha \) be the recursive function defined by

\[
\alpha(x) = \begin{cases} 
  x + 1 & \text{if } x \text{ is even,} \\
  2 \lceil \frac{1}{2} x \rceil & \text{if } x \text{ is odd.}
\end{cases}
\]

It is then easily verified that \( x \in K \oplus \bar{K} \iff \alpha(x) \notin K \oplus \bar{K} \). For (b), note first that \( K \oplus \bar{K} = C_1 - C_2 \) where

\[
C_1 = \{ x : x = 2y + 1 \text{ or } \lceil \frac{1}{2} x \rceil \in K \}, \\
C_2 = \{ x : x = 2y + 1 \text{ and } \lceil \frac{1}{2} x \rceil \in K \},
\]
are both r.e. Hence, if $C_i \leq_m K$ via recursive functions $\beta_i$, $i = 1, 2$, then $K \oplus K \leq_m K \times K$ via the recursive function $\gamma(x) = (\beta_1(x), \beta_2(x))$. Now $K \times K \leq_{br} K$ via the reduction $\varphi_0(x) = ((x_1, x_2), f_0)$ where $f_0$ is the Boolean function satisfying $f_0(0) = 1$ if and only if $a = (1, 0)$. It follows that $K \oplus K \leq_{br} K$ via $\varphi_0 = (\gamma(x))_1, (\gamma(x))_2, f_0)$, where $f_0(0) = f_0(1) = 0$. Hence $K \oplus K \leq_{br} K$ with norm 2. Suppose the minimum (fixed) norm $< 2$. It cannot be 0 since $K, K \leq_m K \oplus K$ and hence $K \oplus K$ is not recursive. If it is 1, then by Theorem 5.4, $K \oplus K$ has $K$-order 1 and hence $K \oplus K \leq_m K$ or $K \oplus K \leq_m K$, which implies $K \leq_m K$ or $K \leq_m K$ which is false. Hence 2 is the minimum fixed norm. (The word “fixed” is emphasized here because in this case the norm can be reduced to 1 by removing the restriction that in the reduction $\varphi(x) = (y, j_x)$, the Boolean function $f_x$ be the same for all $x$; clearly $K \subseteq \mathbb{E}$ via $\varphi(x) = ([1], f_x)$ where $f_x$ is the identity if $x$ is even and $f_x$ is the identity if $x$ is odd.)

It follows from Lemma 5.5(b) and Theorem 5.4 that if $\varphi$ is a btt-reduction of $K \oplus K$ to $K$ of norm 2 then $f_\varphi(0) = f_\varphi(1)$. We now show that there are such btt-reductions $\varphi$ for which $f_\varphi(0) = f_\varphi(1) = 0$ and others for which $f_\varphi(0) = f_\varphi(1) = 1$. The btt-reduction $\varphi_0 = (\gamma(x))_1, (\gamma(x))_2, f_0)$ given in the proof of Lemma 5.5(b) satisfies $f_0(0) = f_0(1) = 0$. Now $\varphi_0 = (\gamma(x))_1, (\gamma(x))_2, f_0)$ is a btt-reduction of $K \oplus K$ to $K$ of norm 2. Combining this with Lemma 5.5(a) then gives the desired reduction. More precisely, if $\alpha$ is as in the proof of Lemma 5.5(a), then $\varphi(x) = (\gamma(x))_1, (\gamma(x))_2, f_0)$ is a btt-reduction of $K \oplus K$ to $K$ of norm 2 for which $f_\alpha(0) = f_\alpha(1) = 1$. This construction evidently depends on the symmetry of $K \oplus K$ and its complement. If the sets $B, \overline{B}$ are very asymmetric one might expect to strengthen the information yielded by Theorem 5.4 about btt-reductions of minimum norm. It will be shown in the next section that this is the case when $B \leq_{br} K$ and $B$ is an index set.

6. Btt-reductions of index sets to $K$

Let $\{W_x\}_{x \in \mathbb{N}}$ be a standard indexing of all r.e. sets; if $C$ is a class of r.e. sets, the index set $\theta C$ of $C$ is $\{x : W_x \in C\}$. We may set $A = K$ in the results of the previous section since $K$ is evidently a $p$-yylinder. In the case where $\theta C \leq_{br} K$ much more complete information can then be obtained about btt-reductions of minimum norm. (Theorem 6.2 and 6.4). We shall require the following results from [4]:

**Lemma 6.1.** Let $\theta C$ be any class of r.e. sets, $n \geq 0$,

(a) If $\emptyset \in \theta C$ then $\theta C \leq_m K_{n+1}$ implies $\theta C \leq_m K_n$;
(b) If $N \in \theta C$ then $\theta C \leq_m K_{2n+1}$ implies $\theta C \leq_m K_{2n}$ and $\theta C \leq_m K_{2n+2}$ implies $\theta C \leq_m K_{2n+1}$;
(c) If $\theta C \leq_m K_{n+1}$ and $\theta C \leq_m K_{n+1}$ then $\theta C \leq_m K_n$ or $\theta C \leq_m K_{n}$. 

**Proof.** (a) and (b) are proved by combining Theorem 2, Theorem 3, Lemma 13 and Lemma 15 of [4] (noting that the sets $K_{2n-1}, K_{2n-2}$ as defined in the previous
section are denoted in \([4]\) by \(Z_{2n}, Z_{2n+1}\) respectively.) Part (c') is Theorem 3(g) of \([4]\).

The following information can now be added to that yielded by Theorem 5.4:

**Theorem 6.2.** Assume \(\theta \in \leq_{\text{bt}} K\) with minimum norm \(n > 0\), and let \(\{f_i\}_{i=0}^n\) be the set of Boolean functions \(f_i\) determined by all possible \(\varphi\) of norm \(n\) which reduce \(\theta \in\) to \(K\). The following conditions are then equivalent:

(a) \(f_i(0) = 0\) for every \(i\);

(b) \(f_i(0) = 0\) for some \(i\);

(c) \(\theta \in \leq_m K;\)

(d) \(\theta \in \leq_m \overline{K};\)

(e) \(\emptyset \notin \mathcal{C}\).

**Proof.** If \(\theta \in \leq_{\text{bt}} K\) with minimum norm \(n > 0\) then by Theorem 5.4, \(\theta \in\) has \(K\)-order \(n\) and hence \(\theta \in \leq_m K_n\) or \(\overline{K};\), \(\theta \in \leq_m K_{n-1}\), \(\theta \in \leq_m \overline{K}_{n-1}\). The implication (a) \(\rightarrow\) (b) is evidently trivial.

(b) \(\rightarrow\) (c): If \(f_i(0) = 0\) for some \(i\), then by Lemma 5.3 either \(\theta \in \leq_m K_{2k}\) for some \(k, 1 \leq k \leq [\frac{1}{2} n]\) or \(\theta \in \leq_m K_{2k+1}\) for some \(k, 0 \leq k \leq [\frac{1}{2} (n - 1)]\); in either case \(\theta \in \leq_m K_{n'}\) for some \(n' \leq n\) and hence by Lemma 5.2(a) \(\theta \in \leq_m K_n\).

(c) \(\rightarrow\) (d): Assume \(\theta \in \leq_m K_n\); since \(\theta \in \leq_m K_{n-1}\) and \(\theta \in \not\leq_m \overline{K}_{n-1}\) it follows from Lemma 6.1(c) that \(\theta \in \leq_m \overline{K}_{n}\).

(d) \(\rightarrow\) (e): Assume \(\theta \in \leq_m \overline{K}_{n}\). Then \(\theta \in \leq_m K_{n}\), and since \(\theta \in \not\leq_m K_{n-1}\) it follows from Lemma 6.1(a) that \(\emptyset \notin \mathcal{C}\).

(e) \(\rightarrow\) (a): Assume \(\emptyset \notin \mathcal{C}\). If \(f_i(0) = 1\) for some \(i\), then \(f_i(0) = 0\). But \(\overline{f_i}\) corresponds to a bt-reduction \(\varphi\) of \(\overline{\theta \in}\) to \(K\); hence since (b) \(\rightarrow\) (e) it follows that \(\emptyset \notin \mathcal{C}\) and thus \(\emptyset \in \mathcal{C}\), contrary to assumption. Hence \(f_i(0) = 0\) for all \(i\).

Thus among classes whose index sets are bt-reducible to \(K\) (but non-recursive) those which do not contain \(\emptyset\) are exactly those for which the minimum-norm reductions are “0-preserving”. An easy consequence is the following:

**Corollary 6.3.** If \(\theta \in \leq_{\text{bt}} K\) and \(\mathcal{D} = _m \theta \in\) then \(\emptyset \notin \mathcal{C} \leftrightarrow \emptyset \in \mathcal{D}\).

We note that Corollary 6.3 actually holds under the weaker hypothesis \(\theta \in \leq_{\text{bt}} K\), this was shown in \([5, \text{Theorem 4.5}]\) where a different characterization was given for index sets \(\theta \in \leq_{\text{bt}} K\) for which \(\emptyset \notin \mathcal{C}\).

By using Lemma 6.1(b) the following sharper result can be obtained:

**Theorem 6.4.** Assume \(\theta \in \leq_{\text{bt}} K\) with minimum norm \(n > 0\) and let \(\{f_i\}_{i=0}^n\) be as in Theorem 6.2. Then \(\mathcal{C}\) satisfies exactly one of the following sets of conditions:

I

(a) \(\emptyset \notin \mathcal{C}\) and \(N \in \mathcal{C}\)

(b) \(f_i(0) = 0\) and \(f_i(1) = 1\) for some \(i\)

II

(a) \(\emptyset \in \mathcal{C}\) and \(N \notin \mathcal{C}\)

(b) \(f_i(0) = 1\) and \(f_i(1) = 0\) for some \(i\).
Ve note that the hypothesis of Corollary 6.5 cannot be weakened to \( \mathcal{E} \subseteq \tau K \) (as was the case for Corollary 6.3), as shown by the following example: Let \( \mathcal{E} = \{ W : W, \mathcal{E}, P \mathcal{E} \text{ and the least element of } WX \text{ is even} \} \)

\[ \mathcal{E} = \{ W : W, \mathcal{E}, Z \mathcal{E} \text{ and the least element of } WX \text{ is odd} \} \]

Then \( \mathcal{E} \subseteq \mathcal{F} \) and \( \mathcal{E} \not\subseteq \mathcal{F} \). That \( \mathcal{E} \subseteq \tau K \) and \( \mathcal{E} \not\subseteq \tau K \) is easily seen from the fact that membership in \( \mathcal{E} \), \( \mathcal{F} \) can be tested by asking finitely many questions of the form: Is \( W, = 0 \)? Does \( W, \mathcal{E} \)? But \( \mathcal{E} \not\subseteq \mathcal{F} \) since if we define

\[ W_{\psi(x)} = \{ n + 1 : n \in W, \} \]

then \( \psi \) \( m \)-reduces \( \mathcal{E} \) to \( \mathcal{F} \) and \( \mathcal{F} \not\subseteq \mathcal{E} \).

As an application of the “canonical” minimum norm reductions provided by Theorem 6.2, we sketch a proof of the following theorem, first proved by Morris [7].
Theorem 6.6 (Morris). If $A$ is r.e. then $\{x : W_x \cap \tilde{A} \neq \emptyset\} \leq_{bt} K$ if and only if $A$ is recursive.

**Proof.** The “if” part is obvious. For the “only if” part, let $H_{\tilde{A}}$ denote $\{x : W_x \cap \tilde{A} \neq \emptyset\}$ and assume $H_{\tilde{A}} \leq_{bt} K$ with minimum (fixed) norm $n$. If $n = 0$ then $H_{\tilde{A}}$ is recursive which evidently implies $A$ is recursive. Assume $A$ is non-recursive; then $n > 0$ and by Theorem 6.2 $H_{\tilde{A}} \preceq_{m} K_{n}$, i.e.. there is a recursive function

$$\varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$$

such that

$$x \in H_{\tilde{A}} \iff [\{i : \varphi_i(x) \in K\}] \text{ is odd.}$$

Let $\psi(x, y)$ be a recursive function such that

$$W_{\psi(x, y)} = W_x \cup \{y\}$$

for all $x, y$ and note that if $y \in A$ then

$$x \in H_{\tilde{A}} \iff \psi(x, y) \in H_{\tilde{A}}.$$  

Hence if $y \in A$, the “reduction” $\varphi$ may be applied equivalently to $x$ or to $\psi(x, y)$. Now if $x \in H_{\tilde{A}}$, then $[\{i : \varphi_i(x) \in K\}]$ is odd, hence $\geq 1$ so that $\{i : \varphi_i(x) \in K\} \neq \emptyset$. If $x \notin H_{\tilde{A}}$ we claim that $\{i : \varphi_i(\psi(x, y)) \in K\} \neq \emptyset$ for some $y \in A$; if not,

$$y \in \tilde{A} \iff W_{\psi(x, y)} \cap \tilde{A} \neq \emptyset \iff \{i : \varphi_i(\psi(x, y)) \in K\} \neq \emptyset$$

so that $\tilde{A}$ is r.e. and $A$ is recursive, contrary to hypothesis. The “reduction” $\varphi$ may then be replaced by a “reduction” $\varphi'$ of smaller norm as follows: To compute $\varphi'(x)$, simultaneously enumerate $K$ and

$$E_x = \{\varphi(x)\} \cup \{\varphi(\psi(x, y)) : y \in A\}$$

and for each $z \in E_x$, $z = (z_1, \ldots, z_n)$, look to see if $z_i \in K$ for some $i$. By the above, some such $z$ must appear, and let $z$ be the first such; if $z = (z_1, \ldots, z_n)$, let $z_i$ be the first component appearing in $K$. Then if

$$\varphi'(x) = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$$

evidently

$$x \in H_{\tilde{A}} \iff [\{i : \varphi'_i(x) \in K\}] \text{ is even}$$

$$\iff \varphi'(x) \in K_{n-1}.$$  

But this contradicts Theorem 5.4; hence $A$ is recursive.

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References