

Available online at www.sciencedirect.com

SCIENCE @ DIRECT®

Applied Mathematics Letters 19 (2006) 1100–1106

**Applied
Mathematics
Letters**www.elsevier.com/locate/aml

Stability of impulsive infinite delay differential equations[☆]

Yu Zhang, Jitao Sun^{*}*Department of Applied Mathematics, Tongji University, Shanghai 200092, PR China*

Received 8 June 2004; received in revised form 12 January 2005; accepted 12 September 2005

Abstract

In this work, we consider the stability of impulsive infinite delay differential equations. By using Lyapunov functions and the Razumikhin technique, we get some results that are more general than ones given before. And in using the Razumikhin technique, we use a new technique that has been given by Shunian Zhang; we extend this technique to study impulsive systems. An example is also discussed in this work to illustrate the advantage of the results obtained.

© 2005 Elsevier Ltd. All rights reserved.

Keywords: Impulsive infinite delay differential equations; Uniform stability; Lyapunov function; Razumikhin technique

1. Introduction

It is known that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems exhibit the impulse effect. Thus impulsive differential equations, that is, differential equations involving impulse effects, appear as a natural description of observed evolution phenomena for several real word problems. In recent years, qualitative properties of the mathematical theory of impulsive differential equations have been developed by a large number of mathematicians; see [1–12].

Since time delay exists in many fields in our society, systems with time delay have received significant attention in recent years. In [4–6], the authors considered the stability of impulsive differential equations with finite delay, and got some results. Systems with infinite delay deserve study because they describe a kind of system present in the real world. For example, in a predator–prey system the predation decreases the average growth rate of the prey species, linearly, with an infinite delay—for the predator cannot hunt prey when the predators are infants, and predators have to mature for a duration of time (which for simplicity in the mathematical analysis has been assumed to be infinite) before they are capable of decreasing the average growth rate of the prey species. And there are some results on systems with infinite delay; see [13,14] and references therein. However, to the best of the authors' knowledge, results for impulsive infinite delay differential equations are rare.

[☆] This work was supported by the National Natural Science Foundation of China (60474008).

^{*} Corresponding author. Fax: +86 21 65982341.

E-mail address: [sunjt@sh163.net](mailto:sunj@sh163.net) (J. Sun).

In this note, we consider impulsive infinite delay differential equations, by using Lyapunov functions and the Razumikhin technique; we get some results that are more general than the ones given in [4]. We extend the new technique developed in [1] to study impulsive systems.

This work is organized as follows. In Section 2, we introduce some basic definitions and notation. In Section 3, we get some criteria for uniform stability for impulsive infinite delay differential equations; an example is also discussed in this section to illustrate the advantage of the results obtained. Finally, concluding remarks are given in Section 4.

2. Preliminaries

Consider the following impulsive infinite delay differential equations:

$$\begin{aligned}\dot{x}(t) &= f(t, x(t), x(t - \tau(t))), \quad t \geq t_0, \quad t \neq \tau_k, \\ \Delta x(t) &\triangleq x(t) - x(t^-) = I_k(x(t^-)), \quad t = \tau_k, \quad k = 1, 2, \dots\end{aligned}\quad (1)$$

where $t \in R^+$, $f \in C[R^+ \times R^n \times PC((-\infty, 0], R^n), R^n]$, $PC((-\infty, 0], R^n)$ denotes the space of piecewise right continuous functions $\phi : (-\infty, 0] \rightarrow R^n$ with the sup-norm $\|\phi\| = \sup_{-\infty < s \leq 0} |\phi(s)|$, $|\cdot|$ is a norm in R^n , $f(t, 0, 0) \equiv 0$, $I_k(0) = 0$, $t \geq \tau(t) \geq 0$, $0 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_k < \dots$, $\tau_k \rightarrow \infty$ for $k \rightarrow \infty$, $x(t^+) = \lim_{s \rightarrow t^+} x(s)$, and $x(t^-) = \lim_{s \rightarrow t^-} x(s)$. The functions $I_k : R^n \rightarrow R^n$, $k = 1, 2, \dots$, are such that if $\|x\| < H$ and $I_k(x) \neq 0$, then $\|x + I_k(x)\| < H$, where $H = \text{const.} > 0$.

The initial condition for system (1) is given by

$$x_\sigma = \varphi \quad (2)$$

where $\varphi \in PC((-\infty, 0], R^n)$.

We assume that a solution for the initial problem (1) and (2) does exist and is unique. Since $f(t, 0, 0) = 0$, then $x(t) = 0$ is a solution of (1), which is called the zero solution.

Let

$$PC(\rho) = \{\phi \in PC((-\infty, 0], R^n) : \|\phi\| < \rho\}.$$

For $\varphi \in PC(\rho)$, we define

$$\|\varphi\| = \|\varphi\|^{(-\infty, t]} = \sup_{-\infty < s \leq t} |\varphi(s)|.$$

For convenience, we define

$$|x| = \max_{1 \leq i \leq n} |x_i|, \quad \text{for } x \in R^n.$$

We introduce some definitions as follows:

Definition 1 ([4]). The zero solution of (1) and (2) is said to be stable if for any $\sigma \geq t_0$ and $\varepsilon > 0$ there is a $\delta = \delta(\sigma, \varepsilon) > 0$ such that $[\varphi \in PC(\delta), t \geq \sigma]$ implies that $|x(t, \sigma, \varphi)| \leq \varepsilon$. The zero solution is said to be uniformly stable if δ is independent of σ .

Definition 2 ([1]). A continuous function $w : R^+ \rightarrow R^+$ is called a wedge function if $w(0) = 0$ and $w(s)$ is (strictly) increasing.

In what follows, we will split $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)^T \in PC(\rho)$ into several vectors, say, $(\varphi_1^{(1)}, \varphi_2^{(1)}, \dots, \varphi_{n_1}^{(1)})^T, (\varphi_1^{(2)}, \varphi_2^{(2)}, \dots, \varphi_{n_2}^{(2)})^T, \dots, (\varphi_1^{(m)}, \varphi_2^{(m)}, \dots, \varphi_{n_m}^{(m)})^T$ such that $n_1 + n_2 + \dots + n_m = n$ and $\{\varphi_1^{(1)}, \dots, \varphi_{n_1}^{(1)}, \varphi_1^{(2)}, \dots, \varphi_{n_2}^{(2)}, \dots, \varphi_1^{(m)}, \dots, \varphi_{n_m}^{(m)}\} = \{\varphi_1, \varphi_2, \dots, \varphi_n\}$.

For convenience, we define

$$\varphi^{(j)} = (\varphi_1^{(j)}, \varphi_2^{(j)}, \dots, \varphi_{n_j}^{(j)}), \quad j = 1, 2, \dots, m$$

and

$$\varphi = (\varphi^{(1)}, \varphi^{(2)}, \dots, \varphi^{(m)})^T.$$

For $x = (x_1, x_2, \dots, x_n)^T \in R^n$, we adopt notation similar to that used before, as for $\varphi \in PC(\rho)$.
Let

$$|x^{(j)}| = \max_{1 \leq k \leq n_j} |x_k^{(j)}|, \quad j = 1, 2, \dots, m$$

and thus,

$$|x| = \max_{1 \leq j \leq m} |x^{(j)}|.$$

Correspondingly,

$$|\varphi^{(j)}(s)| = \max_{1 \leq k \leq n_j} |\varphi_k^{(j)}(s)|, \quad j = 1, 2, \dots, m, \quad \text{and} \quad |\varphi(s)| = \max_{1 \leq j \leq m} |\varphi^{(j)}(s)|.$$

Let

$$\|\varphi^{(j)}\| = \|\varphi^{(j)}\|^{(-\infty, t]} = \sup_{-\infty < s \leq t} |\varphi^{(j)}(s)|, \quad j = 1, 2, \dots, m,$$

$$PC^{(j)}(t) = \{\varphi^{(j)} : (-\infty, t] \rightarrow R^{n_j} \mid \varphi^{(j)} \text{ is continuous and bounded}\},$$

and

$$PC_\rho^{(j)}(t) = \{\varphi^{(j)} \in PC^{(j)}(t) \mid \|\varphi^{(j)}\| < \rho\}.$$

3. Main results

For simplicity, we start with the case of $m = 2$, and first establish the following result on the uniformly stability.

Theorem 1. Let $\phi_j : R^+ \rightarrow R^+$ be continuous, $\phi_j \in L'[0, \infty)$, $\phi_j(t) \leq K_j$ for $t \geq 0$ with some constant K_j ($j = 1, 2$) and w_{ij} ($i = 1, 2, 3, 4; j = 1, 2$) be wedge functions. If there exist continuous functionals $V_j : [0, \infty) \times PC_H^{(i)}(t) \rightarrow R^+$ ($i = 1, 2$) such that

- (i) $w_{1j}(|\varphi^{(j)}(t)|) \leq V_j(t, \varphi^{(j)}(t)) \leq w_{2j}(|\varphi^{(j)}(t)|) + w_{3j}[\int_{-\infty}^t \phi_j(t-s)w_{4j}(|\varphi^{(j)}(s)|)ds]$, $j = 1, 2$;
- (ii) when $V_1(t, x^{(1)}(t)) \geq V_2(t, x^{(2)}(t))$, it holds that

$$V_1'(t, x^{(1)}(t)) \leq 0, \quad \text{if } V_1(t - \tau(t), x^{(1)}(t - \tau(t))) \leq V_1(t, x^{(1)}(t)),$$

when $V_1(t, x^{(1)}(t)) \leq V_2(t, x^{(2)}(t))$, it holds that

$$V_2'(t, x^{(2)}(t)) \leq 0, \quad \text{if } V_2(t - \tau(t), x^{(2)}(t - \tau(t))) \leq V_2(t, x^{(2)}(t));$$

- (iii) $V_j(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \leq (1 + b_k)V_j(\tau_k^-, x(\tau_k^-))$, $j = 1, 2$, $k = 1, 2, \dots$, for which $b_k \geq 0$, and $\sum_{k=1}^{\infty} b_k < \infty$;

where $x(t) = (x^{(1)}(t), x^{(2)}(t))$ is a solution of (1) and (2), then the zero solution of (1) and (2) is uniformly stable.

Proof. Since $b_k \geq 0$, and $\sum_{k=1}^{\infty} b_k < \infty$, it follows that $\prod_{k=1}^{\infty} (1 + b_k) = M$ and $1 \leq M < \infty$.

Define a function $V(t)$ as follows:

$$\begin{aligned} V(t) &= V_1(t, x^{(1)}(t)) \quad \text{if } V_1(t, x^{(1)}(t)) \geq V_2(t, x^{(2)}(t)), \\ V(t) &= V_2(t, x^{(2)}(t)) \quad \text{if } V_1(t, x^{(1)}(t)) \leq V_2(t, x^{(2)}(t)). \end{aligned} \quad (3)$$

Obviously, $V(t)$ is continuous for all $t \in R^+$.

In the following, we denote, for the sake of brevity,

$$V_j(t) = V_j(t, x^{(j)}(t)) \quad V_j'(t) = V_j'(t, x^{(j)}(t)), \quad j = 1, 2.$$

First, we prove that for any $t \in R^+$,

$$\begin{aligned} \frac{w_{11}(|x^{(1)}(t)|) + w_{12}(|x^{(2)}(t)|)}{2} &\leq V(t) \leq w_{21}(|x^{(1)}(t)|) + w_{22}(|x^{(2)}(t)|) \\ &+ w_{31} \left[\int_{-\infty}^t \phi_1(t-s)w_{41}(|x^{(1)}(s)|) ds \right] + w_{32} \left[\int_{-\infty}^t \phi_2(t-s)w_{42}(|x^{(2)}(s)|) ds \right]. \end{aligned} \quad (4)$$

In fact, if $V_1(t) \geq V_2(t)$, then by (3) and condition (i),

$$V(t) = V_1(t) \geq \frac{[V_1(t) + V_2(t)]}{2} \geq \frac{[w_{11}(|x^{(1)}(t)|) + w_{12}(|x^{(2)}(t)|)]}{2}$$

whereas, if $V_1(t) \leq V_2(t)$, we also have

$$V(t) = V_2(t) \geq \frac{[V_1(t) + V_2(t)]}{2} \geq \frac{[w_{11}(|x^{(1)}(t)|) + w_{12}(|x^{(2)}(t)|)]}{2}.$$

On the other hand, the right-hand inequality in (4) obviously holds.

Next, we show that for each $t \geq t_0$, the right-hand and the left-hand derivatives of $V(t)$, both denoted by $V'(t)$, satisfy

$$V'(t) \leq 0, \quad \text{if } V(t - \tau(t)) \leq V(t). \quad (5)$$

Indeed, suppose $V_1(s_0) \geq V_2(s_0)$ and there exists some $s_1 > s_0$ such that

$$V_1(t) \geq V_2(t) \quad \text{for } t \in [s_0, s_1].$$

Then by (3),

$$V(t) = V_1(t) \quad \text{for } t \in [s_0, s_1].$$

If $V_1(t - \tau(t)) \geq V_2(t - \tau(t))$, then $V(t - \tau(t)) = V_1(t - \tau(t))$, and hence $V(t - \tau(t)) \leq V(t)$ implies $V_1(t - \tau(t)) \leq V_1(t)$; while if $V_1(t - \tau(t)) \leq V_2(t - \tau(t))$, then $V(t - \tau(t)) = V_2(t - \tau(t))$, and also $V(t - \tau(t)) \leq V(t)$ implies $V_1(t - \tau(t)) \leq V_2(t - \tau(t)) \leq V(t) = V_1(t)$.

Therefore, in any case we have

$$V'(t) = V'_1(t) \leq 0, \quad \text{if } V(t - \tau(t)) \leq V(t).$$

If $V_1(t) \leq V_2(t)$ for $t \in [s_0, s_1]$, like before we can also prove that (5) holds.

For any given $\varepsilon > 0$ ($\varepsilon < H$), let $M\varepsilon^* = \min\{w_{11}(\varepsilon), w_{12}(\varepsilon)\}$, we may choose a $\delta(\varepsilon) > 0$ such that

$$\delta < \varepsilon, \quad w_{2j}(\delta) < \frac{\varepsilon^*}{8} \quad \text{and} \quad w_{3j}(J_j w_{4j}(\delta)) < \frac{\varepsilon^*}{8} \quad j = 1, 2$$

where $J_j = \int_0^\infty \phi_j(s) ds$ ($j = 1, 2$).

For any $\sigma \geq t_0$, $\varphi \in PC(\delta)$, $\sigma \in [\tau_{l-1}, \tau_l]$ for some positive integer l . Define $x(t) = x(t, \sigma, \varphi)$. Then by (4) we have

$$\begin{aligned} V(t, x(t)) &= V(t, \varphi(t - \sigma)) \leq w_{21}(\delta) + w_{22}(\delta) \\ &\quad + w_{31}(J_1 w_{41}(\delta)) + w_{32}(J_2 w_{42}(\delta)) < \frac{\varepsilon^*}{2} \quad \text{for } t \in [0, \sigma]. \end{aligned}$$

We prove that

$$\frac{w_{11}(|x^{(1)}(t)|) + w_{12}(|x^{(2)}(t)|)}{2} \leq V(t) \leq \frac{\varepsilon^*}{2} \quad \text{for } \sigma \leq t < \tau_l. \quad (6)$$

If this does not hold, then there is a $\hat{t} \in (\sigma, \tau_l)$ such that

$$V(\hat{t}) > \frac{\varepsilon^*}{2} \quad \text{and} \quad V'(\hat{t}) > 0, \quad V(t) \leq V(\hat{t}) \quad \text{for } t \in [\sigma, \hat{t}].$$

Since $t \geq \tau(t) \geq 0$, we have $V(\hat{t} - \tau(\hat{t})) \leq V(\hat{t})$. From (5) we have $V'(\hat{t}) \leq 0$. This is a contradiction. So (6) holds.

If $V_1(\tau_l) \geq V_2(\tau_l)$, then $V(\tau_l) = V_1(\tau_l)$; from inequality (6) and condition (iii) we have

$$\begin{aligned} V(\tau_l) &= V(\tau_l, x(\tau_l^-) + I_k(x(\tau_l^-))) = V_1(\tau_l, x(\tau_l^-) + I_k(x(\tau_l^-))) \\ &\leq (1 + b_l)V_1(\tau_l^-, x(\tau_l^-)) \leq (1 + b_l)\frac{\varepsilon^*}{2}. \end{aligned}$$

If $V_1(\tau_l) < V_2(\tau_l)$, then $V(\tau_l) = V_2(\tau_l)$; from inequality (6) and condition (iii) we have

$$\begin{aligned} V(\tau_l) &= V(\tau_l, x(\tau_l^-) + I_k(x(\tau_l^-))) = V_2(\tau_l, x(\tau_l^-) + I_k(x(\tau_l^-))) \\ &\leq (1 + b_l)V_2(\tau_l^-, x(\tau_l^-)) \leq (1 + b_l)\frac{\varepsilon^*}{2}. \end{aligned}$$

So in either case we have proved that

$$V(\tau_l) \leq (1 + b_l)\frac{\varepsilon^*}{2}.$$

Next, we prove that

$$V(t) \leq (1 + b_l)\frac{\varepsilon^*}{2} \quad \text{for } \tau_l \leq t < \tau_{l+1}. \quad (7)$$

If inequality (7) does not hold, then there is a $\hat{s} \in (\tau_l, \tau_{l+1})$ such that

$$V(\hat{s}) > (1 + b_l)\frac{\varepsilon^*}{2} \quad \text{and} \quad V'(\hat{s}) > 0, \quad V(t) \leq V(\hat{s}) \quad \text{for } t \in [\tau_l, \hat{s}].$$

Since $t \geq \tau(t) \geq 0$, we have $V(\hat{s} - \tau(\hat{s})) \leq V(\hat{s})$. From (5) we have $V'(\hat{s}) \leq 0$. This is a contradiction. So (7) holds.

If $V_1(\tau_{l+1}) \geq V_2(\tau_{l+1})$, then $V(\tau_{l+1}) = V_1(\tau_{l+1})$; from inequality (7) and condition (iii) we have

$$\begin{aligned} V(\tau_{l+1}) &= V(\tau_{l+1}, x(\tau_{l+1}^-) + I_k(x(\tau_{l+1}^-))) = V_1(\tau_{l+1}, x(\tau_{l+1}^-) + I_k(x(\tau_{l+1}^-))) \\ &\leq (1 + b_{l+1})V_1(\tau_{l+1}^-, x(\tau_{l+1}^-)) \leq (1 + b_{l+1})(1 + b_l)\frac{\varepsilon^*}{2}. \end{aligned}$$

If $V_1(\tau_{l+1}) < V_2(\tau_{l+1})$, then $V(\tau_{l+1}) = V_2(\tau_{l+1})$; from inequality (7) and condition (iii) we have

$$\begin{aligned} V(\tau_{l+1}) &= V(\tau_{l+1}, x(\tau_{l+1}^-) + I_k(x(\tau_{l+1}^-))) = V_2(\tau_{l+1}, x(\tau_{l+1}^-) + I_k(x(\tau_{l+1}^-))) \\ &\leq (1 + b_{l+1})V_2(\tau_{l+1}^-, x(\tau_{l+1}^-)) \leq (1 + b_{l+1})(1 + b_l)\frac{\varepsilon^*}{2}. \end{aligned}$$

So, in either case we have proved that

$$V(\tau_{l+1}) \leq (1 + b_{l+1})(1 + b_l)\frac{\varepsilon^*}{2}.$$

By simple induction, we can prove that, in general

$$V(t) \leq (1 + b_{l+i+1}) \cdots (1 + b_l)\frac{\varepsilon^*}{2} \quad \text{for } \tau_{l+i} \leq t \leq \tau_{l+i+1}.$$

Taking this together with (4) and (6) and $\prod_{k=1}^{\infty} (1 + b_k) = M$, we have

$$\frac{w_{11}(|x^{(1)}(t)|) + w_{12}(|x^{(2)}(t)|)}{2} \leq V(t) \leq M\frac{\varepsilon^*}{2} \quad \text{for } t \geq \sigma. \quad (8)$$

Since $M\varepsilon^* = \min\{w_{11}(\varepsilon), w_{12}(\varepsilon)\}$, we have

$$w_{11}(|x^{(1)}(t)|) \leq w_{11}(\varepsilon), \quad w_{12}(|x^{(2)}(t)|) \leq w_{12}(\varepsilon).$$

Therefore,

$$|x(t)| = \max(|x^{(1)}(t)|, |x^{(2)}(t)|) \leq \varepsilon.$$

Thus the zero solution of (1) and (2) is uniformly stable.

Corollary 1. Suppose there exist continuous Lyapunov functions $V_j : (-\infty, \infty) \times B_H^{(j)} \rightarrow R^+$ with $B_H^j = \{x^{(j)} \in R^{n_j} \mid |x^{(j)}| < H\}$ ($j = 1, 2$) and wedge functions w_{1j}, w_{2j} ($j = 1, 2$) such that

- (i) $w_{1j}(|\varphi^{(j)}(t)|) \leq V_j(t, x^{(j)}(t)) \leq w_{2j}(|\varphi^{(j)}(t)|)$;

(ii) when $V_1(t, x^{(1)}(t)) \geq V_2(t, x^{(2)}(t))$, it holds that

$$V_1'(t, x^{(1)}(t)) \leq 0, \quad \text{if } V_1(t - \tau(t), x^{(1)}(t - \tau(t))) \leq V_1(t, x^{(1)}(t)),$$

when $V_1(t, x^{(1)}(t)) \leq V_2(t, x^{(2)}(t))$, it holds that

$$V_2'(t, x^{(2)}(t)) \leq 0, \quad \text{if } V_2(t - \tau(t), x^{(2)}(t - \tau(t))) \leq V_2(t, x^{(2)}(t));$$

(iii) $V_j(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \leq (1 + b_k)v_j(\tau_k^-, x(\tau_k^-))$, $j = 1, 2$, $k = 1, 2, \dots$, in which $b_k \geq 0$, and $\sum_{k=1}^{\infty} b_k < \infty$;

where $x(t) = (x^{(1)}(t), x^{(2)}(t))$ is a solution of (1) and (2), then the zero solution of (1) and (2) is uniformly stable.

Remark. We can easily see that Corollary 1 which is a special case of Theorem 1 is an extension of the result for finite delay equations in [4] (Theorem 1). Since in our result $\tau(t)$ may be ∞ , the result that we have obtained is more general than that given in [4].

Theorem 2. Suppose there exist continuous functionals $V_i : [0, \infty) \times PC_H^{(i)}(t) \rightarrow R^+$ ($i = 1, 2, \dots, m$) such that w_{1j}, w_{2j} ($j = 1, 2, \dots, m$) are wedge functions,

(i) $w_{1j}(|\varphi^{(j)}(t)|) \leq V_j(t, \varphi^{(j)}(t)) \leq w_{2j}(|\varphi^{(j)}(t)|) + w_{3j}[\int_{-\infty}^t \phi_j(t-s)w_{4j}(|\varphi^{(j)}(s)|)ds]$, $j = 1, 2, \dots, m$;

(ii) if $V_j(t, x^{(j)}(t)) = \max\{V_l(t, x^{(l)}(t)) \mid 1 \leq l \leq m\}$, it holds that

$$V_j'(t, x^{(j)}(t)) \leq 0, \quad \text{if } V_j(t - \tau(t), x^{(j)}(t - \tau(t))) \leq V_j(t, x^{(j)}(t));$$

(iii) $V_j(\tau_k, x(\tau_k^-) + I_k(x(\tau_k^-))) \leq (1 + b_k)V_j(\tau_k^-, x(\tau_k^-))$, $j = 1, 2, \dots, m$, $k = 1, 2, \dots$, for which $b_k \geq 0$, and $\sum_{k=1}^{\infty} b_k < \infty$;

where $x(t) = (x^{(1)}(t), x^{(2)}(t), \dots, x^{(m)}(t))$ is a solution of (1) and (2), then the zero solution of (1) and (2) is uniformly stable.

Proof. Like in the proof of Theorem 1, for $x(t) = (x^{(1)}(t), x^{(2)}(t), \dots, x^{(m)}(t))$, we can define

$$V(t) = V_k(t, x^{(k)}(t)) \quad \text{if } V_k(t, x^{(k)}(t)) = \max\{V_j(t, x^{(j)}(t)) \mid 1 \leq j \leq m\};$$

and, instead of (4), similarly we can prove that

$$\frac{\left[\sum_{j=1}^m w_{1j}(|x^{(j)}(t)|) \right]}{m} \leq V(t) \leq \sum_{j=1}^m w_{2j}(|x^{(j)}(t)|) + \sum_{j=1}^m w_{3j} \left[\int_{-\infty}^t \phi_j(t-s)w_{4j}(|x^{(j)}(s)|) ds \right], \quad \text{for } t \geq \sigma.$$

And by the same process as in the proof of the Theorem 1, we can prove that the zero solution of (1) and (2) is uniformly stable.

Example. Consider the following impulsive infinite delay differential equations:

$$\begin{aligned} x_1'(t) &= -a_1x_1(t) + a_2x_2(t) + a_3x_1(t - \tau(t)) \quad t \geq t_0, t \neq \tau_k \quad x_1(\tau_k) = cx_1(\tau_k^-) \\ x_2'(t) &= b_1x_1(t) - b_2x_2(t) + b_3x_2(t - \tau(t)) \quad t \geq t_0, t \neq \tau_k \quad x_2(\tau_k) = cx_2(\tau_k^-) \end{aligned} \quad (9)$$

where $k = 1, 2, \dots, t \geq \tau(t) \geq 0, a_1 > 0, a_2 > 0, a_3 > 0, b_1 > 0, b_2 > 0, b_3 > 0, 0 < c < 1, a_2 + a_3 \leq a_1, b_1 + b_3 \leq b_2$ and $x_j(0) = 0$ $j = 1, 2$.

Let $V_j(t, x_j(t)) = \frac{1}{2}[x_j(t)]^2$ ($j = 1, 2$); obviously condition (i) of the Theorem 1 holds, and moreover when $V_1(t, x_1(t)) \geq V_2(t, x_2(t))$, i.e. $|x_1(t)| \geq |x_2(t)|$, if $V_1(t - \tau(t), x_1(t - \tau(t))) \leq V_1(t, x_1(t))$, i.e. $|x_1(t - \tau(t))| \leq |x_1(t)|$, we have

$$\begin{aligned} V_1'(t, x_1(t)) &= x_1(t)x_1'(t) = -a_1x_1^2(t) + a_2x_1(t)x_2(t) + a_3x_1(t)x_1(t - \tau(t)) \\ &\leq (-a_1 + a_2 + a_3)x_1^2(t) \leq 0; \end{aligned}$$

when $V_1(t, x_1(t)) \leq V_2(t, x_2(t))$, i.e. $|x_1(t)| \leq |x_2(t)|$, if $V_2(t - \tau(t), x_2(t - \tau(t))) \leq V_2(t, x_2(t))$, i.e. $|x_2(t - \tau(t))| \leq |x_2(t)|$, we have

$$V_2'(t, x_2(t)) = x_2(t)x_2'(t) = b_1x_1(t)x_2(t) - b_2x_2^2(t) + b_3x_2(t)x_2(t - \tau(t)) \leq (b_1 - b_2 + b_3)x_2^2(t) \leq 0.$$

And

$$V_j(x_j(\tau_k^-) + I_k(x_j(\tau_k^-))) = V_j(cx_j(\tau_k^-)) = \frac{1}{2}c^2x_j^2(\tau_k^-) < \frac{1}{2}x_j^2(\tau_k^-) = V_j(x_j^2(\tau_k^-)), \quad j = 1, 2.$$

Let $b_k = 0$, $k = 1, 2, \dots$. Then conditions (ii) and (iii) of [Theorem 1](#) hold. Therefore the zero solution of (9) is uniformly stable.

Since in this example $\tau(t)$ may be ∞ , by the previous theory we cannot obtain this stability result.

4. Conclusion

In this work, we have considered the stability of impulsive infinite delay differential equations. By using Lyapunov functions and the Razumikhin technique, we have obtained some more general results. When using the Razumikhin technique, we used a new technique that has been given in [1], and we extended this technique to study impulsive infinite delay differential systems. We can see that impulses do contribute to the system's stability behavior.

Acknowledgements

The authors would like to thank the editor and the anonymous reviewer for their constructive comments and suggestions for improving the quality of the work.

References

- [1] S.N. Zhang, A new technique in stability of infinite delay differential equations, *Comput. Math. Appl.* 44 (2002) 1275–1287.
- [2] T. Yang, Impulsive control, *IEEE Trans. Automat. Control* 44 (1999) 1081–1083.
- [3] T. Yang, *Impulsive Systems and Control: Theory and Applications*, Nova Science Publishers, Inc., Huntington, NY, 2001.
- [4] J.H. Shen, Razumikhin techniques in impulsive functional differential equations, *Nonlinear Anal.* 36 (1999) 119–130.
- [5] X.Z. Liu, G. Ballinger, Uniform asymptotic stability of impulsive delay differential equations, *Comput. Math. Appl.* 41 (2001) 903–915.
- [6] Z.G. Luo, J.H. Shen, New Razumikhin type theorems for impulsive functional differential equations, *Appl. Math. Comput.* 125 (2002) 375–386.
- [7] A.A. Soliman, Stability criteria of impulsive differential systems, *Appl. Math. Comput.* 134 (2003) 445–457.
- [8] J.T. Sun, Y.P. Zhang, Stability analysis of impulsive control systems, *IEEE Proc. Control Theory Appl.* 150 (4) (2003) 331–334.
- [9] J.T. Sun, Y.P. Zhang, Impulsive control of a nuclear spin generator, *J. Comput. Appl. Math.* 157 (1) (2003) 235–242.
- [10] Y. Zhang, J.T. Sun, Boundedness of the solutions of impulsive differential systems with time-varying delay, *Appl. Math. Comput.* 154 (1) (2004) 279–288.
- [11] J.T. Sun, Y.P. Zhang, Q.D. Wu, Less conservative conditions for asymptotic stability of impulsive control systems, *IEEE Trans. Automat. Control* 48 (5) (2003) 829–831.
- [12] C.H. Kou, S.N. Zhang, S.J. Wu, Stability analysis in terms of two measures for impulsive differential equations, *J. London Math. Soc.* 66 (1) (2002) 142–152.
- [13] C. Cuevas, M. Pinto, Asymptotic properties of solutions to nonautonomous Volterra difference systems with infinite delay, *Comput. Math. Appl.* 42 (3–5) (2001) 671–685.
- [14] J. Liang, T.J. Xiao, J. van Casteren, A note on semilinear abstract functional differential and integrodifferential equations with infinite delay, *Appl. Math. Lett.* 17 (4) (2004) 473–477.