Discrete Mathematics 124 (1994) 241–250
North-Holland

Two-cacti with minimum number of spanning trees

Preben Dahl Vestergaard

Institut for Electronic Systems, Aalborg University, Fredrik Bajers Vej 7, DK-9220 Aalborg, Denmark

Received 2 June 1990
Revised 22 July 1991

Abstract

Zelinka (1978) proved that the spanning trees of a 2-cactus partition into at least 3 isomorphism classes. Here we examine the structure of these 2-cacti for which the spanning trees partition into exactly 3 isomorphism classes.

1. Notation and preliminaries

A *tree* is a connected graph with no circuit. A *rooted tree* (T, v) is a tree T together with a distinguished vertex v, the *root*. When it is clear which root v is intended, we may for short write the rooted tree T.

A *unicyclic graph* is a connected graph G with exactly one circuit C. Let the vertices on C be $v_1, v_2, \ldots, v_n$ in that order, $C=(v_1, v_2, \ldots, v_n)$. The connected components of $G-E(C)$ are trees $T_1, T_2, \ldots, T_n$ and we can choose notation such that $v_i \in V(T_i)$, $1 \leqslant i \leqslant n$. We say that the rooted tree (T_i, v_i) is attached to C at v_i and we write $G=C((T_1, v_1), (T_2, v_2), \ldots, (T_n, v_n))$ or for short $G=C(T_1, T_2, \ldots, T_n)$. $L(T_1, T_2, \ldots, T_n)$ denotes the graph $C(T_1, T_2, \ldots, T_n)-(v_1, v_n)$.

A *cactus* is a connected graph which contains at least one circuit and which has the property that any pair of its circuits have at most a vertex in common. An *n-cactus* is a cactus with exactly n circuits, $n \geqslant 1$.

Let G be a 2-cactus with the two circuits $D_1=(u_1, u_2, \ldots, u_m)$ and $D_2=(v_1, v_2, \ldots, v_n)$ where notation can be chosen such that u_1 and v_1 are the two vertices in D_1 and D_2 respectively, having minimum distance in G, where $u_1=v_1$ may occur. G contains a connected graph H consisting of D_1, D_2 and a path joining u_1 to v_1. If $u_1=v_1$ then the path is that single vertex. The connected components of $G-E(H)$ are trees. For $i=2,3,\ldots,m$ denote by (U_i, u_i) that rooted tree which is a connected component of $G-E(H)$ and contains u_i. We say that U_i is attached to D_1 at u_i or to H at u_i. Analogously we say that $V_2, V_3, \ldots, V_n$ are trees attached to D_2, or to H, at $v_2, v_3, \ldots, v_n$, respectively.

SSDI 0012-365X(92)00064-N

$\hat{D}_1$ denotes that connected component of $G-u_1$ which contains D_1-u_1. Analogously $\hat{D}_2$ is that connected component of $G-v_1$ which contains D_2-v_1. $G'\cong G''$ means that the graphs G' and G'' are isomorphic. $(G',v')\dot{\cong}(G'',v'')$ means that the rooted graphs (G',v') and (G'',v'') are root-isomorphic, i.e., there exists an isomorphism from G' onto G'' which maps v' onto v''. If it is clear which roots are intended, we may write $G'\dot{\cong}G''$ for short.

A *spanning tree* for G is a subgraph of G which is a tree and which contains all vertices of G. For a connected graph G we denote by $\tau(G)$ the number of isomorphism classes into which all the spanning trees of G partition.

The *distance* between two vertices x and y in a connected graph is denoted $d(x,y)$. The distance between two vertex-subsets A and B in a connected graph is $d(A,B)=\min\{d(x,y)\mid x\in A, y\in B\}$.

For a vertex x in a tree T we let

$$\delta_T(x)=\delta(x)=\sum_{y\in V(T)} d(x,y)$$

denote the *deviation* of x w.r.t. T. A vertex q in T with minimum deviation is called a *median* of T and $\delta(q)=\delta(T)$ is called the *deviation of* T.

In [6] it was proved that each tree has either exactly one median or exactly two medians which are joined by an edge.

If v is a vertex of a tree T and e is an edge of T incident with v, then all vertices which belong to paths from v with the first edge e form a subgraph which is called a branch of T with the *knag* v. The branch of T with the knag v with the maximal number of vertices is called a *weight branch* and its number of vertices is called the *weight* at v. In [6] it was proved that a vertex of a tree has minimum weight, if and only if it is a median of this tree.

Below in (1)–(3) we give adaptions of the characterization of graphs with exactly one isomorphism class of spanning trees ([1,2,4,7]) and the characterization of graphs with exactly two isomorphism classes of spanning trees ([3, Theorem 2]).

Let F denote a unicyclic graph obtained from a 2-cactus by the deletion of one edge. We then have the following.

$\tau(F)=1$ *if and only if*

(1) $F=C(A,B;\ A,B,\dots;A,B)$ *or* $F=C(A,A,\dots,A)$.

$\tau(F)=2$ *if and only if*

either (2) $F=C(A,B,B;A,B,B;\dots;A,B,B;\dots;A,B,B)$, $A\dot{\ncong} B$,

or (3) $F=C(A,B,C,B;A,B,C,B;\dots;A,B,C,D;\dots;A,B,C,B)$, $A\dot{\ncong} C$.

2. Statement of Theorem

Theorem. *The spanning trees of a 2-cactus G partition into exactly three isomorphism classes if and only if G can be obtained by one of the following constructions:*

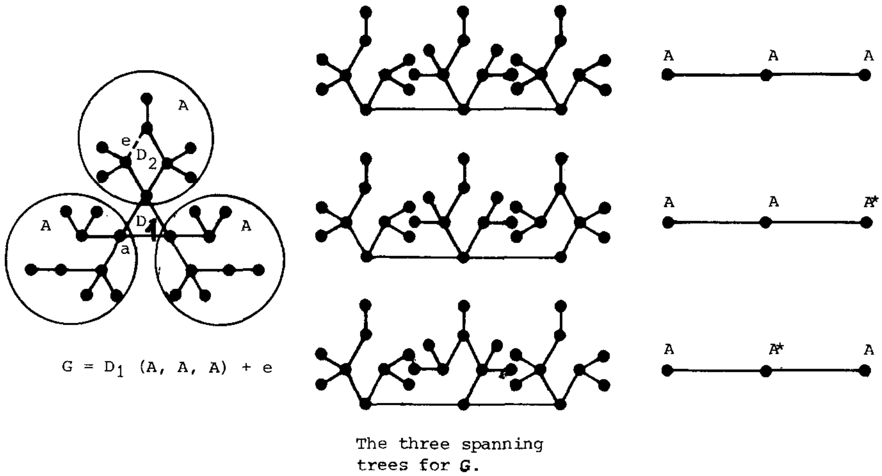


Fig. 1. Construction 1.

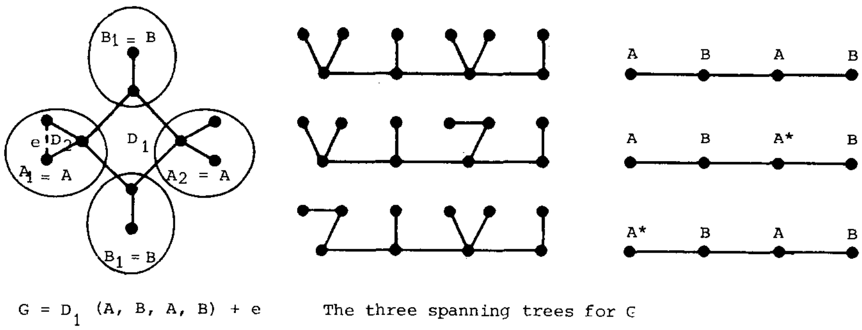


Fig. 2. Construction 2.

Construction 1 (*see Fig.* 1): (I) *Let U be the unicyclic graph $U=D_1(A,A,A)$,*
(II) *Add an edge e to one of the pendant trees A, such that:*
- (i) *the circuit D_2 in $A+e$ has length* 3 *or* 4 *and*
- (ii) *the edges of D_2 partition into exactly two $\dot{\cong}$-isomorphism classes in the rooted graph $(A+e,a)$, where the root a is the attachment-vertex of A to D_1 in the graph U.*

Construction 2 (*see Fig.* 2): (I) *Let U be the unicyclic graph*

$$U=D_1((A_1,a_1),(B_1,b_1)(A_2,a_2),(B_2,b_2))$$

where

$$(A_1,a_1)\dot{\cong}(A_2,a_2) \text{ and } (B_1,b_1)\dot{\cong}(B_2,b_2).$$

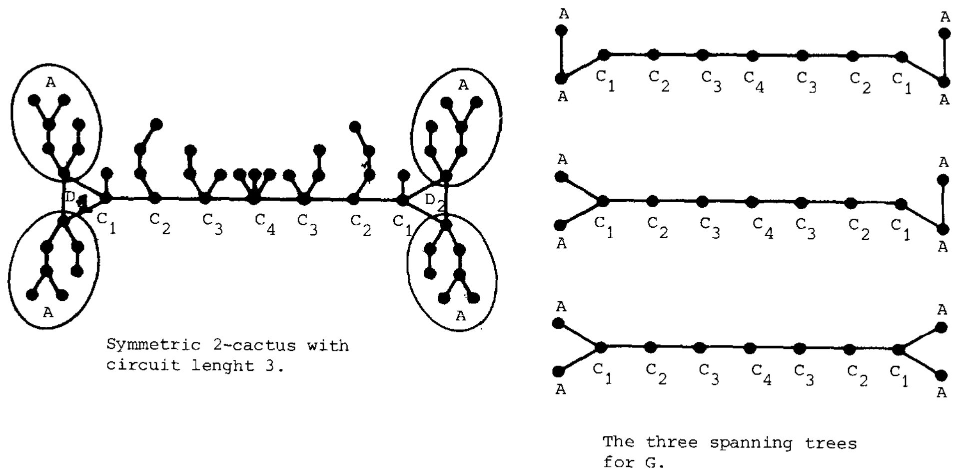


Fig. 3. Construction 3.

(II) *Add an edge e to one of the pendant trees, say A_1 such that:*
(i) *the circuit D_2 in A_1+e has length* 3 *or* 4 *and*
(ii) *the edges of D_2 partition into exactly two $\dot{\cong}$-isomorphism classes in the rooted graph (A_1+e, a_1).*

Construction 3 (*see Fig.* 3): *Let H be a* 2-*cactus consisting of the two circuits $D_1=(u_1,u_2,u_3), D_2=(v_1,v_2,v_3)$ and a path joining u_1 to v_1. Attach a copy of the rooted tree (A,a) to H at each of the vertices u_2,u_3,v_2,v_3.*

Attach symmetrically a rooted tree at each vertex of the u_1v_1-path:

$$C_1,C_2,C_3,\dots,C_k,C_{k+1},C_k,\dots,C_3,C_2,C_1$$

or

$$C_1,C_2,C_3,\dots,C_{k-1},C_k,C_k,C_{k-1},\dots,C_3,C_2,C_1.$$

Construction 4 (*see Fig.* 4): *Let H be a* 2-*cactus consisting of the two circuits $D_1=(u_1,u_2,u_3,u_4), D_2=(v_1,v_2,v_3,v_4)$ and a path joining u_1 to v_1. Attach a copy of the rooted tree (A,a) to H at each of the vertices u_2,u_4,v_2,v_4. Attach a copy of the rooted tree (B,b) to H at each of the vertices u_3,v_3.*

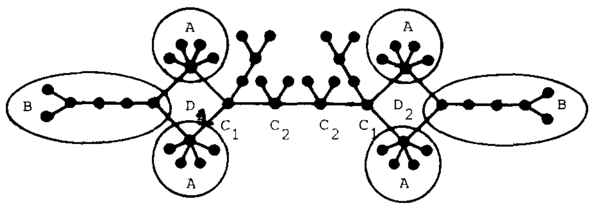
Attach symmetrically a rooted tree at each vertex of the u_1v_1-path:

$$C_1,C_2,C_3,\dots,C_k,C_{k+1},C_k,\dots,C_3,C_2,C_1$$

or

$$C_1,C_2,C_3,\dots,C_{k-1},C_k,C_k,C_{k-1},\dots,C_3,C_2,C_1.$$

Definition. We shall call a 2-cactus *symmetric* if it can be obtained by Construction 3 or 4.



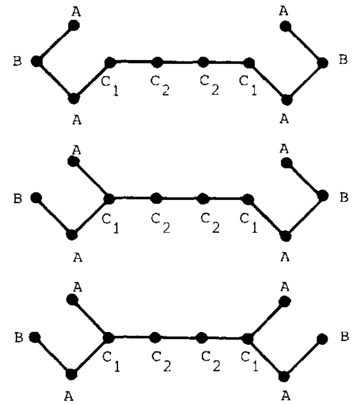


Fig. 4. Construction 4.

Remark. (ii) implies (i) in both of Constructions 1 and 2: If the edges of a rooted circuit partition into exactly two $\dot{\cong}$-isomorphism classes then the circuit must have length 3 or 4. Further, (ii) implies that trees attached to the circuit equidistant from the root (resp. a and a_1) must be $\dot{\cong}$-isomorphic.

3. Proof of the Theorem

If G has been obtained by Constructions 1, 2, 3 or 4, then it can be verified by inspection that $\tau(G)=3$. Conversely, let G be a 2-cactus with $\tau(G)=3$. We shall then prove that G can be obtained by one of Constructions 1, 2, 3 or 4.

Let for the remainder of this paper G denote a 2-cactus with $\tau(G)=3$ with circuits $D_1=(u_1,u_2,\dots,u_m)$ and $D_2=(v_1,v_2,\dots,v_n)$. Let $d(D_1,D_2)=d(u_1,v_1)$ and let $\hat{D}_1$ (resp. $\hat{D}_2$) denote that connected component of $G-u_1$ (resp. $G-v_1$) which contains D_1-u_1 (resp. D_2-v_1). We may suppose $|V(\hat{D}_1)|\geqslant|V(\hat{D}_2)|$. This hypothesis implies by [8] that for any spanning tree of G the intersection of its median with $\hat{D}_2$ will be empty.

Lemma 1. *With notation and hypotheses about G as above we have*

$$\forall e\in E(D_2): \tau(G-e)\leqslant 2.$$

Proof of Lemma 1. Suppose that an edge $e\in E(D_2)$ exists such that $\tau(G-e)\geqslant 3$. We shall then prove that $\tau(G)\geqslant 4$. Among all spanning trees of $G-e$ choose one, say T, for which the deviation, $\delta(T)$, is maximum. If $e\notin\{(v_1,v_2),(v_1,v_n)\}$ then the spanning tree $T^*=T+e-(v_1,v_2)$ for G will have the same median, $M(T^*)=M(T)$, and therefore $\delta(T^*)>\delta(T)$, but then T^* cannot be isomorphic to any spanning tree of $G-e$, and we have that $\tau(G)\geqslant 4$. If $e\in\{(v_1,v_2),(v_1,v_n)\}$ then we may analogously obtain $\tau(G)\geqslant 4$ by considering a spanning tree T of $G-e$ with minimum deviation and $T^*=T+e-(v_2,v_3)$. In either case the hypothesis $\tau(G)=3$ is contradicted, so Lemma 1 is proven. □

For $i=1,2$ let $E_i(D_2)=\{e\in E(D_2)\mid\tau(G-e)=i\}$. Then $E(D_2)=E_1(D_2)\cup E_2(D_2)$ is a partitioning of $E(D_2)$.

Lemma 2. *Notation and hypotheses are as above. Let G be a 2-cactus with* $\tau(G)=3$, $|V(\hat{D}_1)|\geqslant|V(\hat{D}_2)|$, *and* $E(D_2)=E_1(D_2)\cup E_2(D_2)$.

Then we have:

(i) *If* $e_1,e_2\in E_1(D_2)$ *then* $(G-e_1,u_1)\dot{\cong}(G-e_2,u_1)$,

(ii) $0\leqslant|E_1(D_2)|\leqslant 2$.

(iii) *The length of* D_2 *is either* 3 *or* 4.

Proof of Lemma 2. If $|E_1(D_2)|\geqslant 2$ then let $e_1,e_2\in E_1(D_2)$, $e_1\neq e_2$. It is proven in e.g. [4] for $i=1,2$ that $\tau(G-e_i)=1$ implies that the pendant trees in $G-e_i$ to D_1 at u_1 and u_3, respectively, are root-isomorphic. But the pendant tree at u_3 remains unchanged from $G-e_1$ to $G-e_2$, therefore the pendant trees to D_1 at u_1 in $G-e_1$ and $G-e_2$, respectively, are root-isomorphic.

This proves (i).

We shall prove (ii) by showing that the root-isomorphism in (i) can occur for at most one pair of distinct edges in $E_1(D_2)$. For any pair of distinct edges e_1,e_2 in $E_1(D_2)$ the position of e_1 and D_2 will uniquely determine that of e_2. This is because the deviation of u_1 w.r.t. S, $\delta_S(u_1)=\sum_{x\in V(S)}d(u_1,x)$, must by (i) remain the same integer whether S is the pendant tree rooted at u_1 in $G-e_1$ or in $G-e_2$. This proves that $|E_1(D_2)|\leqslant 2$ and (ii) is finished. By (ii) we have that $E_2(D_2)\neq\emptyset$. Let $f\in E_2(D_2)$, then by definition of $E_2(D_2)$ we have that $\tau(G-f)=2$. Let T_1 and T_2 denote two non-isomorphic spanning trees for $G-f$, and hence also for G. If $|E(D_2)|\geqslant 5$ then we can by arguing as in the proof of Lemma 1 construct another two spanning trees T_3,T_4 for G such that their deviations satisfy $\delta(T_3)\neq\delta(T_4)$ and $\{\delta(T_1),\delta(T_2)\}\cap\{\delta(T_3),\delta(T_4)\}=\emptyset$. Hence we have $\tau(G)\geqslant 4$, a contradiction.

This proves (iii) and Lemma 2 is proven. □

We shall now demonstrate that not only D_2 but also D_1 has length 3 or 4.

We shall consider $E_1(D_2)\neq\emptyset$ and $E_1(D_2)=\emptyset$ separately. In both cases the main tools in our analysis will be (1)–(3) from the preliminaries.

Lemma 3. *Notation and hypotheses are as above. Let G be a 2-cactus with* $\tau(G)=3$, $|V(\hat{D}_1)|\geqslant|V(\hat{D}_2)|$, *and* $E(D_2)=E_1(D_2)\cup E_2(D_2)$.

If $E_1(D_2)\neq\emptyset$ *then we have:*

either (i) *the length of* D_1 *is* 3 *and there exists an edge e on* D_2 *such that we can express G in the form:* $G=D_1((A+e,u_1),(A,u_2),(A,u_3))$, *or* (ii) *the length of* D_1 *is* 4 *and there exists an edge* e *on* D_2 *such that we can express* G *in the form*: $G=D_1((A+e,u_1),(B,u_2),(A,u_3),(B,u_4))$, $(A\dot{\cong}B$ *may occur*).

Obviously, we have both in (i) and (ii) that $D_2\subseteq A+e$, and we shall see below that we can choose any edge as e from $E_1(D_2)$.

Proof of Lemma 3. Since $E_1(D_2)\neq\emptyset$ there exist by Lemmas 1 and 2(ii) edges $e\in E_1(D_2)$ and $f\in E_2(D_2)$ such that $\tau(G-e)=1$ and $\tau(G-f)=2$. We shall compare the structure of $G-e$ with that of $G-f$ by help of (1)–(3).

If (1) and (2) hold for $G-e$ and $G-f$, respectively, then by (2) the length of D_1 is a multiple of 3, and we must in fact have $|E(D_1)|=3$. Because, suppose otherwise $|E(D_1)|\geqslant 6$ and let U_i, $1\leqslant i\leqslant m$, denote the rooted tree attached to D_1 at u_i in $G-e$ and let U_1^* denote the rooted tree $U_1^*=U_1+e-f$ attached to D_1 at u_1 in $G-f$.

From (1) we obtain $U_1\dot{\cong}U_3$, $U_2\dot{\cong}U_4$ and from (2) we obtain $U_2\dot{\cong}U_3$, $U_1^*\dot{\cong}U_4$. This yields $U_1\dot{\cong}U_1^*$, but obviously $\tau(G-e)\neq\tau(G-f)$ implies that $U_1\dot{\ncong}U_1^*$. This contradiction proves that D_1 has length 3.

By (1) this implies that $U_1\dot{\cong}U_2\dot{\cong}U_3\dot{\cong}A$ and $G-e=D_1(A,A,A)$ or $G=D_1(A+e,A,A)$ as described in (i).

If (1) and (3) hold, then similarly we obtain (ii), because $|E(D_1)|$ must by (3) be a multiple of 4 and $|E(D_1)|\geqslant 8$ would for $e\in E_1(D_2)$, $f\in E_2(D_2)$ imply that $U_1\dot{\cong}U_3\dot{\cong}U_5$ by (1) and $U_1^*\dot{\cong}U_5$ by (3) in contradiction to $U_1\dot{\ncong}U_1^*$. Thus $|E(D_1)|=4$ and we have by (1) that $G-e=D_1(A,B,A,B)$ or $G=D_1(A+e,B,A,B)$ as desired in (ii). This proves Lemma 3. □

Lemma 4. *Notation and hypotheses are as above. Let* G *be a* 2*-cactus with* $\tau(G)=3$, $|V(\hat{D}_1)|\geqslant|V(\hat{D}_2)|$, *and* $E(D_2)=E_1(D_2)\cup E_2(D_2)$.

If $E_1(D_2)=\emptyset$ *then*:

(i) D_1 *has length either* 3 *or* 4

(ii) *the trees attached to* $D_1=(u_1,u_2,\ldots,u_m)$ *at* u_2 *and* u_m *are root-isomorphic to each other.*

Proof of Lemma 4. Let $f_1,f_2\in E_2(D_2)$ such that the two trees attached to D_1 at u_1 in $G-f_1$, $G-f_2$ respectively, are not $\dot{\cong}$-isomorphic. This is for instance the case if f_1, but not f_2, is incident with v_1.

For $i=1,2$ the graph $G-f_i$ consists of D_1 with trees attached in repeated series of length 3 or 4 as expressed in (2) or (3), respectively.

There can only be one sequence of 3 or 4 trees round D_1 because the tree attached to D_1 at u_1 otherwise would be repeated in the next sequence. But that would imply that the trees attached to D_1 at u_1 in $G-f_1$ and $G-f_2$, respectively, should be root-isomorphic and that contradicts the choice of f_1, f_2. Hence D_1 must have length 3 or 4. This proves (i).

We shall split the proof of (ii) into two cases.

Case 1: $|E(D_1)|=3$.

Suppose (ii) is false, then we have from (2) that $U_2 \dot{\ncong} U_3$ for the trees U_2, U_3 attached to D_1 to u_2, u_3, respectively, and also that the trees U_1, U_1^* attached to D_1 at u_1 in $G-f_1$, $G-f_2$, respectively, must be pairwise root-isomorphic to U_2 and U_3, say $U_1 \dot{\cong} U_2$, $U_1^* \dot{\cong} U_3$. But then we find that $G-f_1$ has two non-isomorphic spanning trees $T_1=L(U_1, U_2, U_3)$ and $T_2=L(U_1, U_3, U_2)$ and $G-f_2$ has two non-isomorphic spanning trees $T_3=L(U_1^*, U_2, U_3)$ and $T_4=L(U_1^*, U_3, U_2)$.

These four trees T_1, T_2, T_3, T_4 are, in fact, pairwise non-isomorphic. T_1 has the property that it contains two adjacent edges such that their deletion leaves the three connected components $U_1, U_2, U_3 (U_1 \dot{\cong} U_2)$ but neither in T_3, nor in T_4 can such two adjacent edges be found. Hence $T_1 \dot{\ncong} T_3$ and $T_1 \dot{\ncong} T_4$. Analogously we can show that $T_2 \dot{\ncong} T_3$ and $T_2 \dot{\ncong} T_4$. This yields the contradiction that $\tau(G) \geqslant 4$ and we have proved that (ii) holds in Case 1.

Case 2: $|E(D_1)|=4$.

Suppose (ii) is false, then we have $U_2 \dot{\ncong} U_4$ for the trees U_2, U_4 attached to D_1 at u_2, u_4, respectively, and by (3) the tree U_3 attached to D_1 at u_3 must be root-isomorphic to the tree U_1 attached to D_1 at u_1 in the graph $G-f_1$, but U_3 must also be root-isomorphic to U_1^*, which is the tree attached to D_1 at u_1 in the graph $D-f_2$. This contradicts the choice of f_1, f_2 and we proved that (ii) also holds in Case 2. This completes the proof of Lemma 4. □

Lemma 5. *Notation and hypotheses are as above. Let G be a 2-cactus with $\tau(G)=3$, $|V(\hat{D}_1)| \geqslant |V(\hat{D}_2)|$, and $E(D_2)=E_1(D_2) \cup E_2(D_2)$.*

If $E_1(D_2)=\emptyset$ then either $|E(D_1)|=|E(D_2)|=3$ or $|E(D_1)|=|E(D_2)|=4$.

Proof of Lemma 5. The lengths of D_1, D_2 must by Lemmas 2(iii) and 4(i) be either 3 or 4. We shall prove that the circuits cannot have different lengths. Suppose, say, that D_1 has length 4 and D_2 has length 3. The case of interchanged lengths can be treated analogously.

On D_2 choose edges f_1, f_2 such that f_1 is incident with v_1 but f_2 is not. Let U_1, U_1^* denote the trees rooted at u_1 in $G-f_1$ and $G-f_2$, respectively. Obviously $U_1 \dot{\ncong} U_1^*$.

From Lemma 4(ii) we have that the trees U_2, U_4 attached to D_1 at u_2, u_4, respectively, satisfy $U_2 \dot{\cong} U_4$ while U_3, the tree attached to D_1 at u_3, satisfies $U_3 \dot{\ncong} U_1$ and $U_3 \dot{\ncong} U_1^*$.

G has the four spanning trees:

$$T_1=L(U_1, U_2, U_3, U_4),$$

$$T_2=L(U_3, U_2, U_1, U_4),$$

$$T_3=L(U_1^*, U_2, U_3, U_4),$$

$$T_4=L(U_3, U_2, U_1^*, U_4), (U_2 \dot{\cong} U_4).$$

We have from (3) that $T_1 \not\cong T_2$ and $T_3 \not\cong T_4$. We can prove that $T_1 \not\cong T_3$ and $T_1 \not\cong T_4$, and similarly that $T_2 \not\cong T_3$, $T_2 \not\cong T_4$, by deletion of edges analogously to the proof of Case 1 in Lemma 4. Thus we obtain the contradiction $\tau(G) \geqslant 4$ and Lemma 5 is proven. □

Lemma 6. *Notation and hypothesis are as above. Let G be a 2-cactus with* $\tau(G)=3$, $|V(\hat{D}_1)| \geqslant |V(\hat{D}_2)|$, *and* $E(D_2)=E_1(D_2) \cup E_2(D_2)$.

If $E_1(D_2)=\emptyset$ *then G is a symmetric cactus, i.e., G can be obtained either by Construction* 3 *or by Construction* 4.

Proof of Lemma 6. Consider the following four spanning trees of G (see Fig. 5):

$$T_1 = G - \{(u_1, u_2), (v_1, v_2)\},$$

$$T_2 = G - \{(u_2, u_3), (v_1, v_2)\},$$

$$T_3 = G - \{(u_1, u_2), (v_2, v_3)\},$$

$$T_4 = G - \{(u_2, u_3), (v_2, v_3)\}.$$

We have from $(v_1, v_2) \in E_2(D_2)$ that $T_1 \not\cong T_4$ and from $(v_2, v_3) \in E_2(D_2)$ that $T_3 \not\cong T_4$. We have that $T_1 \not\cong T_3$ and $T_2 \not\cong T_4$ because T_1 and T_3 (resp. T_2 and T_4) have the same median but $\delta(T_1) > \delta(T_3)$ (resp. $\delta(T_2) > \delta(T_4)$).

We shall now prove that $T_1 \not\cong T_4$. We shall consider $|E(D_1)| = |E(D_2)| = 3$ and $|E(D_1)| = |E(D_2)| = 4$ separately.

Case 1: $|E(D_1)| = 3$.

$\hat{D}_1$ is by Lemma 4(ii) symmetric w.r.t. u_1, therefore $M(T_4) \cap \hat{D}_1 = \emptyset$ and by [8] the hypothesis $|V(\hat{D}_1)| \geqslant |V(\hat{D}_2)|$ implies that $M(T_4) \cap \hat{D}_2 = \emptyset$.

In T_4 we thus have that $M(T_4) \subseteq G - (\hat{D}_1 \cup \hat{D}_2)$, and in T_1 we can then easily show that $M(T_1)$ must be the corresponding same one vertex or two vertices because

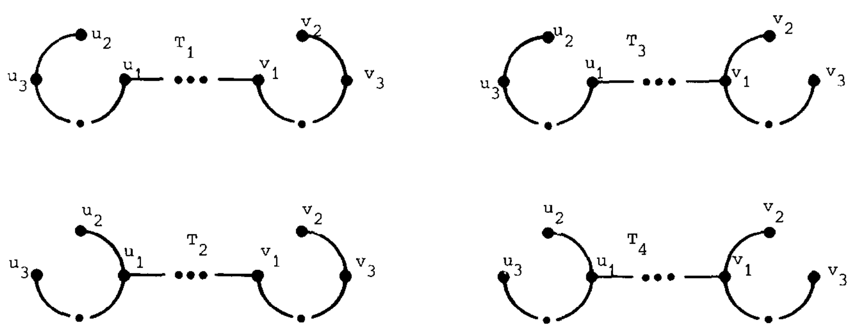


Fig. 5. Four spanning trees of G. The common length of circuits D_1, D_2 may be 3 or 4, this is indicated by a dot.

internal edge-exchanges within a weight branch do not affect the weight of a vertex. But now we have that $\delta(T_1) > \delta(T_4)$ and thus $T_1 \ncong T_4$.

Case 2: $|E(D_1)| = 4$.

As in Case 1 it can be proved that in T_4 we have that $M(T_4) \subseteq G - (\hat{D}_1 \cup \hat{D}_2)$ if and only if in T_1 and T_4 we have that $M(T_1)$ and $M(T_4)$ are the corresponding same one or two vertices. We can then use $\delta(T_1) > \delta(T_4)$ to establish $T_1 \ncong T_4$.

If $M(T_4 \nsubseteq G - (\hat{D}_1 \cup \hat{D}_2)$ then it is not difficult, but somewhat tedious, to show by case examination that we also obtain $T_1 \ncong T_4$. We thus have that T_1, T_2, T_4 are three pairwise non-isomorphic spanning trees for G. Since $\tau(G) = 3$ and $T_3 \ncong T_1$, $T_3 \ncong T_4$ it then follows that $T_2 \cong T_3$. This in turn will lead to the desired conclusion that G is symmetric.

We shall sketch the argument for $|E(D_1)| = 3$, the case $|E(D_1)| = 4$ can be treated similarly.

Suppose $|E(D_1)| = 3$. By symmetry of u_2, u_3 in T_2 we see that $M(T_2) \subseteq G - (\hat{D}_1 \cup \hat{D}_2)$. Therefore the medians of T_2 and T_3 are corresponding vertices and we can by comparison of connected components of $T_2 - M(T_2)$ and $T_3 - M(T_3)$ establish that G is symmetric as desired. This proves Lemma 6. □

The proof of the Theorem can now be concluded:

For $E_1(D_2) \neq \emptyset$ Lemmas 1–3 prove (I) and II(i) in the statements that G can be obtained by Construction 1 or 2. The proof of statement II(ii) concerning Constructions 1 and 2 is not difficult and is left to the reader.

For $E_1(D_2) = \emptyset$ the proof of the Theorem follows from Lemmas 4–6.

This concludes the proof of the Theorem. □

References

[1] R. Fischer, Über Graphen mit isomorphen Gerüsten, Monatsh. Math. 77 (1973) 24–30.

[2] L. Friess, Graphen worin je zwei Gerüste isomorph sind, Math. Ann. 204 (1973) 65–71.

[3] B.L. Hartnell, On graphs with exactly two isomorphism classes of spanning trees, Utilitas Math. 6 (1974) 121–137.

[4] P.D. Vestergaard, Finite and infinite graphs whose spanning trees are pairwise isomorphic, Ann. Discrete Math. 41 (1989) 421–436.

[5] P.D. Vestergaard, The least number of spanning trees in a cactus, to appear.

[6] B. Zelinka, Medians and peripherians of trees, Arch. Math. Brno 4 (1968) 87–95.

[7] B. Zelinka, Grafy, jejichž všechny kostry jsou spolu isomorfni, Časopis Pěst. Mat. 96 (1971) 33–40. (Title translated: Graphs, all of whose spanning trees are isomorphic to each other.)

[8] B. Zelinka, The number of isomorphism classes of spanning trees of a graph, Math. Slovaca 28 (1978) 385–388.