Absolute matrix summability methods

H.S. Özarslan *, T. Ari

Department of Mathematics, Erciyes University, 38039 Kayseri, Turkey

Abstract

In this paper, a general theorem dealing with the \( \varphi - |A; \delta|_k \) summability method has been proved. This theorem also includes some known results.

1. Introduction

Let \( \sum a_n \) be a given infinite series with the partial sums \( (s_n) \). Let \( (p_n) \) be a sequence of positive numbers such that

\[
P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad (n \to \infty), \quad (p_{-i} = p_{-i} = 0, \ i \geq 1).
\]

The sequence-to-sequence transformation

\[
t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v
\]

defines the sequence \( (t_n) \) of the Riesz means of the sequence \( (s_n) \), generated by the sequence of coefficients \( (p_n) \) (see [1]). The series \( \sum a_n \) is said to be summable \( |R, p_n|_k \), \( k \geq 1 \), if (see [2])

\[
\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty.
\]

Let \( A = (a_{mv}) \) be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then \( A \) defines the sequence-to-sequence transformation, mapping the sequence \( s = (s_n) \) to \( As = (A_n(s)) \), where

\[
A_n(s) = \sum_{v=0}^{n} a_{mv} s_v, \quad n = 0, 1, \ldots.
\]

The series \( \sum a_n \) is said to be summable \( |A|_k \), \( k \geq 1 \), if (see [3])

\[
\sum_{n=1}^{\infty} n^{k-1} |\Delta A_n(s)|^k < \infty.
\]
where
\[ \tilde{A}_n(s) = A_n(s) - A_{n-1}(s). \]

If we take \( a_{nv} = \frac{p_n}{q_n} \), then \( |A|_k \) summability is the same as \( |R, p_n|_k \) summability.

Let \( (\phi_n) \) be a sequence of positive real numbers. We say that the series \( \sum a_n \) is summable \( \varphi - |A; \delta|_k \), \( k \geq 1 \) and \( \delta \geq 0 \), if
\[
\sum_{n=1}^{\infty} \phi_n^{k+1} |\tilde{A}_n(s)|^k < \infty. \tag{6}
\]

If we take \( \delta = 0 \) and \( \phi_n = n \) for all values of \( n \), then \( \varphi - |A; \delta|_k \) summability is the same as \( |A|_k \) summability.

Before stating the main theorem we must first introduce some further notations.

Given an normal matrix \( A = (a_{nv}) \), we associate two lower semimatrices \( \bar{A} = (\bar{a}_{nv}) \) and \( \hat{A} = (\hat{a}_{nv}) \) as follows:
\[
\bar{a}_{nv} = \begin{cases} n - v & \text{if } n = 0, v = 0, 1, \ldots \\ a_{nv} & \text{if } n, v = 1, 2, \ldots. \end{cases}
\]

and
\[
\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots, \tag{8}
\]

It may be noted that \( \bar{A} \) and \( \hat{A} \) are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have
\[
A_n(s) = \sum_{v=0}^{n} a_{nv} s^v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{9}
\]

and
\[
\tilde{A}_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{10}
\]

If \( A \) is a normal matrix, then \( A' = (a'_{nv}) \) will denote the inverse of \( A \). Clearly if \( A \) is normal then, \( \hat{A} = (\hat{a}_{nv}) \) is normal and it has two-sided inverse \( \hat{A}' = (\hat{a}'_{nv}) \), which is also normal (see [4]).

The following result dealing with the relative strength of two absolute summability methods was given by Bor [2].

**Theorem A.** Let \( k > 1 \). In order that
\[
|R, p_n|_k \Rightarrow |R, q_n|_k \tag{11}
\]

it is necessary that
\[
\frac{q_n}{p_n} \frac{P_n}{Q_n} = O(1). \tag{12}
\]

If we suppose that
\[
\sum_{n=1}^{\infty} n^{k-1} q_n^k \frac{Q_n}{Q_{n-1}} = O \left( \frac{q^k \cdot Q^{-1}}{Q^k} \right) \tag{13}
\]

then (12) is also sufficient.

**Remark.** If we take \( k = 1 \), then condition (13) is obvious.

2. **Main theorem**

The aim of this paper is to generalize Theorem A for the \( \varphi - |A; \delta|_k \) and \( \varphi - |B; \delta|_k \) summabilities. Therefore we shall prove the following theorem.

**Theorem.** Let \( k > 1 \), \( A = (a_{nv}) \) and \( B = (b_{nv}) \) be two positive normal matrices. In order that
\[
\varphi - |A; \delta|_k \Rightarrow \varphi - |B; \delta|_k \tag{14}
\]

it is necessary that
\[
b_{nm} = O(a_{nm}). \tag{15}
\]
If we suppose that
\[ b_{n-1,v} \geq b_{nv}, \quad \text{for } n \geq v + 1, \]  \tag{16}
\[ \bar{a}_{n0} = 1, \quad \tilde{b}_{n0} = 1, \quad n = 0, 1, 2, \ldots, \]
\[ a_{nv} - a_{v+1,v} = O(a_{nv} a_{v+1,v+1}), \]
\[ \sum_{v=1}^{n-1} (b_{nv} \tilde{b}_{n,v+1}) = O(b_{nn}), \]  \tag{19}
\[ \sum_{n=v+1}^{m+1} \varphi_n^{\delta k-k-1} b_{nm} \tilde{b}_{n,v+1} = O(\varphi_v^{\delta k-k-1} b_{nv}^{k-1}), \]  \tag{20}
\[ \sum_{n=v+1}^{m+1} \varphi_n^{\delta k-k-1} a_{nm}^{k-1} |\Delta_v \tilde{b}_{nv}| = O(\varphi_v^{\delta k-k-1} b_{nv}^k), \]  \tag{21}
\[ \sum_{v=r+2}^{n} b_{nv} |\tilde{a}_{vr}'| = O(\tilde{b}_{n,r+1}), \]  \tag{22}
then (15) is also sufficient.

It should be noted that if we take \( \delta = 0, a_{nv} = \frac{a_v}{a_v}, b_{nv} = \frac{b_v}{b_v} \) and \( \varphi_v = n \) for all values of \( n \) in this theorem, then we get Theorem A.

We need the following lemma for the proof of our theorem.

**Lemma ([2])**. Let \( k \geq 1 \) and let \( A = (a_{nv}) \) be an infinite matrix. In order that \( A \in (\hat{f}; \hat{f}) \) it is necessary that
\[ a_{nv} = O(1) \quad (\text{all } n, v). \]  \tag{23}

### 3. Proof of the theorem

**Necessity.** Now, let \( (x_n) \) and \( (y_n) \) be denote the A-transform and B-transform of the series \( \sum a_n \), respectively. Then we have, by (9) and (10)
\[ \tilde{\Delta}x_n = \sum a_{nv} \bar{a}_v \quad \text{and} \quad \tilde{\Delta}y_n = \sum \hat{b}_{nv} \hat{a}_v, \]
which implies that
\[ a_v = \sum_{r=0}^{v} \hat{a}_{vr}' \tilde{\Delta}x_r. \]  \tag{24}

In this case
\[ \tilde{\Delta}y_n = \sum_{v=0}^{n} \hat{b}_{nv} \bar{a}_v = \sum_{v=0}^{n} \hat{b}_{nv} \sum_{r=0}^{v} \hat{a}_{vr}' \tilde{\Delta}x_r. \]

On the other hand, since
\[ \hat{b}_{n0} = \hat{b}_{n0} - \hat{b}_{n-1,0} \]
by (17), we have that
\[ \tilde{\Delta}y_n = \sum_{r=0}^{n} \hat{b}_{nr} \left( \sum_{v=0}^{r} \hat{a}_{vr}' \tilde{\Delta}x_r \right) \]
\[ = \sum_{v=0}^{n} \hat{b}_{nv} \hat{a}_v \tilde{\Delta}x_v + \sum_{v=1}^{n} \hat{b}_{nv} \hat{a}_{v-1} \tilde{\Delta}x_{v-1} + \sum_{v=0}^{n-2} \hat{b}_{nv} \hat{a}_v \tilde{\Delta}x_r \]
\[ = \hat{b}_{nn} \hat{a}'_{nn} \tilde{\Delta}x_n + \sum_{r=1}^{n-1} (\hat{b}_{nv} \hat{a}'_{vr} + \hat{b}_{n,v+1} \hat{a}'_{v+1,r}) \tilde{\Delta}x_r + \sum_{r=0}^{n-2} \tilde{\Delta}x_r \sum_{v=r+2}^{n} \hat{b}_{nv} \hat{a}_{vr}' \]  \tag{25}
by considering the equality
\[ \sum_{k=v}^{n} \hat{a}_{nk}' \hat{a}_{kv} = \delta_{nv}, \]
where $\delta_{m}$ is the Kronecker delta, we have that
\[
\hat{b}_{m,v} \hat{a}_{v} + \hat{b}_{n,v+1} \hat{a}_{v+1} = \hat{b}_{m,v} + \hat{b}_{n,v+1} \left( -\frac{\hat{a}_{v+1,v}}{\hat{a}_{v+1,v+1}} \right)
\]
\[
= \Delta_{v} \hat{b}_{m,v} + \hat{b}_{n,v+1} \frac{a_{v+1,v}}{a_{v+1,v+1}} - a_{v+1,v+1}
\]
and so
\[
\Delta_{y_{n}} = \frac{b_{m,n}}{a_{n,m}} \Delta_{x_{n}} + \sum_{v=1}^{n-1} \frac{\Delta_{v} \hat{b}_{m,v}}{a_{n,m}} \Delta_{x_{v}} + \sum_{v=1}^{n-1} \hat{b}_{n,v+1} \frac{a_{v+1,v}}{a_{v+1,v+1}} \Delta_{x_{v}} + \sum_{v=1}^{n-2} \Delta_{x_{v}} \sum_{v+1}^{n} \hat{b}_{n,v} \hat{a}_{v+1}
\]
= $T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}$, say.

Now, from (23) we can write down the matrix transforming $(\varphi_{n+\frac{1}{2}} \Delta x_{n})$ into $(\varphi_{n+\frac{1}{2}} \Delta y_{n})$. The assertion (14) is equivalent to the assertion that this matrix $\epsilon (\hat{\delta} ; \hat{\Phi})$. Hence, by the lemma, a necessary condition for (14) is that the elements of this matrix should be bounded, and this gives the result that (15) is necessary.

**Sufficiency.** Suppose the conditions are satisfied. Then, since
\[
|T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}|^{k} \leq 4^{k} (|T_{n,1}|^{k} + |T_{n,2}|^{k} + |T_{n,3}|^{k} + |T_{n,4}|^{k})
\]
to complete the proof of the theorem, it is sufficient to show that
\[
\sum_{n=1}^{\infty} \varphi_{n}^{k} |T_{n,i}|^{k} < \infty \quad \text{for } i = 1, 2, 3, 4.
\]

Firstly, we have
\[
\sum_{n=1}^{m} \varphi_{n}^{k} |T_{n,i}|^{k} = \sum_{n=1}^{m} \varphi_{n}^{k} \left| \frac{b_{m,n}}{a_{n,m}} \Delta_{x_{n}} \right|^{k}
\]
\[
= O(1) \sum_{n=1}^{m} \varphi_{n}^{k} \Delta_{x_{n}}^{k}
\]
\[
= O(1) \text{ as } m \to \infty,
\]
in view of the hypotheses of the theorem.

Applying Hölder’s inequality with indices $k$ and $k'$, where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have that
\[
\sum_{n=1}^{m+1} \varphi_{n}^{k} |T_{n,2}|^{k} = O(1) \sum_{n=2}^{m+1} \varphi_{n}^{k} \left( \sum_{v=1}^{n-1} \frac{|\Delta_{v} \hat{b}_{m,v}|}{a_{v,m}} \right)^{k-1}
\]
\[
= O(1) \sum_{n=2}^{m+1} \frac{|\Delta_{x_{v}}|^{k}}{a_{v,m}} \sum_{v=1}^{m+1} \varphi_{n}^{k} \frac{b_{m,v}}{a_{v,m}} \left| \Delta_{x_{v}} \right|^{k}
\]
\[
= O(1) \sum_{v=1}^{m} \frac{|\Delta_{x_{v}}|^{k}}{a_{v,m}} \left( \frac{\hat{b}_{v,v}}{a_{v,v}} \right)^{k} \left| \Delta_{x_{v}} \right|^{k}
\]
\[
= O(1) \text{ as } m \to \infty,
\]
by virtue of the hypotheses of the theorem.

Also
\[
\sum_{n=1}^{m+1} \varphi_{n}^{k} |T_{n,3}|^{k} = O(1) \sum_{n=2}^{m+1} \varphi_{n}^{k} \left( \sum_{v=1}^{n-1} \hat{b}_{n,v+1} |\Delta_{x_{v}}| \right)^{k}
\]
\[
= O(1) \sum_{n=2}^{m+1} \varphi_{n}^{k} \left( \sum_{v=1}^{n-1} \hat{b}_{n,v+1} |\Delta_{x_{v}}| \hat{b}_{v,v+1} \right)^{k-1}
\]
\[
= O(1) \text{ as } m \to \infty,
\]
\[
\begin{align*}
&= O(1) \sum_{n=2}^{m+1} \psi_n^{\delta k + k - 1} b_{mn}^{k-1} \sum_{v=1}^{n-1} \hat{b}_{n,v+1} \bar{x}_v^k b_{vv}^{1-k} \\
&= O(1) \sum_{v=1}^{m} b_{vv}^{1-k} |\bar{x}_v|^k \sum_{n=v+1}^{m} \psi_n^{\delta k + k - 1} b_{mn}^{k-1} \hat{b}_{n,v+1} \\
&= O(1) \sum_{v=1}^{m} \psi_n^{\delta k + k - 1} |\bar{x}_v|^k \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{align*}
\]

by virtue of the hypotheses of the theorem.

Finally, as in \(T_{n,3}\), we have that

\[
\begin{align*}
\sum_{n=2}^{m+1} \psi_n^{\delta k + k - 1} |T_{n,4}|^k &= O(1) \sum_{n=2}^{m+1} \psi_n^{\delta k + k - 1} \left\{ \sum_{r=0}^{n-2} |\bar{x}_r| \sum_{v=r+2}^{n} \hat{b}_{nv} |\hat{a}'_v| \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \psi_n^{\delta k + k - 1} \left\{ \sum_{r=0}^{n-2} |\bar{x}_r| \hat{b}_{r,r+1} \right\}^k \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{align*}
\]

by virtue of the hypotheses of the theorem.

Therefore, we have that

\[
\sum_{n=1}^{m} \psi_n^{\delta k + k - 1} |T_{n,i}|^k = O(1) \quad \text{as} \quad m \to \infty, \quad \text{for} \quad i = 1, 2, 3, 4.
\]

This completes the proof of the theorem.

References