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The crossed product by a partial endomorphism and the covariance algebra

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Abstract

Given a local homeomorphism $\sigma : U \to X$ where $U \subseteq X$ is clopen and X is a compact and Hausdorff topological space, we obtain the possible transfer operators L_ρ which may occur for $\alpha : C(X) \to C(U)$ given by $\alpha(f) = f \circ \sigma$. We obtain examples of partial dynamical systems (X_A, σ_A) such that the construction of the covariance algebra $C^*(X_A, \sigma_A)$, proposed by B.K. Kwasniewski, and the crossed product by a partial endomorphism $\mathcal{O}(X_A, \alpha, L)$, recently introduced by the author and R. Exel, associated to this system are not equivalent, in the sense that there does not exist an invertible function $\rho \in C(U)$ such that $\mathcal{O}(X_A, \alpha, L_\rho) \cong C^*(X_A, \sigma_A)$.
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1. Introduction

We begin with a summary of the construction of the crossed product by a partial endomorphism. Details may be seen in [3]. A partial C^*-dynamical system (A, α, L) consists of a (closed) ideal I of a C^*-algebra A, an idempotent self-adjoint ideal J of I (not necessarily closed), a $*$-homomorphism $\alpha : A \to M(I)$, where $M(I)$ is the multiplier algebra of I, and a linear positive map (which preserves $*$) $L : J \to A$ such that $L(a\alpha(b)) = L(a)b$ for each $a \in J$ and $b \in A$. The map L is called *transfer operator*, as in [2]. Define in J an inner product

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(which may be degenerated) by $(x, y) = L(x^*y)$. Then we obtain an inner product $\langle , \rangle$ in the quotient $J_0 = J/\{x \in J: L(x^*x) = 0\}$ defined by $\langle \tilde{x}, \tilde{y} \rangle = L(x^*y)$, which induces a norm $\| \|$. Define $M = \overline{J_0}^{\| \|}$, which is a right Hilbert A-module and also a left A-module, where the left multiplication is defined by the $*$-homomorphism $\varphi : A \to L(M)$ (the adjointable operators in M), where $\varphi(a)(\tilde{x}) = \widetilde{ax}$ for each $x \in J$. The Toeplitz algebra associated to (A, α, L) is the universal C^*-algebra $\mathcal{T}(A, \alpha, L)$ generated by $A \cup M$ with the relations of A, of M, the bi-module products and $m^*n = \langle m, n \rangle$. A redundancy in $\mathcal{T}(A, \alpha, L)$ is a pair $(a, k) \in A \times \widehat{K_1}$, $(\widehat{K_1} = \overline{\text{span}}\{mn^*,\ m, n \in M\})$, such that $am = km$ for every $m \in M$. The *Crossed Product by a Partial Endomorphism* $\mathcal{O}(A, \alpha, L)$ is the quotient of $\mathcal{T}(A, \alpha, L)$ by the ideal generated by all the elements $(a - k)$ where (a, k) is a redundancy and $a \in \ker(\varphi)^\perp \cap \varphi^{-1}(K(M))$. The algebra $\mathcal{O}(A, \alpha, L)$ is the Cuntz–Pimsner algebra associated to the C^*-correspondence $\varphi : A \to L(M)$.

In [3] it was introduced the crossed product by a partial endomorphism $\mathcal{O}(X, \alpha, L)$ associated to the C^*-dynamical system $(C(X), \alpha, L)$. This system is induced by a local homeomorphism $\sigma : U \to X$, where U is an open subset of a compact topological Hausdorff space X. More specifically,

$$\begin{aligned} \alpha : C(X) &\to C^b(U), \\ f &\mapsto f \circ \sigma, \end{aligned}$$

where $C^b(U)$ is the space of all continuous bounded functions in U and $L : C_c(U) \to C(X)$ ($C_c(U)$ is the set of the continuous functions with compact support in U) is defined by

$$L(f)(x) = \begin{cases} \sum_{y \in \sigma^{-1}(x)} f(y) & \text{if } x \in \sigma(U), \\ 0 & \text{otherwise}, \end{cases}$$

for every $x \in X$ and $f \in C_c(U)$.

In [4] it was defined the algebra $C^*(X, \alpha)$, called covariance algebra. This algebra is also constructed from a partial dynamical system, that is, a continuous map $\sigma : U \to X$ where X is a topological compact Hausdorff space, U is a clopen subset of X and $\sigma(U)$ is open.

If we suppose that $\sigma : U \to X$ is a local homeomorphism, U clopen (and so $\sigma(U)$ is always open) then (X, σ) gives rise to two C^*-algebras, the covariance algebra $C^*(X, \sigma)$, proposed in [4], and the crossed product by a partial endomorphism $\mathcal{O}(X, \alpha, L)$, proposed in [3].

In this paper we identify the transfer operators L_ρ which may occur for α. Moreover, we show that the constructions of the covariance algebra and the crossed product by a partial endomorphism are not equivalent in the following sense: we obtain examples of partial dynamical systems (X_A, σ_A) such that there does not exist an invertible function ρ such that $\mathcal{O}(X_A, \alpha, L_\rho) \cong C^*(X, \alpha)$.

2. Transfer operators of X for α

Let $\sigma : U \to X$ be a local homeomorphism and U an open subset of the compact Hausdorff space X. This local homeomorphism induces the $*$-homomorphism

$$\begin{aligned} \alpha : C(X) &\to C^b(U), \\ f &\mapsto f \circ \sigma. \end{aligned}$$

Given a positive function $\rho \in C(U)$, for all $f \in C_c(U)$ we may define

$$L_\rho(f)(x) = \begin{cases} \sum_{y \in \sigma^{-1}(x)} \rho(y) f(y) & \text{if } x \in \sigma(U), \\ 0 & \text{otherwise}, \end{cases}$$

for each $x \in X$. Note that $L_\rho(f) = L(\rho f)$, and since $\rho f \in C_c(U)$ and $L(\rho f) \in C(X)$ (see [3]) then $L_\rho(f)$ in fact is an element of $C(X)$. In this way we may define the map $L_\rho : C_c(U) \to C(X)$, which is linear and positive (by the fact that ρ is positive). It is easy to see that $L_\rho(f\alpha(g)) = L_\rho(f)g$ for each $f \in C_c(U)$ and $g \in C(X)$. The following proposition shows that if U is clopen in X then every transfer operator for α is of the form L_ρ for some $\rho \in C(U)$.

Proposition 2.1. *Let $L : C_c(U) \to C(X)$ (U clopen in X) be a transfer operator for α, that is, L is linear, positive, preserves *, and $L(g\alpha(f)) = L(g)f$ for each $f \in C(X)$ and $g \in C_c(U)$. Then there exists $\rho \in C(U)$ such that $L = L_\rho$.*

Proof. Let $\{V_i\}_{i=1}^n$ be an open cover of U such that $\sigma|_{V_i}$ is a homeomorphism (such a cover exists because U is compact and σ is a local homeomorphism). For each i take an open subset $U_i \subseteq V_i$ such that $\overline{U_i} \subseteq V_i$ and $\{U_i\}_i$ is also a cover for U. Consider the partition of unity $\{\varphi_i\}_i$ subordinated to $\{U_i\}_i$ and define $\xi_i = \sqrt{\varphi_i}$. Since ξ_i is positive for each i then $L(\xi_i)$ is a positive function. Define $\rho = \sum_{i=1}^n \alpha(L(\xi_i))\xi_i$ which is also positive. Given $f \in C_c(U)$ define for each i,

$$g_i(x) = \begin{cases} \xi_i(\sigma^{-1}(x)) f(\sigma^{-1}(x)), & x \in \sigma(V_i), \\ 0 & \text{otherwise.} \end{cases}$$

Claim 1. *$g_i \in C(X)$ for all i.*

Let $x_j \to x$. Suppose $x \in \sigma(V_i)$. Since $\sigma(V_i)$ is open we may suppose that $x_j \in \sigma(V_i)$ for each j. Since $\sigma|_{V_i}$ is a homeomorphism then $\sigma^{-1}(x_j) \to \sigma^{-1}(x)$ in V_i and so $g_i(x_j) = (\xi_i f)(\sigma^{-1}(x_j)) \to (\xi_i f)(\sigma^{-1}(x)) = g_i(x)$. If $x \notin \sigma(V_i)$ then $x \notin \sigma(\overline{U_i})$, which is closed. Therefore we may suppose that $x_j \notin \sigma(\overline{U_i})$ and so $g_i(x_j) = 0 = g_i(x)$.

Claim 2. *$\xi_i\alpha(g_i) = \varphi_i f$.*

If $x \notin U_i$ then $(\xi_i\alpha(g_i))(x) = 0 = (\varphi_i f)(x)$. If $x \in U_i$ then $\alpha(g_i)(x) = g_i(\sigma(x)) = \xi_i(x) f(x)$ and so $\xi_i(x)\alpha(g_i)(x) = \xi^2(x) f(x) = \varphi(x) f(x)$.

Since φ is a partition of unity then $f = \sum_{i=1}^n \varphi_i f = \sum_{i=1}^n \xi_i\alpha(g_i)$, where the last equality follows by Claim 2. Then

$$L(f) = \sum_{i=1}^n L\big(\xi_i\alpha(g_i)\big) = \sum_{i=1}^n L(\xi_i) g_i.$$

We show that $L = L_\rho$. If $x \notin \sigma(U)$ then $L_\rho(f)(x) = 0 = L(f)(x)$ by definition.

Given $x \in \sigma(U)$,

$$\begin{aligned} L_\rho(f)(x) &= \sum_{y \in \sigma^{-1}(x)} \rho(y) f(y) = \sum_{y \in \sigma^{-1}(x)} \sum_{i=1}^n \alpha\big(L(\xi_i)\big)(y)\xi_i(y) f(y) \\ &= \sum_{y \in \sigma^{-1}(x)} \sum_{i:\ y \in U_i} L(\xi_i)(x)\xi_i(y) f(y). \end{aligned}$$

On the other hand,

$$L(f)(x) = \sum_{i=1}^n L(\xi_i)(x) g_i(x) = \sum_{i:\ x \in \sigma(U_i)} L(\xi_i)(x)\xi_i\big(\sigma^{-1}(x)\big) f\big(\sigma^{-1}(x)\big).$$

To see that

$$\sum_{y\in\sigma^{-1}(x)}\ \sum_{i:\ y\in U_i} L(\xi_i)(x)\xi_i(y)f(y)=\sum_{i:\ x\in\sigma(U_i)} L(\xi_i)(x)\xi_i\big(\sigma^{-1}(x)\big)f\big(\sigma^{-1}(x)\big)$$

note that summands of each side are the same. □

Denote by M_ρ the Hilbert bi-module generated by $C_c(U)$ with the inner product given by L_ρ and by $\widehat{K_{1\rho}}$ the algebra generated by nm^* in $\mathcal{T}(X,\alpha,L_\rho)$. Moreover, denote by $\varphi_\rho : C(X)\to L(M_\rho)$ the ∗-homomorphism given by the left product of A by M_ρ.

Lemma 2.2. *Let $\rho,\rho'\in C(U)$ positive functions. If* $\ker(\rho)=\ker(\rho')$ *then* $\ker(\varphi_\rho)=\ker(\varphi_{\rho'})$.

Proof. Let $f\in C(X)$. Then $f\in\ker(\varphi_\rho)$ if and only if $fm=0$ for each $m\in M_\rho$, if and only if $\widetilde{fg}=f\tilde g=0$ for each $g\in C_c(U)$. It is easy to check that $\widetilde{fg}=0$ in M_ρ if and only if $\rho fg=0$. Then $f\in\ker(\varphi_\rho)$ if and only if $\rho fg=0$ for each $g\in C_c(U)$. In the same way, $f\in\ker(\varphi'_\rho)$ if and only if $\rho' fg=0$ for each $g\in C_c(U)$. Since $\ker(\rho)=\ker(\rho')$ then $\rho fg=0$ if and only if $\rho' fg=0$ for each $g\in C_c(U)$. □

Proposition 2.3. *If ρ and ρ' are elements of $C(U)$ such that there exists a positive function $r\in C(U)$ such that $r(x)\neq 0$ for each $x\in U$ and $\rho=r\rho'$ then $\mathcal{O}(X,\alpha,L_\rho)$ and $\mathcal{O}(X,\alpha,L_{\rho'})$ are ∗-isomorphic.*

Proof. Let us define a ∗-homomorphism from $\mathcal{O}(X,\alpha,L_\rho)$ to $\mathcal{O}(X,\alpha,L_{\rho'})$. Define

$$\begin{aligned}\psi_1 : C(X) &\to \mathcal{T}(X,\alpha,L_{\rho'}),\\ f &\mapsto f.\end{aligned}$$

Let $\xi=\sqrt{r}$, and note that for each $g\in C_c(U)$,

$$\|\tilde g\|_\rho^2=\big\|L_\rho(g^*g)\big\|=\big\|L(\rho g^*g)\big\|=\big\|L(r\rho' g^*g)\big\|=\big\|L_{\rho'}\big((\xi g)^*\xi g\big)\big\|=\|\widetilde{\xi g}\|_{\rho'}^2,$$

where $\|\ \|_\rho$ is the norm in M_ρ. So we may define $\psi_2 : M_\rho\to\mathcal{T}(X,\alpha,L_{\rho'})$ by $\psi_2(\tilde g)=\widetilde{\xi g}$. Let $\psi_3=\psi_1\cup\psi_2$. We show that ψ_3 extends to $\mathcal{T}(X,\alpha,L_\rho)$. For each $f\in C(X)$ and $g\in C_c(U)$ we have

$$\psi_3(f)\psi_3(\tilde g)=f\widetilde{\xi g}=\widetilde{\xi fg}=\psi_3(\widetilde{fg})$$

and

$$\psi_3(\tilde g)\psi_3(f)=\widetilde{\xi g}f=\widetilde{\xi g\alpha(f)}=\psi_3(\widetilde{g\alpha f}).$$

Moreover, if $h\in C_c(U)$ then

$$\psi_3(\tilde g)^*\psi_3(\tilde h)=\widetilde{\xi g}^*\widetilde{\xi h}=L_{\rho'}\big((\xi g)^*\xi h\big)=L_{\rho' r}(g^*h)=L_\rho(g^*h)=\psi_3\big(L_\rho(g^*h)\big).$$

So ψ_3 extends to $\mathcal{T}(X,\alpha,L_\rho)$. Let $(f,k)\in C(X)\times\widehat{K_{1\rho}}$ a redundancy with $f\in\ker(\varphi_\rho)^\perp\cap\varphi_\rho^{-1}(K(M_\rho))$. Since $\psi_3(M_\rho)\subseteq M_{\rho'}$ it follows that $\psi_3(k)\in\widehat{K_{1\rho'}}$ and so $(\psi_3(f),\psi_3(k))\in C(X)\times\widehat{K_{1\rho'}}$. Moreover, given $g\in C_c(U)$ then $\xi^{-1}g\in C_c(U)$ and $\psi_3(\widetilde{\xi^{-1}g})=\tilde g$ from where $\psi_3(M_\rho)$ is dense in $M_{\rho'}$, and so, since $fm=km$ for each $m\in M_\rho$ then $\psi_3(f)n=\psi_3(k)n$ for every $n\in M_{\rho'}$. Therefore $(\psi_3(f),\psi_3(k))$ is a redundancy. Since $f\in\ker(\varphi_\rho)^\perp$, by the previous lemma, $\psi_3(f)\in\ker(\varphi_{\rho'})^\perp$. Then, since $(\psi_3(f),\psi_3(k))$ is a redundancy of $\mathcal{T}(X,\alpha,L)$ then by

[3, 2.6], $\psi_3(f) \in \varphi^{-1}(K(M_{\rho'}))$. So $\psi_3(f) \in \ker(\varphi_{\rho'})^{\perp} \cap \varphi_{\rho'}^{-1}(K(M_{\rho'}))$. This shows that if ϕ is the quotient $*$-homomorphism from $\mathcal{T}(X,\alpha,L)$ in $\mathcal{O}(X,\alpha,L)$ then $\phi \circ \psi_3 : \mathcal{T}(X,\alpha,L_\rho) \to \mathcal{O}(X,\alpha,L)$ is a homomorphism which vanishes on all the elements of the form $(a-k)$ where (a,k) is a redundancy and $a \in \varphi_\rho^{-1}(K(M_\rho)) \cap \ker(\varphi_\rho)^{\perp}$. So we obtain a $*$-homomorphism

$$\begin{aligned} \psi : \mathcal{O}(X,\alpha,L_\rho) &\to \mathcal{O}(X,\alpha,L_{\rho'}), \\ f &\mapsto f, \\ \tilde{g} &\mapsto \widetilde{\xi g}. \end{aligned}$$

In the same way we may define the $*$-homomorphism

$$\begin{aligned} \psi_0 : \mathcal{O}(X,\alpha,L_{\rho'}) &\to \mathcal{O}(X,\alpha,L_\rho), \\ f &\mapsto f, \\ \tilde{g} &\mapsto \widetilde{\xi^{-1} g}. \end{aligned}$$

Note that ψ_0 is the inverse of ψ, showing that the algebras are $*$-isomorphic. □

Corollary 2.4. *If $\rho \in C(U)$ is a positive function such that $\rho(x) \neq 0$ for all $x \in U$ then $\mathcal{O}(X,\alpha,L_\rho)$ is $*$-isomorphic to $\mathcal{O}(X,\alpha,L)$.*

Proof. Note that the transfer operator L associated to the algebra $\mathcal{O}(X,\alpha,L)$ is the operator L_{1_U}. Since $\rho = 1_U$ is invertible, taking $r = \rho^{-1}$, by the previous proposition follows the corollary. □

3. Relationship between the covariance algebra and the crossed product by a partial endomorphism

We show here that given a partial dynamical system $\sigma : U \to X$, where U is clopen, there exists another partial dynamical system $\tilde{\sigma} : \widetilde{U} \to \widetilde{X}$ (called in [4] the σ-extension of X) such that $C^*(X,\sigma) \cong \mathcal{O}(\widetilde{X},\alpha,L)$. Moreover, if σ is injective then $C^*(X,\sigma) \cong \mathcal{O}(X,\alpha,L)$.

3.1. The covariance algebra as a crossed product by a partial endomorphism

Let us start with a summary of the construction of the covariance algebra. Let $\sigma : U \to X$ be a continuous map, $U \subseteq X$ clopen, X compact Hausdorff and $\sigma(U)$ open. Denote $\sigma(U) = U_{-1}$. Consider the space $X \cup \{0\}$, where $\{0\}$ is a symbol, which we define to be clopen. So $X \cup \{0\}$ is a compact and Hausdorff space.

Define $\widetilde{X} \subset \prod_{i=0}^{\infty} X \cup \{0\}$,

$$\widetilde{X} = \bigcup_{N=0}^{\infty} X_N \cup X_\infty,$$

where

$$X_N = \big\{(x_0, x_1, \ldots, x_N, 0, 0, \ldots)\colon\ \sigma(x_i) = x_{i-1} \text{ and } x_N \notin U_{-1}\big\}$$

and

$$X_\infty = \big\{(x_0, x_1, \ldots)\colon\ \sigma(x_i) = x_{i-1}\big\}.$$

In $\widetilde{X}$ we consider the product topology induced from $\prod_{i=0}^{\infty} X \cup \{0\}$.

By [4, 2.2], $\widetilde{X}$ is compact. Define

$$\Phi : \widetilde{X} \to X,$$
$$(x_0, x_1, x_2, \dots) \mapsto x_0,$$

which is continuous and surjective. Consider the clopen subsets $\widetilde{U} = \Phi^{-1}(U)$ and $\widetilde{U_{-1}} = \Phi^{-1}(U_{-1})$ and the continuous map

$$\tilde{\sigma} : \widetilde{U} \to \widetilde{U_{-1}},$$
$$(x_0, x_1, x_2, \dots) \mapsto \big(\sigma(x_0), x_0, x_1, \dots\big).$$

Those maps satisfies the relation

$$\Phi\big(\tilde{\sigma}(\tilde{x})\big) = \sigma\big(\Phi(\tilde{x})\big).$$

Note that $\tilde{\sigma}$ is in fact a homeomorphism. This homeomorphism induces the $*$-isomorphism

$$\begin{aligned} \theta : C(\widetilde{U_{-1}}) &\to C(\widetilde{U}), \\ f &\mapsto f \circ \tilde{\sigma}. \end{aligned}$$

So we may consider the partial crossed product $C(\widetilde{X}) \rtimes_\theta \mathbb{Z}$ (see [1]).

Definition 3.1. [4, 4.2] The covariance algebra associated to the partial dynamical system (X, σ) is the algebra $C(\widetilde{X}) \rtimes_\theta \mathbb{Z}$ and will be denoted $C^*(X, \sigma)$.

Lemma 3.2. *If $\sigma : U \to X$ is injective, U clopen and U_{-1} open then $C(X) \rtimes_\theta \mathbb{Z} \cong \mathcal{O}(X, \alpha, L)$, where $\theta : C(U_{-1}) \to C(U)$ is given by $\theta(f) = f \circ \sigma$.*

Proof. Define $\psi_1 : C(X) \cup M \to C(X) \rtimes_\theta \mathbb{Z}$ by $\psi_1(f) = f\delta_0$ and $\psi_1(\widetilde{1_U}) = 1_U\delta_1$. It is easy to check that ψ_1 extends to $\mathcal{T}(X, \alpha, L)$. We show that Ψ_1 vanishes on the redundancies. Let (f, k) be a redundancy with $f \in \ker(\varphi)^\perp \cap \varphi^{-1}(K(M))$. By [3, 2.6], $f \in C(U)$. Then $\psi_1(\tilde{f})\psi_1(\widetilde{1_U})^* = f\delta_1 1_{U_{-1}}\delta_{-1} = \theta(\theta^{-1}(f)1_{U_{-1}})\delta_0 = \psi_1(f)$. Take $(k_n)_n \subseteq \widehat{K_1}$, $k_n = \sum_i m_{ni} l_{ni}^*$ where $m_{ni}, l_{ni} \in M$. Then

$$\begin{aligned} &\big(\psi_1(f) - \psi(k)\big)\big(\psi_1(f) - \psi(k)\big)^* \\ &\quad = \big(\psi_1(f) - \psi_1(k)\big)\psi_1(f - k) = \psi_1(f - k)\big(\psi_1(\tilde{f}\widetilde{1_U}^*) - \psi_1(k)\big)^* \\ &\quad = \psi(f - k)(\widetilde{1_U}\tilde{f}^* - k) = \lim_{n\to\infty} \psi(f - k)(\widetilde{1_U}\tilde{f}^* - k_n) = 0. \end{aligned}$$

The last equality follows by the fact that $(f - k)m = 0$ for each $m \in M$. So, by passage to the quotient we may consider $\psi : \mathcal{O}(X, \alpha, L) \to C(X) \rtimes_\theta \mathbb{Z}$. On the other hand, define

$$\begin{aligned} \psi_0 : C(X) &\to \mathcal{O}(X, \alpha, L), \\ f &\mapsto f, \end{aligned}$$

which is a $*$-homomorphism. Note that for each $f \in C(U_{-1})$,

$$\widetilde{1_U}\psi_0(f)\widetilde{1_U}^* = \widetilde{1_U}\alpha(f)\widetilde{1_U}^* = 1_U\alpha(f) = \theta(f) = \psi_0\big(\theta(f)\big)$$

and moreover $\widetilde{1_U}$ is a partial isometry such that $\widetilde{1_U}\widetilde{1_U}^* = 1_U$ and $\widetilde{1_U}^*\widetilde{1_U} = 1_{U_{-1}}$. Then, since $(\psi_0, \widetilde{1_U})$ is a covariant representation of $C(X)$ in $\mathcal{O}(X, \alpha, L)$, there exists a $*$-homomorphism $\psi' : C(X) \rtimes_\theta \mathbb{Z} \to \mathcal{O}(X, \alpha, L)$ such that $\psi'(f\delta_n) = f\widetilde{1_U}^n$ (see [1, 5]). The $*$-homomorphisms ψ and ψ' are inverses of each other, and so the algebras are $*$-isomorphic. □

Corollary 3.3. $C^*(X,\sigma) \cong \mathcal{O}(\widetilde{X},\alpha,L)$.

Proof. Follows by the definition of covariance algebras and by the previous lemma. □

By the following proposition, if σ is injective then the constructions of covariance algebra and crossed product by a partial endomorphism are equivalent.

Proposition 3.4. *If $\sigma : U \to X$ is injective then $C^*(X,\sigma) \cong \mathcal{O}(X,\alpha,L)$.*

Proof. By [4, 2.3] the map

$$\begin{aligned} \Phi : \widetilde{X} &\to X, \\ (x_0, x_1, x_2, \ldots) &\mapsto x_0 \end{aligned}$$

is a homeomorphism. Moreover, since $\Phi \circ \tilde{\sigma} = \sigma \circ \Phi$ then $C(\widetilde{X}) \rtimes_{\tilde{\theta}} \mathbb{Z} \cong C(X) \rtimes_{\theta} \mathbb{Z}$. By the previous lemma $C(X) \rtimes_{\theta} \mathbb{Z} \cong \mathcal{O}(X,\alpha,L)$. □

3.2. Cuntz–Krieger algebras

We show examples of partial dynamical systems $\sigma_A : U \to X_A$ such that there does not exist an invertible function $\rho \in C(U)$ such that $\mathcal{O}(X,\alpha,L_\rho)$ and $C^*(X,\alpha)$ are $*$-isomorphic. The examples are based on the Cuntz–Krieger algebras.

Let A be a $n \times n$ matrix with $A(i,j) = A_{i,j} \in \{0,1\}$. Denote by $\mathrm{Gr}(A)$ the directed graph of A, that is, the vertex set is $\{1,\ldots,n\}$ and $A(i,j)$ is the number of oriented edges from i to j. A path is a sequence $x_1,\ldots,x_m$ such that $A(x_i,x_{i+1}) = 1$ for each i. The graph $\mathrm{Gr}(A)$ is transitive if for each i and j there exists a path from i to j, that is, a path $x_1,\ldots,x_m$ such that $x_1 = i$ and $x_m = j$. The graph is a cycle if for each i there exists only one j such that $A(i,j) = 1$.

Let

$$X_A = \left\{x = (x_1,x_2,\ldots) \in \{1,\ldots,n\}^{\mathbb{N}}\colon\ A(x_i,x_{i+1}) = 1\ \forall i\right\} \subseteq \{1,\ldots,n\}^{\mathbb{N}}$$

and

$$\begin{aligned} \sigma_A : X_A &\to X_A, \\ (x_0,x_1,\ldots) &\mapsto (x_1,x_2,\ldots). \end{aligned}$$

Consider the set

$$\overline{X_A} = \left\{(x_i)_{i\in\mathbb{Z}} \in \{1,\ldots,n\}^{\mathbb{Z}}\colon\ A(x_i,x_i+1) = 1\ \forall i\right\} \subseteq \{1,\ldots,n\}^{\mathbb{Z}}$$

and the map $\overline{\sigma_A} : \overline{X_A} \to \overline{X_A}$ defined by $\overline{\sigma_A}((x_i)_{i\in Z}) = (x_{i+1})_{i\in\mathbb{Z}}$. It is showed in [4, 2.8] that there exists a homeomorphism $\Phi : \widetilde{X_A} \to \overline{X_A}$ such that $\Phi \circ \widetilde{\sigma_A} = \overline{\sigma_A} \circ \Phi$. Therefore $\mathcal{O}(\widetilde{X_A},\alpha,L) \cong \mathcal{O}(\overline{X_A},\alpha,L)$ and so $C^*(X_A,\sigma_A) \cong \mathcal{O}(\overline{X_A},\alpha,L)$. So we may analyze the ideal structure of $C^*(X_A,\sigma_A)$ by using the theory developed for $\mathcal{O}(\overline{X_A},\alpha,L)$ in [3]. This theory is based on the $\overline{\sigma_A}$, $\overline{\sigma_A}^{-1}$ invariant open subsets of $\overline{X_A}$. (In a system $\sigma : U \to X$, a subset $V \subseteq X$ is σ, σ^{-1} invariant if $\sigma(U \cap V) \subseteq V$ and $\sigma^{-1}(V) \subseteq V$.)

Proposition 3.5. *If $\mathrm{Gr}(A)$ is transitive and is not a cycle then there exists at least one open nontrivial $\overline{\sigma_A}$, $\overline{\sigma_A}^{-1}$ invariant subset of $\overline{X_A}$.*

Proof. Let $r = x_1, x_2, \dots, x_n$ be an admissible word (that is, $A(x_i, x_{i+1}) = 1$ for each i). Let $V_r = \{x \in \overline{X_A}\colon\ r \in x\}$. Note that V_r is open and $\overline{\sigma_A}$, $\overline{\sigma_A}^{-1}$ invariant. We show that there exists such a nontrivial V_r. Take $x_1 \in \{1, \dots, n\}$. Consider an admissible word $x_1, \dots, x_m$ where $x_j \neq x_1$ for each $j > 1$ and $A(x_m, x_1) = 1$. Such a word exists because $\mathrm{Gr}(A)$ is transitive. Let $r = x_1, \dots, x_m, x_1$. Then

$$y = (\dots, x_m, \mathring{x}_1, x_2, \dots, x_m, x_1, x_2, \dots) \in V_r$$

where $\mathring{x}_1$ means $y_0 = x_1$.

We conclude the proof by showing that $V_r \neq \overline{X_A}$. Suppose that there exists $y_0 \in \{1, \dots, n\}$ with $y_0 \notin \{x_1, \dots, x_m\}$. Let $x_1, y_1, \dots, y_t, y_0, s_1, \dots, s_l$ be an admissible word such that $y_j \neq x_1$ and $s_j \neq x_1$ for each j and $A(s_l, x_1) = 1$. Then

$$(\dots, s_l, \mathring{x}_1, y_1, \dots, y_t, y_0, s_1, \dots, s_l, x_1 \dots) \notin V_r.$$

If $\{x_1, \dots, x_m\} = \{1, \dots, n\}$, since $\mathrm{Gr}(A)$ is not a cycle, for some x_i there exists x_t such that $A(x_i, x_t) = 1$ and $x_t \neq x_{i+1}$ (if $i = m$ consider $x_{i+1} = x_1$). If $x_t = x_1$ (and so $i \neq m$) consider an admissible word $x_1, \dots, x_i, x_1$ and note that

$$(\dots, x_i, \mathring{x}_1, x_2, \dots, x_i, x_1, \dots) \notin V_r.$$

If $x_t \neq x_1$ consider an admissible word $x_1, x_2, \dots, x_i, x_t, y_1, \dots, y_l$ such that $y_j \neq x_1$ and $A(y_l, x_1) = 1$ (if there does not exist $y_1 \neq x_1$ such that $A(x_t, y_1) = 1$ then $y_1, \dots, y_l$ is the empty word) and so

$$(\dots, y_l, \mathring{x}_1, x_2, \dots, x_i, x_t, y_1, \dots, y_l, y_1, \dots) \notin V_r.$$

So $V_r \neq \overline{X_A}$. □

Now we analyze the σ_A, σ_A^{-1} invariant subsets of X_A.

Proposition 3.6. *If* $\mathrm{Gr}(A)$ *is transitive and is not a cycle, then the unique open* σ_A*-invariant subsets of* X_A *are* $\emptyset$ *and* X_A.

Proof. Let $V \subseteq X_A$ be an open nonempty σ_A invariant subset of X_A. Let $x \in V$ and V_m be an open neighbourhood of x, $V_m \subseteq V$,

$$V_m = \{y \in X_A\colon\ x_i = y_i \text{ for each } 1 \leqslant i \leqslant m\}.$$

Given $z \in X_A$ take $r = r_1, \dots, r_t$ a path from x_m to z_1. Then

$$s = (x_1, \dots, x_m, r_2, \dots, r_{t-1}, z_1, z_2, \dots) \in V_m$$

and since V is σ_A invariant then $z = \sigma_A^{m+t-2}(s) \in V$. So $V = X_A$. □

According to [3] a partial dynamical system $\sigma : U \to X$ is topologically free if the closure of $V^{i,j} = \{x \in U\colon\ \sigma^i(x) = \sigma^j(x)\}$ has empty interior for each $i, j \in \mathbb{N}$, $i \neq j$.

Proposition 3.7. *If* $\mathrm{Gr}(A)$ *is transitive and is not a cycle then* (X_A, σ_A) *is topologically free.*

Proof. Suppose that $\overline{V^{i,j}}$ has nonempty interior and $i < j$, $j = i + k$. Let x' be an interior point of $\overline{V^{i,j}}$ and $V_{x'} \subseteq \overline{V^{i,j}}$ be an open neighbourhood of x'. Take $x \in V^{i,j} \cap V_{x'}$. Since $\sigma_A^i(x) =$

$\sigma_A^j(x)$ then $z_{i+t} = z_{j+t}$ for each $t \in \mathbb{N}$ and since $j = i + k$ then $x = (x_1, \ldots, x_{i-1}, r, r, \ldots)$ where $r = x_i x_{i+1} \ldots x_{i+k-1}$. Consider the open subset

$$V_m = \{z \in X_A\colon\ z_i = x_i,\ 1 \leqslant i \leqslant m\},$$

where m is such that $m \geqslant i + k$ and $V_m \subseteq V_{x'}$. Then, if $y \in V_m$ with $y \in V^{i,j}$ then $y = x$. Therefore $V_m = \{x\}$. We show that there exists $z \in V_m$ with $z \neq x$, and that will be a contradiction. Suppose $y_0 \in \{1, \ldots, n\}$ and $y_0 \notin \{x_i, \ldots, x_{i+k-1}\}$. Take a path $s = s_1, \ldots, s_t$ from x_i to x_{i+k-1} such that $s_j = y_0$ for some j. Then $z = (x_1, \ldots, x_{i-1}, r, r, \ldots, r, s, s, \ldots) \in V_m$ (where r is repeated m times) but $z \neq x$. Suppose $\{1, \ldots, n\} = \{x_i, \ldots, x_{i+k-1}\}$. Since $\mathrm{Gr}(A)$ is not a cycle then for some x_j there exists $x_t \neq x_{j+1}$ (consider $x_{j+1} = x_i$ if $j = i + k - 1$) such that $A(x_j, x_t) = 1$. Let s be a path from x_t to x_{i+k-1} and define $p = x_i, \ldots, x_j, x_t, s$. Then

$$z = (x_1, \ldots, x_{i-1}, r, r, \ldots, r, x_i, \ldots, x_j, x_t, p, p, p, \ldots) \in V_m$$

(where r is repeated m times) and $z \neq x$. So, it is showed that there exists $z \in V_m$, $z \neq x$. Therefore, $\overline{V^{i,j}}$ has empty interior for each i, j. □

Theorem 3.8. *If* $\mathrm{Gr}(A)$ *is transitive and is not a cycle then* $C^*(X_A, \sigma_A)$ *and* $\mathcal{O}(X_A, \alpha, L)$ *are not* $*$*-isomorphic* C^**-algebras.*

Proof. By Lemma 3.2, $C^*(X_A, \sigma_A) \cong \mathcal{O}(\widetilde{X_A}, \alpha, L)$ and since $\mathcal{O}(\widetilde{X_A}, \alpha, L) \cong \mathcal{O}(\overline{X_A}, \alpha, L)$ then $C^*(X_A, \sigma_A) \cong \mathcal{O}(\overline{X_A}, \alpha, L)$. By Proposition 3.5, $\overline{X_A}$ has at least one nontrivial open $\overline{\sigma_A}, \overline{\sigma_A}^{-1}$ invariant subset and by [3, 3.9] $\mathcal{O}(\overline{X_A}, \alpha, L)$ has at least one nontrivial ideal. On the other hand, by Proposition 3.6, (X_A, σ_A) has no open σ_A, σ_A^{-1} invariant subsets and by Proposition 3.7, (X_A, σ_A) is topologically free. By [3, 4.8], $\mathcal{O}(X_A, \alpha, L)$ is simple. So $C^*(X_A, \sigma_A)$ and $\mathcal{O}(X_A, \alpha, L)$ are not $*$-isomorphic. □

Corollary 3.9. *If* $\mathrm{Gr}(A)$ *is transitive and is not a cycle then there does not exist a transfer operator* L_ρ*, with* $\rho(x) \neq 0$ *for each* $x \in U$ *such that* $C^*(X_A, \sigma_A)$ *and* $\mathcal{O}(X_A, \alpha, L)$ *are* $*$*-isomorphic* C^**-algebras.*

Proof. Follows by the previous theorem and by Corollary 2.4. □

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References

[1] R. Exel, Circle actions on C^*-algebras, partial automorphisms and a generalized Pimsner–Voiculescu exact sequence, J. Funct. Anal. 122 (1994) 361–401.
[2] R. Exel, A new look at the crossed-product of a C^*-algebra by an endomorphism, Ergodic Theory Dynam. Systems 23 (2003) 1733–1750.
[3] R. Exel, D. Royer, The crossed product by a partial endomorphism, preprint, 2004, math.OA/0410192.
[4] B.K. Kwasniewski, Covariance algebra of a partial dynamical system, preprint, 2004, math.OA/0407352.