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# Mod $p$ homology of the stable mapping class group

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## Abstract

We calculate the homology  $H_*(\Gamma_{g,n}; \mathbb{F}_p)$  of the mapping class group  $\Gamma_{g,n}$  in the stable range. The calculation is based on Madsen and Weiss' proof of the "Generalised Mumford Conjecture":  $\Gamma_{g,n}$  has the same homology as a component of the space  $\Omega^\infty \mathbb{C}P_{-1}^\infty$  in the stable range.

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## 1. Introduction

Let  $F_{g,n}$  be an oriented surface of genus  $g$  and with  $n$  boundary components. Let  $\text{Diff}(F_{g,n}, \partial)$  be the topological group of orientation preserving diffeomorphisms of  $F_{g,n}$  fixing pointwise a neighbourhood of the boundary. The mapping class group is the group  $\Gamma_{g,n} = \pi_0 \text{Diff}(F_{g,n}, \partial)$  of components. There are group maps

$$\Gamma_{g,n} \rightarrow \Gamma_{g,n-1} \quad \text{and} \quad \Gamma_{g,n} \rightarrow \Gamma_{g+1,n}$$

induced by gluing a disk, resp. a torus with two boundary components, to one of the boundary components of  $F_{g,n}$ . By a theorem of Harer and Ivanov [12,13], these maps induce isomorphisms in  $H_*(-; \mathbb{Z})$  for  $* \leq (g-1)/2$ , and thus there is a stable range in which the group homology  $H_*(\Gamma_{g,n}; \mathbb{Z})$  is independent of  $g$  and  $n$ . In this range it agrees with  $H_*(\Gamma_\infty; \mathbb{Z})$  where  $\Gamma_\infty = \text{colim}_g \Gamma_{g,1}$  is the stable mapping class group.

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1.1. *Madsen–Weiss’ theorem*

The Mumford conjecture predicts that

$$H^*(\Gamma_\infty; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \dots]$$

for certain classes  $\kappa_i \in H^{2i}(\Gamma_\infty)$ . This was recently proved by Madsen and Weiss, but their result gives more ([7], conjectured in [6]). To state the full result, consider the classifying space  $B\Gamma_\infty$ . Its homology is the group homology of  $\Gamma_\infty$ . By the Quillen plus-construction we get a simply connected space  $B\Gamma_\infty^+$  and a map  $B\Gamma_\infty \rightarrow B\Gamma_\infty^+$  inducing an isomorphism in homology. The Madsen–Weiss theorem determines the homotopy type of  $\mathbb{Z} \times B\Gamma_\infty^+$  to be that of  $\Omega^\infty \mathbb{C}P_{-1}^\infty$ . The space  $\Omega^\infty \mathbb{C}P_{-1}^\infty$  (to be defined below), can be examined by methods from stable homotopy theory. In particular it is easy to calculate  $H^*(\Omega^\infty \mathbb{C}P_{-1}^\infty; \mathbb{Q})$ . This implies the Mumford conjecture.

In this paper we calculate  $H_*(\Omega^\infty \mathbb{C}P_{-1}^\infty; \mathbb{F}_p)$  for any prime  $p$  and hence by the above  $H_*(\Gamma_{g,n}; \mathbb{F}_p)$  for  $* \leq (g - 1)/2$ .

1.2. *Outline and statement of results*

Let  $p$  be a prime number, and let  $H_* = H_*(-; \mathbb{F}_p)$ .

Let  $L$  be the canonical complex line bundle over  $\mathbb{C}P^\infty$  and let  $\mathbb{C}P_{-1}^\infty = \text{Th}(-L)$  be the Thom spectrum of the  $-2$ -dimensional virtual bundle  $-L$ . Inclusion of a point in  $\mathbb{C}P^\infty$  induces a map  $S^{-2} \rightarrow \mathbb{C}P_{-1}^\infty$  and the zero section of the line bundle  $L$  induces a map

$$\mathbb{C}P_{-1}^\infty = \text{Th}(-L) \rightarrow \text{Th}(-L + L) = \Sigma^\infty \mathbb{C}P_+^\infty.$$

These fit together into a cofibration sequence

$$S^{-2} \rightarrow \mathbb{C}P_{-1}^\infty \rightarrow \Sigma^\infty \mathbb{C}P_+^\infty$$

and there is an induced fibration sequence of infinite loop spaces

$$\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty \xrightarrow{\omega} Q(\Sigma \mathbb{C}P_+^\infty) \xrightarrow{\hat{\omega}} QS^0, \tag{1.1}$$

where we write  $Q = \Omega^\infty \Sigma^\infty$ . This fibration sequence is the starting point for the calculation of the mod  $p$  homology of  $\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty$  and  $\Omega^\infty \mathbb{C}P_{-1}^\infty$ .

The mod  $p$  homology of  $Q\Sigma \mathbb{C}P_+^\infty$  and  $QS^0$  is completely known [1,3,4], as is the induced map  $\hat{\omega}_*$  in homology [8]. The first main result of this paper is a calculation of the Hopf algebra  $H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty; \mathbb{F}_p)$ . We need to introduce the following notation to state the results (see Section 2 for further details). All Hopf algebras will be commutative and cocommutative. The Hopf algebra co-kernel and kernel of a map  $f : A \rightarrow B$  of Hopf algebras will be denoted  $B//f$  and  $A \backslash\backslash f$ , respectively.  $PA$  is the vectorspace of primitive elements, and  $QA$  is the vectorspace of indecomposable elements. For a graded vectorspace  $V$ ,  $s^{-1}V$  will denote the desuspension of  $V$ :  $(s^{-1}V)_{i-1} = V_i$ . We also need to introduce the following functors from vectorspaces to algebras. Let  $V$  be a non-negatively graded vectorspace and let  $B \subseteq V_0$  be a basis for the degree zero part of  $V$ . Let  $V = V^+ \oplus V^-$  be the splitting of  $V$  into even and odd dimensional parts. Then  $E[V^-]$  is the exterior algebra generated by  $V^-$  and  $\mathbb{F}_p[V^+]$  is the polynomial algebra generated by  $V^+$ . Furthermore  $\mathbb{F}_p[B, B^{-1}]$  is the algebra of Laurent polynomials in the elements of  $B$  and  $\mathbb{F}_p[V^+, B^{-1}] = \mathbb{F}_p[V^+] \otimes_{\mathbb{F}_p[B]} \mathbb{F}_p[B, B^{-1}]$  is the polynomial algebra generated by  $V^+$ , with the elements of  $B$  inverted. The *free commutative algebra* generated by  $V$  is  $S[V] = E[V^-] \otimes \mathbb{F}_p[V^+, B^{-1}]$ .

The calculations use the theory of *homology operations*. These are defined on the mod  $p$  homology of infinite loop spaces, and are natural with respect to infinite loop maps, cf. [1,3,4]. The basic operations are

$$\begin{aligned} \beta^\varepsilon Q^s &: H_n(X) \rightarrow H_{n+2s(p-1)-\varepsilon}(X) & (p > 2), \\ Q^s &: H_n(X) \rightarrow H_{n+s}(X) & (p = 2), \end{aligned}$$

where  $\varepsilon \in \{0, 1\}$  and  $s \in \mathbb{Z}_{\geq \varepsilon}$ . Given a sequence  $I = (\varepsilon_1, s_1, \dots, \varepsilon_k, s_k)$  (with all  $\varepsilon_i = 0$  if  $p = 2$ ) there is an iterated operation  $Q^I = \beta^{\varepsilon_1} Q^{s_1} \dots \beta^{\varepsilon_k} Q^{s_k}$ . The mod  $p$  homology of  $QS^0$  then has the following form: let  $\iota \in H_0(QS^0)$  be the image of the non-basepoint in  $S^0$ . Then  $H_*(QS^0)$  is the free commutative algebra on the set

$$\mathbf{T} = \{Q^I \iota \mid I \text{ admissible, } e(I) + b(I) > 0\} \tag{1.2}$$

(see Section 3 for the definition of  $e(I)$ ,  $b(I)$  and the notion of admissibility).

As a step towards calculating  $H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)$  we determine the Hopf algebra cokernel of  $\partial_* : H_*(Q\Sigma \mathbb{C}P_+^\infty) \rightarrow H_*(QS^0)$ . In the following theorem,  $Q_0S^0 \subseteq QS^0$  is the basepoint component. Then  $H_*(QS^0) = H_0(QS^0) \otimes H_*(Q_0S^0)$ .

**Theorem 1.1.** *Let  $\mathbf{T}$  be as in (1.2). Let  $H_*(QS^0)^{(0)}$  denote the subalgebra of  $H_*(QS^0)$  generated by the set*

$$\begin{aligned} \{Q^I \iota \in \mathbf{T} \mid \text{all } \varepsilon_i = 0\}, & \quad (p > 2), \\ \{Q^I \iota \in \mathbf{T} \mid \text{all } s_i \text{ even}\}, & \quad (p = 2). \end{aligned}$$

Then the composite

$$H_*(QS^0)^{(0)} \rightarrow H_*(QS^0) \rightarrow H_*(QS^0) // \partial_*$$

is an isomorphism. In particular the Hopf algebra  $H_*(QS^0) // \partial_*$  is concentrated in degrees  $\equiv 0 \pmod{2(p-1)}$ . Similarly for  $H_*(Q_0S^0) // \partial_*$ . Furthermore the dual algebra  $H^*(Q_0S^0) \backslash \backslash \partial^*$  is a polynomial algebra.

**Theorem 1.2.** *The sequence*

$$\mathbb{F}_p \rightarrow H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty) \backslash \backslash \omega_* \rightarrow H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty) \xrightarrow{\omega_*} H_*(Q\Sigma \mathbb{C}P_+^\infty) \backslash \backslash \partial_* \rightarrow \mathbb{F}_p$$

is an exact sequence of Hopf algebras. It is split but not canonically. Furthermore there is a canonical isomorphism

$$H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty) \backslash \backslash \omega_* \cong E[s^{-1}P(H_*(QS^0) // \partial_*)].$$

In particular,  $H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)$  is primitively generated and for  $p > 2$  it is free commutative.

Theorem 1.1 is an algebraic consequence of the known structure of  $H_*(Q\Sigma \mathbb{C}P_+^\infty)$  and  $H_*(QS^0)$  and the induced map  $\partial_*$  in homology. The proof of Theorem 1.2 uses the Eilenberg–Moore spectral sequence of the fibration sequence (1.1).

Next we calculate  $H_*(\Omega^\infty \mathbb{C}P_{-1}^\infty)$ .

**Theorem 1.3.** *For  $p = 2$ , the sequence*

$$\mathbb{F}_2 \rightarrow H_*(\Omega^\infty \mathbb{C}P_{-1}^\infty) \xrightarrow{\Omega\omega_*} H_*(Q\mathbb{C}P_+^\infty) \xrightarrow{\Omega\hat{\omega}_*} H_*(\Omega QS^0) \rightarrow \mathbb{F}_2$$

*is exact. In particular  $\Omega\omega_*$  induces an isomorphism*

$$H_*(\Omega^\infty \mathbb{C}P_{-1}^\infty) \xrightarrow{\cong} H_*(Q\mathbb{C}P_+^\infty) \setminus \setminus \Omega\hat{\omega}_*.$$

The short exact sequence in Theorem 1.3 determines  $H_*(\Omega^\infty \mathbb{C}P_{-1}^\infty; \mathbb{F}_2)$  completely. Indeed, as part of the proof of Theorem 1.3 we determine  $H_*(\Omega QS^0)$  and the induced map  $\Omega\hat{\omega}_*$  in homology. The result can be summarised by the diagram

$$\begin{array}{ccc} QH_*(Q\mathbb{C}P_+^\infty) & \xrightarrow{Q(\Omega\hat{\omega}_*)} & QH_*(\Omega QS^0) \\ \downarrow \cong & & \downarrow \cong \\ PH_*(Q\Sigma\mathbb{C}P_+^\infty) & \xrightarrow{P(\hat{\omega}_*)} & PH_*(QS^0), \end{array} \tag{1.3}$$

where the vertical isomorphisms in (1.3) are the homology suspensions. The homology  $H_*(\Omega_0 QS^0)$  of the basepoint component of  $\Omega QS^0$  is a *divided power algebra*, i.e. its dual is a primitively generated polynomial algebra.

For odd primes  $p$  our results are less precise in that  $H_*(\Omega^\infty \mathbb{C}P_{-1}^\infty; \mathbb{F}_p)$  is only determined up to algebra isomorphism. The main technical theorem is the following.

**Theorem 1.4.** *For odd primes  $p$ , the homology suspension*

$$\sigma_* : QH_*(\Omega^\infty \mathbb{C}P_{-1}^\infty) \rightarrow PH_*(\Omega^\infty \Sigma\mathbb{C}P_{-1}^\infty)$$

*is surjective.*

Proving Theorem 1.4 is the most difficult part of the paper. It uses that  $\sigma_*$  commutes with the homology operations  $\beta^s Q^s$ .

**Corollary 1.5.** *Let  $Y \subseteq H_*(\Omega^\infty \mathbb{C}P_{-1}^\infty)$  be a subset such that  $\sigma_*(Y)$  is a basis of  $PH_*(\Omega^\infty \Sigma\mathbb{C}P_{-1}^\infty)$ . Then  $H_*(\Omega^\infty \mathbb{C}P_{-1}^\infty)$  is the free commutative algebra on the set*

$$Y \cup \{\beta Q^s y \mid y \in Y, \deg(y) = 2s - 1\}.$$

Corollary 1.5 is a formal consequence of Theorem 1.4 and the fact that the Hopf algebra  $H_*(\Omega^\infty \Sigma\mathbb{C}P_{-1}^\infty)$  is primitively generated. The proof uses the “Kudo transgression theorem”, cf. [3], Theorem 1.1(7): If  $\deg(y) = 2s - 1$ , then in the Leray–Serre spectral sequence we have that  $\sigma_*(y)$  transgresses to  $y$  and that  $(\sigma_* y)^{p-1} \otimes y$  transgresses to  $-\beta Q^s y$ .

## 2. Recollections

In this introductory section, we collect the results we need later in the paper. We start by recalling some important results on the structure of Hopf algebras from [9] and proceed to review the functor  $\text{Cotor}$  and the closely related Eilenberg–Moore spectral sequence, cf. [5,10].

### 2.1. Hopf algebras

Here and elsewhere, the field  $\mathbb{F}_p$  with  $p$  elements is the ground field, and  $\otimes = \otimes_{\mathbb{F}_p}$ . Until further notice,  $p$  is assumed odd. Algebras and coalgebras are as in [9] and in particular they always have units resp. counits.

**Definition 2.1.** When  $A$  is a coalgebra and  $M_A$  and  ${}_A N$  are  $A$ -comodules with structure maps  $\Delta_M : M \rightarrow M \otimes A$  and  $\Delta_N : N \rightarrow A \otimes N$ , the cotensor product is defined by the exact sequence

$$0 \rightarrow M \square_A N \rightarrow M \otimes N \rightarrow M \otimes A \otimes N,$$

where the right-hand morphism is  $\Delta_M \otimes N - M \otimes \Delta_N$ . The functors  $M \square_A -$  and  $- \square_A N$  are left exact functors from  $A$ -comodules to  $\mathbb{F}_p$ -vectorspaces in general, and to  $A$ -comodules when  $A$  is cocommutative.

**Definition 2.2.** For a morphism  $f : A \rightarrow B$  of Hopf algebras, define the *kernel* and *cokernel*

$$A \backslash \! \! / f = A \square_B \mathbb{F}_p, \quad B // f = B \otimes_A \mathbb{F}_p.$$

A priori, the kernel and cokernel are vectorspaces, but when  $A$  and  $B$  are commutative and cocommutative, they become Hopf algebras and are the kernel and cokernel in the categorical sense. Hopf algebras that are both commutative and cocommutative are called *abelian*, and the category of those is an abelian category (this essentially follows from [9, Proposition 4.9]).

The Hopf algebras appearing in this paper will (except for  $R$  and  $\mathcal{R}$  defined below) be of the form  $A = H_*(X; \mathbb{F}_p)$  for  $X$  an infinite loop space. Such Hopf algebras will always be abelian. We will often have that  $H_*(X; \mathbb{F}_p)$  is of finite type, and in this case  $H^*(X; \mathbb{F}_p)$  will also be a Hopf algebra. However, if  $\pi_0(X)$  is infinite,  $H^*(X; \mathbb{F}_p)$  will not be a Hopf algebra (e.g.  $X = QS^0$  with  $\pi_0 X = \mathbb{Z}$ ). Usually it will then be the case that the basepoint component  $X_0 \subseteq X$  is of finite type, and thus we can consider  $H^*(X_0; \mathbb{F}_p)$ . Hopf algebras  $A$  with  $A_i = 0$  for  $i < 0$  and  $A_0 = \mathbb{F}_p$  are called *connected*. In general we will have a natural splitting of Hopf algebras  $H_*(X) = H_*(X_0) \otimes \mathbb{F}_p \{ \pi_0 X \}$  where  $\mathbb{F}_p \{ \pi_0 X \} = H_0(X)$  is the group algebra.

**Definition 2.3.** For an algebra  $A$  with augmentation  $\varepsilon$ , let  $IA = \text{Ker}(\varepsilon : A \rightarrow k)$  and dually for a coalgebra  $A$  with augmentation  $\eta$ , let  $JA = \text{Cok}(\eta : k \rightarrow A)$ . Let  $Q$  and  $P$  be the functors defined by the exact sequences

$$IA \otimes IA \xrightarrow{\varphi} IA \rightarrow QA \rightarrow 0$$

and

$$0 \rightarrow PA \rightarrow JA \xrightarrow{\Delta} JA \otimes JA.$$

$P$  and  $Q$  satisfy  $P(A \otimes B) = PA \oplus PB$  and  $Q(A \otimes B) = QA \oplus QB$ , and as functors from abelian Hopf algebras to vectorspaces,  $Q$  is right exact and  $P$  is left exact ([9, Proposition 4.10]). When  $A$  is connected,  $PA \subseteq A$  is the subset consisting of elements  $x$  satisfying  $\Delta x = x \otimes 1 + 1 \otimes x$ . If  $A = \mathbb{F}_p \{ \pi_0 X \}$  is a group algebra, then  $PA = 0$ .

The functors  $P$  and  $Q$  are related by the short exact sequence of [9, Theorem 4.23]:

**Theorem 2.4.** For an abelian Hopf algebra  $A$ , let  $\xi : A \rightarrow A$  be the Frobenius map  $x \mapsto x^p$  and let  $\lambda : A \rightarrow A$  be the dual of  $\xi : A^* \rightarrow A^*$ . Let  $\xi A \subseteq A$  be the image of  $\xi$  and let  $A \rightarrow \lambda A$  be the coimage of  $\lambda$ . Then there is the following natural exact sequence:

$$0 \rightarrow P\xi A \rightarrow PA \rightarrow QA \rightarrow Q\lambda A \rightarrow 0. \tag{2.1}$$

In particular  $PA \rightarrow QA$  is an isomorphism except possibly in degrees  $\equiv 0 \pmod{2p}$  if  $p > 2$ . For  $p = 2$  it is an isomorphism in odd degrees.

Finally, we recall Borel’s structure theorem [9, Theorem 7.11].

**Theorem 2.5.** Any connected abelian Hopf algebra  $A$  is isomorphic as an algebra to a tensor product of algebras of the form  $E[x]$ ,  $\mathbb{F}_p[x]$  and  $\mathbb{F}_p[x]/(x^{p^n})$ ,  $n \geq 1$ .

**Corollary 2.6.** A connected abelian Hopf algebra  $A$  is isomorphic as an algebra to a polynomial algebra if and only if  $\xi : A \rightarrow A$  is injective. Dually if  $A$  is of finite type,  $A^*$  is polynomial if and only if  $\lambda : A \rightarrow A$  is surjective.

### 2.2. The functor Cotor

When  $A$  is a coalgebra and  $B$  and  $C$  are left resp. right  $A$ -comodules, the functor

$$\text{Cotor}^A(B, C)$$

is defined as the right derived functor of the cotensor product  $\square_A$ . To be explicit (and to fix grading conventions), choose an injective resolution  $0 \rightarrow B \rightarrow I_0 \rightarrow I_{-1} \rightarrow \dots$  of  $B$  in the category of right  $A$ -comodules and set

$$\text{Cotor}_n^A(B, C) = H_n(I_* \square_A C).$$

When  $A, B$  and  $C$  are in the graded category, Cotor gets an inner grading and is thus bigraded with  $\text{Cotor}_{n,m}^A(B, C) = (\text{Cotor}_n^A(B, C))_m$ . When  $A, B, C$  are all positively graded, Cotor is concentrated in the second quadrant.

When  $A, B$  and  $C$  are of finite type over a field, this functor is dual to the more common Tor:

$$\text{Cotor}^A(B, C) = (\text{Tor}^{A^*}(B^*, C^*))^*.$$

This follows immediately from the duality between  $\square_A$  and  $\otimes_{A^*}$ .

We shall consider Cotor as a functor from diagrams of cocommutative coalgebras

$$\mathcal{S} = \left\{ \begin{array}{c} B \\ \downarrow \\ C \rightarrow A \end{array} \right\}$$

to coalgebras. The external product is an isomorphism (see [2, Theorem 3.1, p. 209])

$$\text{Cotor}^A(B, C) \otimes \text{Cotor}^{A'}(B', C') \rightarrow \text{Cotor}^{A \otimes A'}(B \otimes B', C \otimes C')$$

and under this isomorphism the comultiplication in  $\text{Cotor}^A(B, C)$  is given by the comultiplication  $\Delta : \mathcal{S} \rightarrow \mathcal{S} \otimes \mathcal{S}$  in the diagram  $\mathcal{S}$ .

Dually, when  $\mathcal{S}$  is a diagram of Hopf algebras,  $\text{Cotor}^A(B, C)$  is a Hopf algebra with multiplication induced by the multiplication  $\varphi: \mathcal{S} \otimes \mathcal{S} \rightarrow \mathcal{S}$  of the diagram  $\mathcal{S}$ .

Later we will need the structure of  $\text{Cotor}^A(B, \mathbb{F}_p)$  where  $\mathbb{F}_p$  denotes the trivial Hopf algebra and  $f: B \rightarrow A$  is a morphism of Hopf algebras. From the change of rings spectral sequence and [9, Theorem 4.9] we get

**Proposition 2.7.** *For a map  $f: B \rightarrow A$  of Hopf algebras, there is a natural isomorphism of Hopf algebras*

$$\text{Cotor}^A(B, \mathbb{F}_p) \xrightarrow{\cong} B \backslash \! \! \! / f \otimes \text{Cotor}^{A//f}(\mathbb{F}_p, \mathbb{F}_p).$$

To complete the description of  $\text{Cotor}^A(B, \mathbb{F}_p)$  we need to calculate the Hopf algebra  $\text{Cotor}^A(\mathbb{F}_p, \mathbb{F}_p)$ . This is easily done by applying Borel’s structure theorem to the dual algebra  $A^*$  and using Lemma 2.8 below. The Hopf algebra  $\Gamma[x]$  is dual to a polynomial algebra:  $\Gamma[x] = (\mathbb{F}_p[x^*])^*$  and  $s^{-v}$  denotes bigraded desuspension:  $(s^{-v}V)_{-v,n} = V_n$  for a singly graded object  $V$ .

**Lemma 2.8.** *The following isomorphisms hold as Hopf algebras:*

$$\text{Tor}^{E[x]}(\mathbb{F}_p, \mathbb{F}_p) = \Gamma[s^{-1}x],$$

$$\text{Tor}^{\mathbb{F}_p[x]}(\mathbb{F}_p, \mathbb{F}_p) = E[s^{-1}x],$$

$$\text{Tor}^{\mathbb{F}_p[x]/(x^{p^n})}(\mathbb{F}_p, \mathbb{F}_p) = E[s^{-1}x] \otimes \Gamma[s^{-2}x^{p^n}].$$

By the duality between Tor and Cotor we obtain the Hopf algebra structure of  $\text{Cotor}^A(\mathbb{F}_p, \mathbb{F}_p)$  in terms of a set of generators of the dual algebra  $A^*$ .

**Corollary 2.9.** *For any connected Hopf algebra  $A$  of finite type,  $\text{Cotor}^A(\mathbb{F}_p, \mathbb{F}_p)$  is freely generated by the primitive elements in  $\text{Cotor}^A_{-1,*}(\mathbb{F}_p, \mathbb{F}_p)$  and  $\text{Cotor}^A_{-2,*}$ . Choosing generators of  $A^*$  (according to Borel’s structure theorem), the generators of  $\text{Cotor}^A(\mathbb{F}_p, \mathbb{F}_p)$  are in bidegrees*

$$(-1, k) \text{ for } x \in A_k^* \text{ an odd generator,}$$

$$(-1, k) \text{ for } x \in A_k^* \text{ an even generator,}$$

$$(-2, p^m k) \text{ for } x \in A_k^* \text{ an even generator of height } p^m.$$

The primitive elements of  $\text{Cotor}^A(\mathbb{F}_p, \mathbb{F}_p)$  are in bidegrees

$$p^n(-1, k) \text{ for } x \in A_k^* \text{ an odd generator,}$$

$$(-1, k) \text{ for } x \in A_k^* \text{ an even generator,}$$

$$p^n(-2, p^m k) \text{ for } x \in A_k^* \text{ an even generator of height } p^m.$$

More functorially, one defines for  $p > 2$  the functor  $\hat{P}A = PCotor_{-2,*}^A(\mathbb{F}_p, \mathbb{F}_p)$ . Then the result in Corollary 2.9 is that  $Cotor^A(\mathbb{F}_p, \mathbb{F}_p) \cong S[s^{-1}PA] \otimes S[s^{-2}\hat{P}A]$  combined with the facts that  $\hat{Q}(\mathbb{F}_p[x]) = \hat{Q}(E[x]) = 0$  and that  $\hat{Q}(\mathbb{F}_p[x]/(x^{p^n})) = \mathbb{F}_p \cdot \{x^{p^n}\}$ , where  $\hat{Q}A = (\hat{P}A^*)^*$ .

In particular, the only primitive elements of *odd* total degree are in bidegrees  $(-1, k)$  for even generators  $x \in A_k^*$ .

Finally, we shall need a criterion for left exactness of the functor  $Q$ , namely

**Proposition 2.10.** *Let*

$$\mathbb{F}_p \rightarrow A \rightarrow B \rightarrow C \rightarrow \mathbb{F}_p$$

*be a short exact sequence of abelian Hopf algebras. If  $C$  is a free commutative algebra, then the sequence*

$$0 \rightarrow QA \rightarrow QB \rightarrow QC \rightarrow 0$$

*is short exact.*

**Proof.** Since  $C$  is free, we may split  $B \rightarrow C$  with a map of algebras. Thus  $B \cong A \otimes C$  as an algebra, and  $Q(B)$  depends only on the algebra structure of  $B$ .  $\square$

A peculiar consequence of Corollary 2.6 is that if  $A$  is a Hopf algebra that is free as an algebra, then any Hopf subalgebra of  $A$  is also free as an algebra.

### 2.3. The spectral sequence

In this section, we recall the spectral sequence of [5] and some of its properties.

We consider *homotopy cartesian squares*

$$\mathcal{C} = \left\{ \begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array} \right\}$$

of connected spaces, and with  $B$  simply connected (homotopy cartesian means that  $F \simeq \text{holim}(X \rightarrow B \leftarrow E)$ ). One can always find a model that is a *fibre square*, i.e. where  $E \rightarrow B$  is a fibration, and  $F \rightarrow X$  is the pullback fibration). In the following,  $H_*$  denotes  $H_*(-; \mathbb{F}_p)$ .

**Definition 2.11.** The Eilenberg–Moore spectral sequence  $E^r$  is a functor from fibre squares  $\mathcal{C}$  as above to spectral sequences of coalgebras. It has

$$E^2 = Cotor^{H_*(B)}(H_*(E), H_*(X))$$

and converges as coalgebra to  $H_*F$ .

**Theorem 2.12** ([5, Proposition 16.4]). *The external product induces an isomorphism*

$$E^r(\mathcal{C}) \otimes E^r(\mathcal{C}') \rightarrow E^r(\mathcal{C} \times \mathcal{C}').$$

*Under this isomorphism, the coalgebra structure is induced by the diagonal  $\Delta: \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ .*



Dually, when  $\mathcal{C}$  is a diagram of  $H$ -spaces and  $H$ -maps (here meaning maps commuting *strictly* with the multiplication such as loop spaces and loop maps), there is a multiplication  $m: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  inducing a multiplication  $\varphi = m_*: E^r(\mathcal{C}) \otimes E^r(\mathcal{C}) \rightarrow E^r(\mathcal{C})$ . In this case, the spectral sequence is one of Hopf algebras. Furthermore it is clear that on the  $E^2$ -term, the Hopf algebra structure is the same as the one on Cotor described above.

### 2.4. The loop suspension

We shall use the spectral sequence only in the case when  $X$  is a point. This corresponds to a fibration sequence

$$F \rightarrow E \rightarrow B$$

and the spectral sequence computes homology of the fibre. When  $E$  is also a point, we have the path-loop fibration sequence

$$\Omega X \rightarrow * \rightarrow X.$$

In this case, the line

$$E_{0,*}^2 = \text{Cotor}_{0,*}^{H_*(X)}(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p \square_{H_*(X)} \mathbb{F}_p = \mathbb{F}_p$$

is concentrated in degree 0 and hence there is a “secondary edge homomorphism”

$$H_*(\Omega X) \rightarrow E_{-1,*}^\infty \hookrightarrow E_{-1,*}^2 \cong PH_*X. \tag{2.2}$$

**Proposition 2.13** ([11, Proposition 4.5]). *The morphism in (2.2) is the loop suspension*

$$\sigma_*: QH_*(\Omega X) \rightarrow PH_*X.$$

We shall also need

**Lemma 2.14.** *Let  $C_*$  be a connected differential graded Hopf algebra. If  $x$  is an element of minimal degree with  $dx \neq 0$ , then  $x$  is indecomposable and  $dx$  is primitive.*

**Proof.** Immediate from the Leibniz rules for product and coproduct.  $\square$

**Corollary 2.15.** *Minimal differentials in the spectral sequence of a path-loop fibration correspond to minimal elements in the cokernel of  $\sigma_*$ .*

**Proof.** Since  $dx$  is primitive and not in  $E_{-1,*}^2$  it is of even total degree and  $x$  is of odd total degree. By Corollary 2.9, the only odd dimensional indecomposable elements are in  $E_{-1,*}^2$  and the result follows.  $\square$

### 3. Unstable $R$ -modules

As sketched in the introduction, homology of an infinite loop space has homology operations  $\beta^e Q^s$ . In this section we recall the precise definitions and explain how to express homology of  $QX$  as a

free algebra on certain iterated operations on the homology of  $X$ . We follow the notation from [3]. In this section we consider only  $p > 2$ . Small changes, which we recall later, are needed for  $p = 2$ .

We define several categories of graded vectorspaces with a set of linear transformations  $\{\beta^\varepsilon Q^s \mid \varepsilon \in \{0, 1\}, s \in \mathbb{Z}_{\geq \varepsilon}\}$  of degree  $2s(p - 1) - \varepsilon$ .

$$\begin{array}{ccc}
 Q\text{-unstable } R\text{-modules} & \longrightarrow & Q\text{-unstable } \mathcal{R}\text{-modules} \\
 \downarrow & & \downarrow \\
 \text{unstable } R\text{-modules} & \longrightarrow & \text{unstable } \mathcal{R}\text{-modules} \\
 & & \downarrow \\
 & & \mathcal{R}\text{-modules} \\
 & & \downarrow \\
 & & \text{graded vectorspaces}
 \end{array} \tag{3.1}$$

Here,  $\mathcal{R}$  is the free non-commutative algebra on the set  $\{\beta^\varepsilon Q^s \mid \varepsilon \in \{0, 1\}, s \in \mathbb{Z}_{\geq \varepsilon}\}$ , and the various entries in (3.1) differ in what relations the action of the operations  $\beta^\varepsilon Q^s$  are assumed to satisfy. It is the left part of the diagram that is geometrically relevant, since the homology of an infinite loop space  $X$  is naturally an unstable  $R$ -modules, and so is the vectorspace of primitive elements  $PH_*(X)$ . The space of indecomposable elements  $QH_*(X)$  is naturally a  $Q$ -unstable  $R$ -module.

All of the above forgetful functors to graded vectorspaces have left adjoint “free” functors. From  $\mathcal{R}$ -modules it is the functor  $V \mapsto \mathcal{R} \otimes V$ , and the other four are quotients thereof.

In Section 3.2, we define the algebras  $\mathcal{R}$  and  $R$  and the four categories of unstable modules. In Section 3.3, we construct the four adjoint functors  $\mathcal{D}, \mathcal{D}', D$  and  $D'$ . Finally, in Section 3.4 we recall the computation of  $H_*(QX)$  in terms of  $H_*(X)$ . It should be noted that the algebra  $\mathcal{R}$  and the related categories are needed only in the proof of Theorem 4.4. It is  $R$  that is geometrically relevant but  $\mathcal{R}$  has the property that a submodule of a free  $\mathcal{R}$ -module is again free and similarly for submodules of free ( $Q$ -)unstable  $\mathcal{R}$ -modules. This makes  $\mathcal{R}$  simpler from the viewpoint of homological algebra.

### 3.1. Araki–Kudo–Dyer–Lashof operations

Recall that an *infinite loop space* is a sequence  $E_0, E_1, \dots$  of spaces and homotopy equivalences  $E_i \rightarrow \Omega E_{i+1}$ . One thinks of  $E_0$  as the “underlying space” of the infinite loop space. In particular,  $E_0 = \Omega^2 E_2$  is a homotopy commutative  $H$ -space. Thus, as mentioned in the introduction,  $H_*(E_0)$  is a commutative algebra under the Pontrjagin product. Furthermore  $H_*(E_0)$  naturally carries a set of linear transformations  $\beta^\varepsilon Q^s, \varepsilon \in \{0, 1\}, s \in \mathbb{Z}_{\geq \varepsilon}$ . These linear transformations are commonly called *Dyer–Lashof operations* (or *Araki–Kudo operations*) and are operations

$$\beta^\varepsilon Q^s : H_n(E_0) \rightarrow H_{n+2s(p-1)-\varepsilon}(E_0)$$

natural with respect to infinite loop maps. They measure the failure of chain level commutativity of the Pontrjagin product.

They satisfy a number of relations that makes  $H_*(E_0)$  an *unstable  $R$ -module*, the notion of which is defined below.

3.2. The algebras  $\mathcal{R}$  and  $R$  and categories of unstable modules

**Definition 3.1.** Let  $\mathcal{R}$  be the free (non-commutative) algebra generated by symbols

$$\beta^\varepsilon Q^s, \quad \varepsilon \in \{0, 1\}, \quad s \in \mathbb{Z}_{\geq \varepsilon}$$

and write  $\beta Q^s = \beta^1 Q^s$  and  $Q^s = \beta^0 Q^s$ .  $\mathcal{R}$  is a graded algebra with

$$\deg(\beta^\varepsilon Q^s) = 2s(p - 1) - \varepsilon.$$

It will occasionally be convenient to consider  $\mathcal{R}$  as a *bigraded* algebra with gradings

$$\deg_Q(\beta^\varepsilon Q^s) = 2s(p - 1), \quad \deg_\beta(\beta^\varepsilon Q^s) = -\varepsilon.$$

$\mathcal{R}$  is a cocommutative Hopf algebra with comultiplication

$$\Delta(\beta^\varepsilon Q^s) = \sum_{\substack{\varepsilon_1 + \varepsilon_2 = \varepsilon \\ s_1 + s_2 = s}} \beta^{\varepsilon_1} Q^{s_1} \otimes \beta^{\varepsilon_2} Q^{s_2}.$$

**Remark 3.2.**  $\mathcal{R}$  is a Hopf algebra in the sense of [9], i.e. a monoid object in the category of cocommutative coalgebras. Notice however that  $\mathcal{R}$  is not a group object, since  $Q^0$  is not invertible.

**Definition 3.3.** An  $\mathcal{R}$ -module is called *unstable*, if

$$\beta^\varepsilon Q^s x = 0 \quad \text{whenever} \quad 2s - \varepsilon < \deg(x). \tag{3.2}$$

It is called *Q-unstable* if furthermore

$$Q^s x = 0 \quad \text{whenever} \quad 2s = \deg(x). \tag{3.3}$$

On homology of an infinite loop space we also have the relation

$$Q^s x = x^p \quad \text{whenever} \quad 2s = \deg(x). \tag{3.4}$$

For an infinite loop space  $X$ ,  $H_*(X)$  is naturally an unstable  $\mathcal{R}$ -module and  $QH_*(X)$  is *Q-unstable* because of (3.4). However, the ideal in  $\mathcal{R}$  of elements with universally trivial action is nonzero, and hence the action of  $\mathcal{R}$  on  $H_*X$  factors through a quotient of  $\mathcal{R}$ . This quotient is the *Dyer–Lashof algebra*  $R$ .

**Definition 3.4.** For each  $r, s \in \mathbb{N}$  and  $\varepsilon \in \{0, 1\}$  with  $r > ps$ , define elements in  $\mathcal{R}$

$$\mathcal{A}^{(\varepsilon, r, 0, s)} = \beta^\varepsilon Q^r Q^s - \left( \sum_{i=0}^{r+s} (-1)^{r+i} (pi - r, r - (p - 1)s - i - 1) \beta^\varepsilon Q^{r+s-i} Q^i \right).$$

For  $r \geq ps$  define elements

$$\begin{aligned} \mathcal{A}^{(0, r, 1, s)} = & Q^r \beta Q^s - \left( \sum_{i=0}^{r+s} (-1)^{r+i} (pi - r, r - (p - 1)s - i) \beta Q^{r+s-i} Q^i \right) \\ & - \sum_{i=0}^{r+s} (-1)^{r+i} (pi - r - 1, r - (p - 1)s - i) Q^{r+s-i} \beta Q^i \end{aligned}$$

and

$$\mathcal{A}^{(1,r,1,s)} = \beta Q^r \beta Q^s - \left( -\sum_{i=0}^{r+s} (-1)^{r+i} (pi - r - 1, r - (p - 1)s - i) \beta Q^{r+s-i} \beta Q^i \right),$$

where  $(i, j) = (i + j)! / (i! j!)$ . These elements are the so-called *Adem relations*.

Let  $\mathcal{A} \subseteq \mathcal{R}$  be the  $\mathbb{F}_p$ -span of all Adem elements. This is a bigraded subspace of  $\mathcal{R}$ . Let  $\langle \mathcal{A} \rangle \subseteq \mathcal{R}$  be the two-sided ideal generated by  $\mathcal{A}$ . Let  $\mathcal{J} \subseteq \mathcal{R}$  be the two-sided ideal (or equivalently the left ideal) generated by the relations (3.2) (for  $x \in \mathcal{R}$ ).  $\mathcal{J}$  is the smallest ideal such that  $\mathcal{R} / \mathcal{J}$  is unstable as a left  $\mathcal{R}$ -module.

**Definition 3.5.** The *Dyer–Lashof algebra* is the quotient

$$R = \mathcal{R} / (\langle \mathcal{A} \rangle + \mathcal{J}).$$

The action of  $\mathcal{A}$  and hence  $\langle \mathcal{A} \rangle$  on homology of infinite loop spaces is trivial by results from [3], dual to Adem’s result for the Steenrod algebra. So is the action of  $\mathcal{J}$ . Hence  $H_*(X)$  is an  $R$ -module when  $X$  is an infinite loop space. Conversely [1,4] the map  $R \rightarrow H_*(QS^0)$  induced by acting on the zero-dimensional class  $\iota$ , corresponding to the non-basepoint of  $S^0$ , is an injection, so there are no further relations.

The set of all products of generators form a vector space basis of  $\mathcal{R}$ . To have an explicit basis for  $R$ , we recall the notion of admissible monomials, [3, p. 16].

A sequence

$$I = (\varepsilon_1, s_1, \dots, \varepsilon_k, s_k)$$

of integers  $\varepsilon_i \in \{0, 1\}$  and  $s_i \in \mathbb{Z}_{\geq \varepsilon_i}$  determines the iterated homology operation

$$Q^I = \beta^{\varepsilon_1} Q^{s_1} \dots \beta^{\varepsilon_k} Q^{s_k} \in \mathcal{R}.$$

This sequence is called *admissible* if for all  $i = 2, \dots, k$ ,

$$s_i \leq ps_{i-1} - \varepsilon_{i-1}. \tag{3.5}$$

The corresponding iterated homology operations  $Q^I \in \mathcal{R}$  are called *admissible monomials*. The *length* and *excess* of  $I$  are

$$\ell(I) = k, \quad e(I) = 2s_1 - \varepsilon_1 - \sum_{j=2}^k [2s_j(p - 1) - \varepsilon_j].$$

Furthermore, define

$$b(I) = \varepsilon_1.$$

Using the Adem relations one may rewrite an arbitrary element of  $R$  as a linear combination of admissible monomials. Applying Adem relations does not raise the excess.

There is a natural quotient map  $\mathcal{R} \rightarrow R$ . Thus  $R$ -modules are also  $\mathcal{R}$ -modules.

**Definition 3.6.** An  $R$ -module is called *unstable*, respectively  *$Q$ -unstable*, if it is so as an  $\mathcal{R}$ -module.

### 3.3. Free functors

**Definition 3.7.** For a graded vectorspace  $V$  we define  $\mathcal{D}V$  to be the quotient of  $\mathcal{R} \otimes V$  by the relations (3.2) and  $\mathcal{D}'V$  to be the quotient of  $\mathcal{D}V$  by the relations (3.3). Define also

$$DV = R \otimes_{\mathcal{R}} \mathcal{D}V, \quad D'V = R \otimes_{\mathcal{R}} \mathcal{D}'V.$$

The functor  $\mathcal{D}$  is left adjoint to the forgetful functor from unstable  $\mathcal{R}$ -modules to vectorspaces. Thus  $\mathcal{D}V$  is the “free unstable  $\mathcal{R}$ -module” generated by  $V$ . Similarly,  $D$  is left adjoint to the forgetful functor from unstable  $R$ -modules to graded vectorspaces. Analogous remarks apply to  $\mathcal{D}'$  and  $D'$ . The functors appear in the following exact sequences, natural in  $V$

$$\langle \mathcal{A} \rangle \otimes_{\mathcal{R}} \mathcal{D}V \rightarrow \mathcal{D}V \rightarrow DV \rightarrow 0, \tag{3.6}$$

$$\langle \mathcal{A} \rangle \otimes_{\mathcal{R}} \mathcal{D}'V \rightarrow \mathcal{D}'V \rightarrow D'V \rightarrow 0. \tag{3.7}$$

When  $V = \mathbb{F}_p \iota$  for a homogeneous element  $\iota$ ,  $DV$  has basis

$$\{Q^I \iota \mid I \text{ admissible, } e(I) \geq \deg(\iota)\}.$$

Together with the additivity of  $D$ , this describes  $DV$  as an  $\mathbb{F}_p$ -vectorspace. Since  $R \cong D\mathbb{F}_p$  as a left  $R$ -module, we also have a basis of  $R$  over  $\mathbb{F}_p$ .

### 3.4. Homology of $QX$

Here we recall the computation of  $H_*(QX)$ . It can be expressed as a functor of  $H_*(X)$  which is left adjoint to a suitable forgetful functor, forgetting the Pontrjagin product and the  $R$ -action, see [3]. We shall give a nonfunctorial description in terms of a basis of  $JH_*(X)$ .

**Theorem 3.8.** *Let  $B \subseteq JH_*(X)$  be a basis consisting of homogeneous elements. Then  $H_*(QX)$  is the free commutative algebra on the set*

$$\mathbf{T} = \{Q^I x \mid x \in B, I \text{ admissible, } e(I) + b(I) > \deg(x)\}.$$

**Corollary 3.9.** *The natural map*

$$\varphi_Q : D'JH_*(X) \rightarrow QH_*(QX)$$

*is an isomorphism of  $Q$ -unstable  $R$ -modules.*

*If  $X$  is connected and  $H_*(X)$  has trivial comultiplication (e.g. if  $X$  is a suspension), then the natural map*

$$\varphi_P : DJH_*(X) \rightarrow PH_*(QX)$$

*is an isomorphism of unstable  $R$ -modules.*

**Remark 3.10.**  $QX$  is connected if and only if  $X$  is connected. More generally the group of components of  $QX$  is determined by the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[\pi_0 X] \rightarrow \pi_0(QX) \rightarrow 0,$$

where the first arrow is induced by the inclusion of the basepoint in  $X$ . When  $X$  is nonconnected we are sometimes only interested in homology of the component  $Q_0X$  of the basepoint in  $QX$ . This can be described as follows. Let  $\tau: QX \rightarrow Q_0X$  be the map that on a component  $Q_iX$ ,  $i \in \pi_0(QX)$  multiplies by an element of  $Q_{-i}X$ . This defines a well-defined homotopy class of maps  $QX \rightarrow Q_0X$  which is left inverse to the inclusion. Then we have that  $H_*(Q_0X)$  is the free commutative algebra on the set

$$\tilde{\mathbf{T}} = \{ \tau_* Q^I x \mid x \in B, I \text{ admissible, } e(I) + b(I) > \deg(x), \deg(Q^I x) > 0 \}.$$

#### 4. Homological algebra of unstable modules

The map

$$Q(\partial_*) : QH_*(Q\Sigma CP_+^\infty) \rightarrow QH_*(Q_0S^0)$$

was computed in [8, Theorem 4.5]. The left-hand side is  $D'JH_*(\Sigma CP_+^\infty)$  and the right-hand side is  $D'\mathbb{F}_p$ . The starting point of our calculations is

**Theorem 4.1** ([8]). *Let  $a_s \in H_s(\Sigma CP_+^\infty)$  be the generator,  $s$  odd. Let  $\iota \in JH_0(S^0)$  be the generator. Then*

$$Q(\partial_*)(a_s) = \begin{cases} (-1)^r \beta Q^r \iota & s = 2r(p - 1) - 1 \\ 0 & \text{otherwise} \end{cases}$$

**Proof.** The map  $\partial: \Sigma CP_+^\infty \rightarrow QS^0$  coincides with the universal  $S^1$ -transfer denoted  $t_0$  in [8]. The formula for  $Q(\partial_*)(a_s)$  in the theorem now follows from ignoring all decomposable terms in [8, Theorem 4.5].  $\square$

##### 4.1. Main technical theorems

To state the theorems, recall from Section 3.2 that  $\mathcal{R}$  may be bigraded by  $\deg = \deg_Q + \deg_\beta$ . Since the Adem relations are homogeneous with respect to  $\deg_Q$  and  $\deg_\beta$ , there is an induced bigrading of  $R$ . If  $V$  is bigraded,  $\mathcal{R} \otimes V$  is a bigraded left  $\mathcal{R}$ -module. Since the unstability relations (3.2) are homogeneous, there is an induced bigrading of  $\mathcal{D}V$ . Similarly for  $\mathcal{D}'V$ ,  $DV$  and  $D'V$ . Thus by Corollary 3.9 a bigrading of  $JH_*(X)$  will induce a bigrading of  $QH_*(QX)$  and, for  $X$  a suspension, a bigrading of  $PH_*(QX)$ . However,  $H_*(QX)$  will only have  $\deg_\beta$  well-defined up to multiplication with  $p$  because of the unstability relation (3.4).

For bigraded modules  $V$  with  $\deg = \deg_Q + \deg_\beta$  as above, we shall write  $V^{i,j} = \{x \in V \mid \deg_Q(x) = i, \deg_\beta(x) = j\}$  and  $V^n = \bigoplus_{i+j=n} V^{i,j}$  and  $V^{(n)} = \bigoplus_i V^{i,n}$ . We will only consider gradings in the fourth quadrant, i.e.  $V^{i,j} = 0$  unless  $i \geq 0$  and  $j \leq 0$ . Write  $V^{(-)} = \bigoplus_{n < 0} V^{(n)}$ .

**Theorem 4.2.** *Bigrade  $JH_*(S^0)$  by setting  $\deg_\beta(\iota) = 0$  and give  $QH_*(QS^0)$  the induced bigrading. Then we have*

$$\text{Im}(Q(\partial_*)) = QH_*(QS^0)^{(-)}.$$

**Proof.** The inclusion  $\text{Im}(Q\partial_*) \subseteq QH_*(QS^0)^{(-)}$  is immediate from Theorem 4.1. The other inclusion follows from Lemma 4.3 below. Indeed, the two-sided ideal in  $R$  generated by the set  $\{\beta Q^s \mid s \geq 1\}$  is spanned by operations  $Q^l$  with at least one  $\beta$ . By Lemma 4.3 below, any such operation is also in the left ideal with the same generators, i.e. is a linear combination of elements of the form  $Q^j \beta Q^s$ . In particular, any element in  $QH_*(QS^0)^{(-)}$  is also in  $\text{Im}(Q\partial_*)$  because  $Q\partial_*$  is  $R$ -linear.  $\square$

**Lemma 4.3.** *The left ideal in  $R$  generated by the set  $\{\beta Q^s \mid s\}$  is also a right ideal.*

**Proof.** Write  $I \subseteq R$  for the left ideal generated by  $\{\beta Q^s \mid s\}$ .

For  $r \leq ps$ , consider the Adem relation  $\mathcal{A}^{(0, ps, 1, r-(p-1)s)}$ :

$$Q^{ps} \beta Q^{r-(p-1)s} = \beta Q^r Q^s + \sum_{i>s} \lambda_i \beta Q^{r+s-i} Q^i + \text{terms of form } Q^{r+s-i} \beta Q^i,$$

where we have singled out the term in the Adem relation corresponding to  $i = s$ , and where the  $\lambda_i \in k$  are certain binomial coefficients. This shows that in the left  $R$ -module  $R/I$  we can write  $\beta Q^r Q^s$  as a linear combination of  $\beta Q^a Q^b$  with  $a < r$ . In particular,  $\beta Q^1 Q^s = 0 \in R/I$  and by induction  $\beta Q^r Q^s = 0 \in R/I$ .

Thus we have  $\beta Q^r Q^s \in I$  whenever  $\beta Q^r Q^s$  is admissible. Since a nonadmissible  $\beta Q^r Q^s$  is a linear combination of admissible ones, we have  $\beta Q^r Q^s \in I$  for any  $r, s$ . This shows that  $I$  is invariant under right multiplication with  $Q^s$ . Since it is obviously invariant under right multiplication with  $\beta Q^s$  it follows that  $I$  is a right ideal.  $\square$

The kernel of  $Q\partial_*$  is harder to determine explicitly. The partial information contained in Theorem 4.4 below suffices for the calculation.

Notice that for any  $\mathcal{R}$ -module  $V$ , the augmentation of  $\mathcal{R}$  gives a natural quotient map  $V \rightarrow \mathbb{F}_p \otimes_{\mathcal{R}} V$  identifying  $\mathbb{F}_p \otimes_{\mathcal{R}} V$  with the quotient of  $V$  by the relations  $\beta^\varepsilon Q^s x = 0$  for  $x \in V, \varepsilon \in \{0, 1\}, s \geq \varepsilon$ . The functor  $\mathbb{F}_p \otimes_{\mathcal{R}} -$  agrees with the functor  $\mathbb{F}_p \otimes_R -$  on  $R$ -modules. Thus the vectorspace  $\mathbb{F}_p \otimes_{\mathcal{R}} V = \mathbb{F}_p \otimes_R V$  measures the dimensions of a minimal set of  $R$ -module generators of an unstable  $R$ -module  $V$ .

In the next theorem,  $a_s \in JH_s(\Sigma \mathbb{C}P_+^\infty)$  denotes the generator for  $s$  odd.

**Theorem 4.4.** *Bigrade  $JH_*(\Sigma \mathbb{C}P_+^\infty)$  by concentrating it in  $\text{deg}_\beta = -1$  and give  $QH_*(Q\Sigma \mathbb{C}P_+^\infty)$  the induced bigrading. Then the bigraded vectorspace*

$$\mathbb{F}_p \otimes_R \text{Ker}(Q\partial_*) = \mathbb{F}_p \otimes_R Q(H_*(Q\Sigma \mathbb{C}P_+^\infty) \setminus \setminus \partial_*)$$

*is concentrated in bidegrees  $\text{deg}_\beta = -1$  and  $\text{deg}_\beta = -2$ . In particular  $\text{Ker}(Q\partial_*)$  is generated as an  $R$ -module by the elements  $a_s \in \text{Ker}(Q\partial_*)$  with  $s \not\equiv -1 \pmod{2(p-1)}$  together with elements of degree  $\equiv -1$  and  $\equiv -2 \pmod{2(p-1)}$ .*

**Proof.** The equality  $\text{Ker}(Q\partial_*) = Q(H_*(Q\Sigma \mathbb{C}P_+^\infty) \setminus \setminus \partial_*)$  in the theorem follows from Proposition 2.10 because  $H_*(QS^0)$  is a free commutative algebra.

The last statement of the theorem follows from the first. Indeed the elements  $Q^l a_s$  are all in the kernel of  $Q(\partial_*)$  when  $s \not\equiv -1 \pmod{2(p-1)}$  because  $a_s$  is in the kernel. These elements give rise to one “tautological” element  $a_s \in \mathbb{F}_p \otimes_R \text{Ker}(Q\partial_*)$ . On the span of the  $Q^l a_s$  with  $s \equiv -1 \pmod{2(p-1)}$

the claim about degrees of generators follows since on these elements  $\deg \equiv \deg_\beta \pmod{2(p-1)}$ . Thus we need only prove the first statement of the theorem.

We have the short exact sequence of  $Q$ -unstable  $R$ -modules

$$0 \rightarrow \text{Ker}(Q\partial_*) \rightarrow QH_*(Q\Sigma\mathbb{C}P_+^\infty) \xrightarrow{Q\partial_*} QH_*(QS^0)^{(-)} \rightarrow 0. \tag{4.1}$$

If we were to apply the functor  $\mathbb{F}_p \otimes_R -$  from  $R$ -modules to vectorspaces, we would get a long exact sequence involving  $\text{Tor}_*^R(\mathbb{F}_p, -)$ , and a determination of the map induced by  $Q\partial_*$  in  $\text{Tor}_1$  would give the result. This is more or less what we do, except that it is technically more convenient to replace the functor  $\mathbb{F}_p \otimes_R -$  by  $\mathbb{F}_p \otimes_{\mathcal{R}} -$  and to replace  $\text{Tor}$  by a suitable functor taking unstability into account. We proceed to make these ideas precise.

The category of  $Q$ -unstable  $\mathcal{R}$ -modules is abelian and has enough projectives. The functor  $\mathbb{F}_p \otimes_{\mathcal{R}} -$  from  $Q$ -unstable  $\mathcal{R}$ -modules is right exact, hence the left derived functors  $L_r(\mathbb{F}_p \otimes_{\mathcal{R}} -)$  are defined. These are unstable versions of  $\text{Tor}_r^{\mathcal{R}}(\mathbb{F}_p, -)$ . For brevity, let us write  $T_1^{\mathcal{R}}(\mathbb{F}_p, -) = L_1(\mathbb{F}_p \otimes_{\mathcal{R}} -)$ .

With these definitions, applying the functor  $\mathbb{F}_p \otimes_{\mathcal{R}} -$  to the sequence (4.1) induces the exact sequence

$$0 \rightarrow \text{Cok}(T_1^{\mathcal{R}}(\mathbb{F}_p, Q\partial_*)) \rightarrow \mathbb{F}_p \otimes_R \text{Ker}(Q\partial_*) \rightarrow \text{Ker}(\mathbb{F}_p \otimes_R Q\partial_*) \rightarrow 0. \tag{4.2}$$

**Claim 1.** *The elements  $a_s \in \text{Ker}(Q\partial_*)$  with  $s \not\equiv -1 \pmod{2(p-1)}$  maps in (4.2) to a generating set in  $\text{Ker}(\mathbb{F}_p \otimes_R Q\partial_*)$ .*

**Proof of Claim 1.** This is the kernel of the map

$$\mathbb{F}_p \otimes_R Q\partial_* : \mathbb{F}_p \otimes_R QH_*(Q\Sigma\mathbb{C}P_+^\infty) \rightarrow \mathbb{F}_p \otimes_R QH_*(Q_0S^0)^{(-)}.$$

Clearly, the natural map  $JH_*(\Sigma\mathbb{C}P_+^\infty) \rightarrow k \otimes_R QH_*(Q\Sigma\mathbb{C}P_+^\infty)$  is an isomorphism, and by Lemma 4.3 we get that  $\mathbb{F}_p \otimes_R QH_*(QS^0)^{(-)}$  is spanned by  $\{\beta Q^{s_1} Q^{s_2} \dots \mid s \geq 1\}$ . Thus Claim 1 follows from Theorem 4.1.  $\square$

**Claim 2.**  $\text{Cok}(T_1^{\mathcal{R}}(\mathbb{F}_p, Q\partial_*))$  is concentrated in  $\deg_\beta = -1$  and  $\deg_\beta = -2$ .

**Proof of Claim 2.** We will compute  $T_1^{\mathcal{R}}(\mathbb{F}_p, Q\partial_*)$  using suitable free resolutions. For brevity, write  $V = JH_*(\Sigma\mathbb{C}P_+^\infty)$ . By Corollary 3.9, we may consider  $Q\partial_*$  as a map from  $D'V$  onto  $D'\mathbb{F}_p^{(-)}$ . Let  $W \subseteq (D'\mathbb{F}_p)^{(-)}$  denote the subspace with basis  $\{\beta Q^{s_1} Q^{s_2} \dots Q^{s_k} \mid s_1 \geq 1, s_2, \dots, s_k \geq 0\}$ . In the diagram

$$\begin{array}{ccccccc}
 & & W & & D'V & & \\
 & & \downarrow & & \downarrow Q\partial_* & & \\
 0 & \longrightarrow & (\langle \mathcal{A} \rangle \cdot D'\mathbb{F}_p)^{(-)} & \longrightarrow & D'\mathbb{F}_p^{(-)} & \longrightarrow & 0
 \end{array} \tag{4.3}$$

in which the lower exact sequence is an instance of (3.7), we may choose a lifting  $\rho : W \rightarrow D'V$  since  $Q\partial_*$  is surjective. Writing  $V = V_0 \oplus V_1$  where  $V_0 = \text{span}\{a_s \mid s \equiv -1 \pmod{2(p-1)}\}$  and  $V_1 = \text{span}\{a_s \mid s \not\equiv -1 \pmod{2(p-1)}\}$ , we may choose the lifting  $\rho$  to have  $\rho(W) \subseteq D'V_0$  since  $D'V = D'V_0 \oplus D'V_1$  and since  $Q\partial_*$  vanishes on  $D'V_1$ . We may also choose the lifting to have



$\rho(\beta Q^s) = a_{2s(p-1)-1}$  and extend (4.3) to the following exact diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Ker}(\rho) & \xrightarrow{j} & \mathcal{D}'W & \xrightarrow{\rho} & D'V_0 \longrightarrow 0 \\
 & & \downarrow \sigma & & \downarrow & & \downarrow Q\partial_* \\
 0 & \longrightarrow & (\langle \mathcal{A} \rangle \cdot \mathcal{D}'\mathbb{F}_p)^{(-)} & \xrightarrow{i} & \mathcal{D}'\mathbb{F}_p^{(-)} & \longrightarrow & D'\mathbb{F}_p^{(-)} \longrightarrow 0
 \end{array} \tag{4.4}$$

Note that the middle map in (4.4) is an isomorphism.

Next, we apply the functor  $\mathbb{F}_p \otimes_{\mathcal{R}} -$  to (4.4). This gives a diagram involving the left derived functor  $T_1^{\mathcal{R}}(\mathbb{F}_p, -) = L_1(\mathbb{F}_p \otimes_{\mathcal{R}} -)$ . This functor vanishes on the middle part of (4.4) since these (isomorphic) objects are free. Thus, a part of the induced diagram looks like this

$$\begin{array}{ccccccc}
 0 & \longrightarrow & T_1^{\mathcal{R}}(\mathbb{F}_p, D'V_0) & \longrightarrow & \mathbb{F}_p \otimes_{\mathcal{R}} \text{Ker}\rho & \xrightarrow{j_*} & \mathbb{F}_p \otimes_{\mathcal{R}} \mathcal{D}'W \\
 & & \downarrow T_1^{\mathcal{R}}(\mathbb{F}_p, Q\partial_*) & & \downarrow \sigma_* & & \downarrow \cong \\
 0 & \longrightarrow & T_1^{\mathcal{R}}(\mathbb{F}_p, D'\mathbb{F}_p^{(-)}) & \longrightarrow & \mathbb{F}_p \otimes_{\mathcal{R}} (\langle \mathcal{A} \rangle \cdot \mathcal{D}'\mathbb{F}_p)^{(-)} & \xrightarrow{i_*} & \mathbb{F}_p \otimes_{\mathcal{R}} (\mathcal{D}'\mathbb{F}_p)^{(-)},
 \end{array} \tag{4.5}$$

where a star in subscript is shorthand for  $\mathbb{F}_p \otimes_{\mathcal{R}} -$  on morphisms. Thus we have represented  $T_1^{\mathcal{R}}(\mathbb{F}_p, D'V_0)$  and  $T_1^{\mathcal{R}}(\mathbb{F}_p, \mathcal{D}'\mathbb{F}_p^{(-)})$  as the kernels of  $j_*$  and  $i_*$ , and the map  $T_1^{\mathcal{R}}(\mathbb{F}_p, Q\partial_*)$  as the restriction of  $\sigma_*$ .

To calculate the cokernel of  $T_1^{\mathcal{R}}(\mathbb{F}_p, Q\partial_*)$  and to prove Claim 2, note that

$$(\langle \mathcal{A} \rangle \cdot \mathcal{D}'\mathbb{F}_p)^{(-)} = \mathcal{R}^{(-)} \cdot \mathcal{A}^{(0)} \cdot \mathcal{D}'\mathbb{F}_p^{(0)} + \mathcal{R} \cdot \mathcal{A}^{(-)} \cdot \mathcal{D}'\mathbb{F}_p^{(0)} + \mathcal{R} \cdot \mathcal{A} \cdot \mathcal{D}'\mathbb{F}_p^{(-)}.$$

This is generated over  $\mathcal{R}$  by the subspace

$$\mathcal{R}^{(-)} \cdot \mathcal{A}^{(0)} \cdot \mathcal{D}'\mathbb{F}_p^{(0)} + \mathcal{A}^{(-)} \cdot \mathcal{D}'\mathbb{F}_p^{(0)} + \mathcal{A} \cdot \mathcal{D}'\mathbb{F}_p^{(-)}. \tag{4.6}$$

The corresponding  $\mathcal{R}$ -indecomposable classes will span  $\mathbb{F}_p \otimes_{\mathcal{R}} (\langle \mathcal{A} \rangle \cdot \mathcal{D}'\mathbb{F}_p)^{(-)}$  as a vectorspace, and since the first and the second term in (4.6) has  $\text{deg}_{\beta} \in \{-1, -2\}$ , it suffices to prove that the last term  $\mathcal{A} \cdot \mathcal{D}'\mathbb{F}_p^{(-)}$  does not contribute to the cokernel of  $T_1^{\mathcal{R}}(\mathbb{F}_p, Q\partial_*)$ .

To this end, notice that  $\mathcal{A} \cdot \mathcal{D}'\mathbb{F}_p^{(-)}$  corresponds to  $\mathcal{A} \cdot \mathcal{D}'W$  under the middle isomorphism in (4.4), and that  $\mathcal{A} \cdot \mathcal{D}'W$  is in the kernel of  $\rho$  since the action of  $\mathcal{A}$  is trivial in  $D'V$ . Notice also that  $\mathcal{A} \cdot \mathcal{D}'W$  vanishes under the projection  $\mathcal{D}'W \rightarrow \mathbb{F}_p \otimes_{\mathcal{R}} \mathcal{D}'W$  and thus by exactness of (4.4) and (4.5) the classes corresponding to  $\mathcal{A} \cdot \mathcal{D}'\mathbb{F}_p^{(-)}$  in  $\mathbb{F}_p \otimes_{\mathcal{R}} (\langle \mathcal{A} \rangle \cdot \mathcal{D}'\mathbb{F}_p)^{(-)}$  lifts all the way to  $T_1^{\mathcal{R}}(\mathbb{F}_p, D'V_0)$  and therefore does not contribute to the cokernel of  $T_1^{\mathcal{R}}(\mathbb{F}_p, Q\partial_*)$ .  $\square$

Now Theorem 4.4 follows from the exact sequence (4.2) and the claims above.  $\square$

### 5. Homology of $\Omega^{\infty} \Sigma \mathbb{C}P_{-1}^{\infty}$

The spectral sequence associated to the fibration (1.1) has

$$E^2 = \text{Cotor}^{H_*(Q_0 S^0)}(H_*(Q\Sigma\mathbb{C}P_+^{\infty}), \mathbb{F}_p) \Rightarrow H_*(\Omega^{\infty} \Sigma \mathbb{C}P_{-1}^{\infty}). \tag{5.1}$$

By Proposition 2.7, the  $E^2$ -term splits as

$$E^2 \cong \text{Cotor}^{H_*(Q_0 S^0)/\partial_*}(\mathbb{F}_p, \mathbb{F}_p) \otimes H_*(Q\Sigma\mathbb{C}P_+^{\infty}) \setminus \setminus \partial_*. \tag{5.2}$$

In this section,  $p$  is odd so after localising fibration (1.1), the base-space is simply connected and the spectral sequence converges.

As explained in the introduction, we will first prove Theorem 1.1 about the coalgebra structure on  $H_*(Q_0S^0)//\partial_*$ , or, equivalently the algebra structure of  $H^*(Q_0S^0)\backslash\backslash\partial^*$ , and then use this to prove that the spectral sequence (5.1) collapses. Then a close examination of the  $E^\infty = E^2$  term will prove Theorem 1.2.

5.1. *The Hopf algebra cokernel of  $\partial_*$*

To state the results, let us introduce a bigrading of  $H_*(QS^0)$ . Recall that  $H_*(QS^0)$  is the free commutative algebra on the set

$$\{Q^l \iota \mid I \text{ admissible, } e(I) + b(I) > 0\}.$$

Make it a bigraded algebra by setting  $\deg_\beta(Q^l \iota) = \deg_\beta(Q^l)$ . By the Cartan formula for the coproduct we get that the subalgebra  $H_*(QS^0)^{(0)}$  is a Hopf subalgebra, but notice that  $H_*(QS^0)$  is not a bigraded  $R$ -module because of the relation (3.4). We are now ready to prove Theorem 1.1 in the case  $p > 2$ .

**Proof of Theorem 1.1 (for  $p > 2$ ).** We first prove that the composition

$$H_*(QS^0)^{(0)} \rightarrow H_*(QS^0) \rightarrow H_*(QS^0)//\partial_* \tag{5.3}$$

is an isomorphism of Hopf algebras.

With the bigrading introduced above, we have  $H_*(QS^0) = H_*(QS^0)^{(0)} \oplus H_*(QS^0)^{(-)}$  where the first summand is a subalgebra and the second is an ideal. Since  $\text{Im}(\partial_*) \subseteq \mathbb{F}_p \oplus H_*(QS^0)^{(-)}$ , the composition (5.3) is injective.

To see surjectivity, note that  $Q(H_*(QS^0)//\partial_*) = \text{Cok}(Q\partial_*)$  since  $Q$  is right exact. By Theorem 4.2 we have  $\text{Im}(Q\partial_*) = QH_*(QS^0)^{(-)}$  and hence  $\text{Cok}(Q\partial_*) = (QH_*(QS^0))^{(0)} = Q(H_*(QS^0))^{(0)}$ .

To prove that  $H_*(Q_0S^0)//\partial_*$  is dual to a polynomial algebra, notice that we have  $H_*(Q_0S^0)//\partial_* \cong H_*(Q_0S^0)^{(0)}$  and that it suffices to prove that  $\lambda : H_*(Q_0S^0)^{(0)} \rightarrow H_*(Q_0S^0)^{(0)}$  is surjective.  $\lambda$  is given by the dual Steenrod operations: If  $\deg(x) = 2ps$ ,  $\lambda x = \mathcal{P}_*^s x$ . By the Nishida relations [3, Theorem 1.1 (9)], one gets  $\lambda Q^{ps} = Q^s \lambda$  and thus

$$\lambda(Q^{ps_1} Q^{ps_2} \dots Q^{ps_k} [1] * [-p^k]) = Q^{s_1} Q^{s_2} \dots Q^{s_k} [1] * [-p^k].$$

Thus  $\lambda$  hits the generators of  $H_*(Q_0S^0)^{(0)}$  and since it is a map of algebras, it is surjective.  $\square$

5.2. *The spectral sequence*

We are now ready to compute the  $E^2$ -term of the spectral sequence (5.1) and to prove that it collapses at the  $E^2$ -term.

**Theorem 5.1.** *The spectral sequence collapses at the  $E^2$ -term. The  $E^2$ -term is given by*

$$E^2 = H_*(Q\Sigma CP_+^\infty)\backslash\backslash\partial_* \otimes E[s^{-1}P(H_*(QS^0)//\partial_*)]$$

as a Hopf algebra.

**Proof.** We need to identify the factor  $\text{Cotor}^{H_*(Q_0S^0)/\partial_*}(\mathbb{F}_p, \mathbb{F}_p)$  in splitting (5.2) of the  $E^2$ -term. By Theorem 1.1, the dual algebra  $H^*(Q_0S^0)\backslash\backslash\partial_*$  is polynomial and hence by Corollary 2.9 we get

$$\text{Cotor}^{H_*(Q_0S^0)/\partial_*}(\mathbb{F}_p, \mathbb{F}_p) \cong E[s^{-1}P(H_*(QS^0)/\partial_*)]$$

as claimed.

In this  $E^2$ -term, primitives and generators are concentrated in bidegrees  $(0, *)$  and  $(-1, *)$  and hence by Lemma 2.14 there can be no non-zero differentials in the spectral sequence.  $\square$

**Proof of Theorem 1.2.** By Theorem 5.1, we get that the sequence

$$\mathbb{F}_p \rightarrow H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)\backslash\backslash\omega_* \rightarrow H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty) \xrightarrow{\omega_*} H_*(Q\Sigma \mathbb{C}P_+^\infty)\backslash\backslash\partial_* \rightarrow \mathbb{F}_p$$

is exact (i.e.  $\omega_*$  is onto).

To identify  $H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)\backslash\backslash\omega_*$  recall that the spectral sequence defines a filtration  $F_0 \supseteq F_{-1} \supseteq \dots$  on  $H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)$  and hence on  $H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)\backslash\backslash\partial_*$  and an isomorphism of graded vectorspaces

$$s^{-1}P(H_*QS^0/\partial_*) \rightarrow F_{-1}(H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)\backslash\backslash\partial_*)/F_{-2}.$$

Choosing any lifting

$$s^{-1}P(H_*QS^0/\partial_*) \xrightarrow{l} H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)\backslash\backslash\partial_*,$$

we will get an isomorphism of algebras

$$E[s^{-1}P(H_*QS^0/\partial_*)] \rightarrow H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)\backslash\backslash\partial_*$$

and since  $H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)\backslash\backslash\omega_*$  is a Hopf algebra, Theorem 2.4 defines a unique choice of lifting  $l$  into  $P(H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)\backslash\backslash\omega_*)$ .

The splitting follows from Lemma 5.2 below.  $\square$

**Lemma 5.2.** *Let*

$$\mathbb{F}_p \rightarrow A \rightarrow B \xrightarrow{\pi} C \rightarrow \mathbb{F}_p$$

*be a short exact sequence of Hopf algebras. If either  $A$  or  $C$  is exterior, the sequence is split exact in the category of Hopf algebras.*

**Proof.** Assume  $C$  is exterior. Then by Theorem 2.4 we have that  $PC \cong QC$  and the diagram

$$\begin{array}{ccc} PB & \xrightarrow{P\pi} & PC \longrightarrow 0 \\ \downarrow & & \downarrow \cong \\ QB & \longrightarrow & QC \longrightarrow 0 \end{array}$$

is exact since  $Q(-)$  is right exact. Thus  $PB \rightarrow PC$  is surjective and a choice of splitting  $PC \rightarrow PB$  of  $P\pi$  induces a splitting  $C \cong E[PC] \rightarrow B$  of  $\pi$ .

The case where  $A$  is exterior follows by duality.  $\square$

**Corollary 5.3.** *The vectorspace*

$$\text{Ker}(P\omega_*) = P(H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)\backslash\backslash\partial_*) = Q(H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)\backslash\backslash\omega_*)$$

*is concentrated in degrees  $\equiv -1 \pmod{2(p-1)}$ .*

**Proof.** This follows from Theorems 1.1 and 1.2.  $\square$

### 6. Homology of $\Omega^\infty \mathbb{C}P_{-1}^\infty$

The goal of this section is to prove Theorem 1.4 and Corollary 1.5.

As mentioned in the introduction, we will consider the Eilenberg–Moore spectral sequence of the path-loop fibration over  $\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty$ . From fibration (1.1) one easily gets that  $\pi_1(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty) = \mathbb{Z}$  and therefore we have a homotopy equivalence

$$\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty \simeq S^1 \times \tilde{\Omega}^\infty \Sigma \mathbb{C}P_{-1}^\infty,$$

where  $\tilde{\Omega}^\infty \Sigma \mathbb{C}P_{-1}^\infty \rightarrow \Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty$  is the universal covering map. Furthermore we have  $\Omega(\tilde{\Omega}^\infty \Sigma \mathbb{C}P_{-1}^\infty) = \Omega_0^\infty \mathbb{C}P_{-1}^\infty$ , the basepoint component of  $\Omega^\infty \mathbb{C}P_{-1}^\infty$ . Similarly  $Q\Sigma \mathbb{C}P_+^\infty \simeq S^1 \times \tilde{Q}\Sigma \mathbb{C}P_+^\infty$  and under these splittings the map  $\omega$  in the fibration (1.1) restricts to a map  $S^1 \rightarrow S^1$  of degree 2. Since  $p$  is odd, the effect of replacing  $\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty$  and  $Q\Sigma \mathbb{C}P_+^\infty$  by their universal covering spaces is to remove a factor of  $H_*(S^1) = E[\sigma]$ ,  $\sigma = [S^1] \in H_1(S^1)$ , from each of the terms  $H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)$  and  $H_*(Q\Sigma \mathbb{C}P_+^\infty) \setminus \setminus \partial_*$  in Theorem 1.2.

The Eilenberg–Moore spectral sequence associated to the path-loop fibration over  $\tilde{\Omega}^\infty \Sigma \mathbb{C}P_{-1}^\infty$  is

$$E^2 = \text{Cotor}^{H_*(\tilde{\Omega}^\infty \Sigma \mathbb{C}P_{-1}^\infty)}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow H_*(\Omega_0^\infty \mathbb{C}P_{-1}^\infty) \tag{6.1}$$

and by Theorem 1.2, the  $E^2$ -term splits (non-canonically) as

$$E^2 \cong \text{Cotor}^{H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty) \setminus \setminus \omega_*}(\mathbb{F}_p, \mathbb{F}_p) \otimes \text{Cotor}^{H_*(\tilde{Q}\Sigma \mathbb{C}P_+^\infty) \setminus \setminus \partial_*}(\mathbb{F}_p, \mathbb{F}_p). \tag{6.2}$$

More canonically there is a short exact sequence of Cotor’s, but for the following arguments we will assume that a splitting has been chosen.

I claim it must collapse. As before, we consider a possibly nonzero differential  $dx = y \neq 0$  with  $\text{deg}(x)$  minimal. We will reach a contradiction in a number of steps. The argument is based on Theorem 4.4 and a careful analysis of degrees modulo  $2(p - 1)$  in the spectral sequence.

By Theorem 1.2 and Lemma 2.9, the first factor in (6.2) is a polynomial algebra on generators of total degree  $\equiv -2 \pmod{2(p - 1)}$ . To gain information about the second factor, we map the spectral sequence (6.1) into the spectral sequence of the path-loop fibration over  $\tilde{Q}\Sigma \mathbb{C}P_+^\infty$  via the map  $\omega : \tilde{\Omega}^\infty \Sigma \mathbb{C}P_{-1}^\infty \rightarrow \tilde{Q}\Sigma \mathbb{C}P_+^\infty$ . This is a map  $E^r(\omega)$  of spectral sequences whose restriction to the first factor in the splitting (6.2) is zero, and whose restriction to the second factor in (6.2) is induced by the inclusion  $H_*(\tilde{Q}\Sigma \mathbb{C}P_+^\infty) \setminus \setminus \partial_* \rightarrow H_*(\tilde{Q}\Sigma \mathbb{C}P_+^\infty)$ . The next lemma says that this second factor in (6.2) injects under  $E^2(\omega)$ .

**Lemma 6.1.** *Let  $f : A \rightarrow B$  be an injection of primitively generated Hopf algebras. Then  $\text{Cotor}^f(\mathbb{F}_p, \mathbb{F}_p) : \text{Cotor}^A(\mathbb{F}_p, \mathbb{F}_p) \rightarrow \text{Cotor}^B(\mathbb{F}_p, \mathbb{F}_p)$  is also injective.*

**Proof.** By Theorem 2.4,  $A^*$  and  $B^*$  are tensor products of exterior algebras and polynomial algebras truncated at height  $p$ . Thus we can split  $f^* : B^* \rightarrow A^*$  in the category of algebras (since a splitting can be chosen on the generators of  $A^*$ ). Dually,  $f : A \rightarrow B$  is split injective as a map of coalgebras and thus  $\text{Cotor}^f(\mathbb{F}_p, \mathbb{F}_p)$  is injective.  $\square$

**Corollary 6.2.** *Relative to splitting (6.2), a differential  $dx = y \neq 0$  with  $x$  of minimal degree will have  $x$  in the right factor and  $y$  in the left.*

**Proof.** Recall that  $P$  and  $Q$  are additive:  $P(A \otimes B) = PA \oplus PB$  and  $Q(A \otimes B) = QA \oplus QB$ . Thus  $x$  and  $y$  do not contain products between the two factors in (6.2).

Since  $y$  is primitive and in bidegree  $(\leq -3, *)$ , it must be of even total degree by Corollary 2.9, and thus  $x$  is of odd total degree. By Theorem 1.2 this is only possible if  $x$  is in the right factor.

By Lemma 6.1, the right factor injects into the spectral sequence of  $Q\Sigma\mathbb{C}P_+^\infty$ , and since all differentials vanish in this spectral sequence,  $y$  must map to 0 there, and hence  $y$  is in the left factor.  $\square$

The remaining part of the collapse proof is to eliminate the possibility of differentials from the right factor to the left. This is the hardest part of the proof, the main ingredient of which is Theorem 4.4.

**Theorem 6.3.** *The spectral sequence (6.1) collapses.*

**Proof.** Assume there is a differential  $dx = y \neq 0$  with  $\deg(x)$  minimal. Then  $y$  is a primitive element in  $\text{Cotor}^{H_*(\tilde{Q}^\infty\Sigma\mathbb{C}P_{-1}^\infty)}(\mathbb{F}_p, \mathbb{F}_p)$ . By Corollaries 6.2 and 2.9 it is of the form

$$y = (s^{-1}z)^{p^k}$$

for a  $z \in P(H_*(\tilde{Q}^\infty\Sigma\mathbb{C}P_{-1}^\infty) \setminus \omega_*)$ . By Theorem 1.2 we must have  $\deg(z) \equiv -1 \pmod{2(p-1)}$ . Write

$$\deg(z) = 2n(p-1) - 1.$$

Then

$$\deg y = p^k(2n(p-1) - 2) = 2p^k(n(p-1) - 1) \equiv -2 \pmod{2(p-1)}$$

and thus  $\deg x \equiv -1 \pmod{2(p-1)}$  because the differential has degree  $-1$ . By Proposition 2.14 we get that  $x$  corresponds to a minimal element in the cokernel of  $\sigma_* : QH_*(\Omega_0^\infty\mathbb{C}P_{-1}^\infty) \rightarrow PH_*(\tilde{Q}^\infty\Sigma\mathbb{C}P_{-1}^\infty)$ , of degree  $\equiv 0 \pmod{2(p-1)}$ . By Corollary 6.2,  $x$  is also a minimal element in the cokernel of the composition

$$QH_*(\Omega_0^\infty\mathbb{C}P_{-1}^\infty) \xrightarrow{\sigma_*} PH_*(\tilde{Q}^\infty\Sigma\mathbb{C}P_{-1}^\infty) \xrightarrow{P\omega_*} P(H_*(\tilde{Q}\Sigma\mathbb{C}P_+^\infty) \setminus \partial_*).$$

By minimality this element is not a  $p$ th power and hence maps to a non-zero element of  $Q(H_*(\tilde{Q}\Sigma\mathbb{C}P_+^\infty) \setminus \partial_*)$ . Again by minimality, and because the loop suspension  $\sigma_*$  is  $R$ -linear, this element is  $R$ -indecomposable and hence since  $\sigma_*$  has degree 1,  $x$  will map to a nonzero element of degree  $\equiv 0 \pmod{2(p-1)}$  in

$$\mathbb{F}_p \otimes_R Q(H_*(\tilde{Q}\Sigma\mathbb{C}P_+^\infty) \setminus \partial_*)$$

in contradiction with Theorem 4.3.  $\square$

Theorem 1.4 now follows from Theorem 6.3 and Proposition 2.15.

**Proof of Corollary 1.5.** This is completely analogous to the inductive step in the classical calculations of the homology of  $QX$  or of the cohomology of  $K(\mathbb{F}_p, n)$ . We sketch the details.

Consider the Leray–Serre spectral sequence

$$E^2 = H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty) \otimes H_*(\Omega^\infty \mathbb{C}P_{-1}^\infty) \Rightarrow H_*(point). \tag{6.3}$$

Since  $\sigma_*$  is onto, we can pick a basis  $B \subseteq PH_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)$  and for each  $x \in B$  pick an element  $\tau x \in H_*(\Omega^\infty \mathbb{C}P_{-1}^\infty)$  with  $\sigma_*(\tau x) = x$ . We can now form a model spectral sequence

$$\tilde{E}^2 = \otimes_{x \in B} E^r(x),$$

where, if  $x \in B$  has odd degree,

$$E^2(x) = E[x] \otimes \mathbb{F}_p[\tau x]$$

with the differential determined by requiring that  $x$  transgresses to  $\tau x$ . If  $x$  has even degree  $\deg(x)=2s$ , we set

$$E^2(x) = \mathbb{F}_p\{1, x, \dots, x^{p-1}\} \otimes E[\tau x] \otimes \mathbb{F}_p[\beta Q^s(\tau x)]$$

with the differential determined by requiring that  $x$  transgresses to  $\tau x$  and that  $x^{p-1} \otimes \tau x$  transgresses to  $\beta Q^s(\tau x)$ .

The choices of  $\tau x \in H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)$  determines a map of spectral sequences  $\tilde{E}^r \rightarrow E^r$  and the comparison theorem implies that it is an isomorphism and then Corollary 1.5 follows.  $\square$

**7. The case  $p = 2$**

At the prime 2, the calculation of  $H_*(\Omega^\infty \mathbb{C}P_{-1}^\infty)$  and  $H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty)$  can also be made. Some details are quite different however. In particular, we will use the looped fibration

$$\Omega^\infty \mathbb{C}P_{-1}^\infty \rightarrow Q(\mathbb{C}P_+^\infty) \rightarrow \Omega QS^0 \tag{7.1}$$

to compute  $H_*(\Omega^\infty \mathbb{C}P_{-1}^\infty)$ , instead of the path-loop fibration over  $\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty$ . At  $p = 2$  our base spaces in the fibrations are no longer simply connected. The following lemma deals with this:

**Lemma 7.1.** *As spaces we have*

$$QS^0 \simeq \mathbb{Z} \times \mathbb{R}P^\infty \times \tilde{Q}_0 S^0,$$

$$\Omega QS^0 \simeq \mathbb{Z}/2 \times \mathbb{R}P^\infty \times \tilde{\Omega}_0 QS^0,$$

where  $\tilde{X} \rightarrow X$  denotes the universal covering.

**Proof.** Let  $X$  be an  $(n - 1)$ -connected  $H$ -space with  $\pi_n(X) = G$ . There is an  $H$ -map  $X \rightarrow K(G, n)$  inducing an isomorphism in  $\pi_n$  and with fibre the  $n$ -connected cover  $X\langle n \rangle$ . If one can find a map  $K(G, n) \rightarrow X$  inducing an isomorphism in  $\pi_n$ , this map will give a splitting  $X \simeq X\langle n \rangle \times K(G, n)$ .

For  $n = 0$  this is automatic.

For  $X = Q_2 S^0 \simeq Q_0 S^0$ ,  $\pi_1(X) = \mathbb{Z}/2$  and the definition of the Dyer–Lashof operation  $Q^1 \iota \in H_1(Q_2 S^0; \mathbb{F}_2)$  gives a map

$$\mathbb{R}P^\infty = B\mathbb{Z}/2 \rightarrow Q_2 S^0 \simeq Q_0 S^0,$$

inducing an isomorphism in  $H_1$  and thus by the Hurewicz theorem an isomorphism in  $\pi_1$  and the splitting of  $QS^0$  follows.

For  $X = \Omega_0 Q_0 S^0$ ,  $\pi_1(X) = \mathbb{Z}/2$ . The Hopf map gives an infinite loop map  $\eta: Q(S^1) \rightarrow Q_0 S^0$ . I claim it is nonzero in  $\pi_2$ . To see this it suffices to show that  $(\eta\langle 1 \rangle)_*$  is nonzero in  $H_2$  which can be seen as follows. Let  $\sigma \in H_1(QS^1)$  be the fundamental class. Since  $QS^1 \simeq S^1 \times QS^1\langle 1 \rangle$ , the element  $Q^1\sigma \in H_2(QS^1)$  must be in the image from  $H_*(QS^1\langle 1 \rangle)$ . Since  $\eta_*(Q^1\sigma) = Q^1(Q^1[1] * [-2]) \neq 0$ ,  $\eta\langle 1 \rangle_*$  is indeed nonzero in  $H_2$ .

Hence,  $\Omega_0\eta: Q_0 S^0 \rightarrow \Omega_0 Q_0 S^0$  is nonzero in  $\pi_1$  and thus the composition

$$\mathbb{R}P^\infty \rightarrow Q_0 S^0 \rightarrow \Omega_0 Q_0 S^0$$

is nonzero in  $\pi_1$  and the splitting of  $\Omega Q_0 S^0$  follows.  $\square$

Lemma 7.1 ensures that our spectral sequences has trivial local coefficients and hence that they converge.

### 7.1. Recollections

The Dyer–Lashof algebra is slightly different at  $p=2$ . Let  $\mathcal{R}$  be the free noncommutative algebra on the set  $\{Q^s \mid s \geq 0\}$  with  $\deg(Q^s) = s$ . The Adem relation  $\mathcal{A}^{(0,r,0,s)}$  in Definition 3.4 still makes sense, and we let  $\mathcal{A} \subseteq \mathcal{R}$  be the span of the  $\mathcal{A}^{(0,r,0,s)}$ . The unstability relations at  $p=2$  are

$$Q^s x = \begin{cases} x^2 & \text{if } \deg x = s, \\ 0 & \text{if } \deg x > s \end{cases}$$

and the algebra  $R$  is defined from these data as before. Corresponding to  $I = (s_1, s_2, \dots, s_k)$  there is an iterated operation  $Q^I = Q^{s_1} \dots Q^{s_k}$ , and this operation is called admissible if  $s_i \leq 2s_i$  for all  $i$ . The definition of excess at  $p=2$  is

$$e(I) = s_1 - \sum_{j=2}^k s_j.$$

Given a basis  $B \subseteq JH_*(X)$ , then  $H_*(QX)$  is the polynomial algebra on the set

$$\mathbf{T} = \{Q^I x \mid x \in B, I \text{ admissible, } e(I) > \deg(x)\}$$

and similarly for  $H_*(Q_0 X)$ .

One pleasant feature of  $p=2$  is the following:

**Lemma 7.2.** *The cohomology algebra  $H^*(Q_0 X)$  is polynomial if the Frobenius  $\xi: H^*(X) \rightarrow H^*(X)$  is injective.*

**Proof.** This is because the Nishida relation  $\lambda Q^{2s} = Q^s \lambda$  makes  $\lambda: H_*(Q_0 X) \rightarrow H_*(Q_0 X)$  surjective if  $\lambda: H_*(X) \rightarrow H_*(X)$  is surjective.  $\square$

In particular,  $H^*(Q_0 S^0)$  and  $H^*(Q_0 \mathbb{C}P_+^\infty)$  are both polynomial.

The calculation in Theorem 2.8 is valid with the remark that  $\mathbb{F}_2[x]/(x^2)$  must be interpreted as  $E[x]$  and thus it does not produce generators of Cotor in bidegree  $(-2, *)$ . Only truncations at height  $p^n, n \geq 2$  does that.

An important difference is that for odd primes,  $\text{Cotor}^A(\mathbb{F}_p, \mathbb{F}_p)$  is automatically a free algebra. This is no longer true for  $p = 2$ , since  $\text{Tor}^{\mathbb{F}_2[x]}(\mathbb{F}_2, \mathbb{F}_2) = E[s^{-1}x]$ , and exterior algebras are not free in characteristic 2.

One consequence of the above remarks is the following:

**Proposition 7.3.** *Let  $X$  be a simply connected space with  $H^*(X)$  polynomial. Then  $H_*(\Omega X)$  is an exterior algebra and the suspension*

$$\sigma_* : QH_*(\Omega X) \rightarrow PH_*(X)$$

*is an isomorphism. The spectral sequence*

$$\text{Cotor}^{H_*(X)}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow H_*(\Omega X)$$

*collapses.*

**Proof.** This is because

$$\text{Cotor}^{H_*(X)}(\mathbb{F}_2, \mathbb{F}_2) \cong E[s^{-1}PH_*(X)]$$

has generators and primitives in bidegrees  $(-1, *)$ . Together with Lemma 2.14, this proves the collapse claim and that  $\sigma_*$  is an isomorphism. Dually we have that

$$\sigma^* : QH^*(X) \rightarrow PH^*(\Omega X)$$

is an isomorphism and since the image generates  $H^*(\Omega X)$  as an algebra,  $H^*(\Omega X)$  is primitively generated. By Theorem 2.4 we get that  $H_*(\Omega X)$  is exterior.  $\square$

In particular this applies to  $X = \tilde{Q}_0S^0$ .

Similarly, we have

**Proposition 7.4.** *For any space  $X$ , the spectral sequence*

$$\text{Cotor}^{H_*(\tilde{Q}\Sigma X)}(k, k) \Rightarrow H_*(Q_0X)$$

*collapses and the suspension*

$$\sigma_* : QH_*(QX) \rightarrow PH_*(Q\Sigma X)$$

*is an isomorphism.*

**Proof.**  $\sigma_*$  is surjective since it hits  $JH_*(\Sigma X)$  and since it is  $R$ -linear. Thus by Corollary 2.15, the spectral sequence must collapse. Now  $H_*(Q\Sigma X)$  is primitively generated, so by Theorem 2.4 we get that  $H^*(Q\Sigma X)$  is exterior and hence the spectral sequence has

$$E^2 = \text{Cotor}^{H_*(\tilde{Q}\Sigma X)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[s^{-1}PH_*(\tilde{Q}\Sigma X)].$$

Since this is free as an algebra, there are no extension problems in homology, and since  $QH_*(Q_0X)$  is in linear bijection with  $E_{-1,*}^\infty$ , we get that  $\sigma_*$  is injective.  $\square$



7.2. Homology of  $\Omega^\infty \mathbb{C}P_{-1}^\infty$

The lemmas in Section 7.1 imply the following diagram:

$$\begin{array}{ccc}
 QH_*(Q_0 \mathbb{C}P_+^\infty) & \xrightarrow{Q(\Omega_0 \hat{\partial})_*} & QH_*(\Omega_0 QS^0) \\
 \cong \downarrow & & \downarrow \cong \\
 PH_*(\tilde{Q}\Sigma \mathbb{C}P_+^\infty) & \xrightarrow{P\hat{\partial}_*} & PH_*(\tilde{Q}_0 S^0)
 \end{array} \tag{7.2}$$

in which the vertical isomorphisms are the suspensions and in which  $H_*(\Omega_0 QS^0)$  is an exterior algebra, dual to a polynomial algebra.

The formula for  $\hat{\partial}_*$  has an extra term because of the Hopf map  $\eta$ . We quote the result from [8, Theorem 4.4]

**Theorem 7.5** ([8]). *Let  $a_s \in H_*(\mathbb{C}P_+^\infty)$  be the generator,  $s$  odd. Then*

$$Q(\hat{\partial}_*)(a_s) = Q^{2s+1}\iota + Q^{s+1}Q^s\iota = Q^{2s+1}\iota + Q^{2s}Q^1\iota$$

We shall need a lemma analogous to Lemma 4.3.

**Lemma 7.6.** *The left ideal in  $R$  generated by  $\{Q^{2s+1} \mid s \geq 0\}$  is also a right ideal.*

**Proof.** This is completely analogous to the proof of Lemma 4.3. One uses the Adem relation

$$Q^{2s}Q^{r-s} = Q^rQ^s + \sum_{i>s} \lambda_i Q^{r+s-i}Q^i$$

valid for  $r \leq 2s$ , for  $r$  odd and  $s$  even.  $\square$

**Lemma 7.7.** *Let  $b_{2s+1} \in PH_*(QS^0) = PH_*(Q_0S^0)$  be the unique primitive element with  $b_{2s+1} - Q^{2s+1}\iota$  decomposable. Then  $PH_*(QS^0)$  is generated over  $R$  by the set  $\{b_{2s+1} \mid s \geq 0\}$ .*

**Proof.** Let  $\lambda : QH_*(Q_0S^0) \rightarrow QH_*(Q_0S^0)$  be the dual of the squaring. By the Nishida relation  $\lambda Q^{2s} = Q^s \lambda$ , the coimage of  $\lambda$  has basis

$$\{Q^I \iota \mid I \text{ admissible, } e(I) > 0, 2 \mid I\},$$

where  $2 \mid I$  means that all entries of  $I$  are even. Thus Theorem 2.4 implies that the image of  $PH_*(QS^0) \rightarrow QH_*(QS^0)$  has basis

$$\{Q^I \iota \mid I \text{ admissible, } e(I) > 0, 2 \nmid I\}$$

and by Lemma 7.6, this is generated over  $R$  by the subset

$$\{Q^{2s+1}\iota \mid s \geq 0\}.$$

Thus the subspace of  $PH_*(QS^0)$  generated over  $R$  by  $\{b_{2s+1} \mid s \geq 0\}$  contains all indecomposable primitives. But this generated subspace is clearly preserved by the Frobenius map  $\zeta : x \mapsto x^2$ , so it contains all primitives and the claim follows from Theorem 2.4.  $\square$

We are now ready to prove the mod 2 analogue of Theorem 1.1. The result is much simpler, and the extra term in Theorem 7.5 does not give much trouble.

**Theorem 7.8.** *The map*

$$P\partial_* : PH_*(Q\Sigma\mathbb{C}P_+^\infty) \rightarrow PH_*(QS^0)$$

*is surjective.*

**Proof.** By the previous lemma, it suffices to prove that  $Q\partial_*$  hits the classes  $Q^{2s+1}\iota$ . Indeed, any indecomposable class mapping to  $Q^{2s+1}\iota$  is odd-dimensional and thus by Theorem 2.4 has a unique primitive representative that will map to  $b_{2s+1}$ .

For  $s = 0$ , this is immediate, since  $\partial_*(a_1) = (Q^1\iota) * \iota^{-2}$ . For general  $s$  we use the Adem relation  $Q^{2s}Q^1 = Q^{s+1}Q^s$  to get

$$\begin{aligned} Q(\partial_*)(a_{2s+1}) &= Q^{2s+1}\iota + Q^{s+1}Q^s\iota, \\ &= Q^{2s+1}\iota + Q^{2s}Q^1\iota. \end{aligned}$$

Thus we have

$$Q(\partial_*)(a_s - Q^{2s}a_1) = Q^{2s+1}\iota \quad \square$$

**Remark 7.9.** The claim of [8, Corollary 7.5] that  $\partial_*$  and thus  $P(\partial_*)$  is *injective* is incorrect. The  $Q^lQ^{2r+1}$  of [8, Corollary 7.4] is not necessarily admissible, and in fact an application of the Adem relations shows that

$$\partial_*(Q^3a_1 - Q^2Q^1a_1) = 0.$$

Together with diagram (7.2), Theorem 7.8 makes the spectral sequence

$$\text{Cotor}^{H_*(\tilde{\Omega}QS^0)}(H_*(\tilde{Q}\mathbb{C}P_+^\infty), k) \Rightarrow H_*(\Omega_0^\infty\mathbb{C}P_{-1}^\infty) \tag{7.3}$$

very simple. We can now prove Theorem 1.3.

**Proof of Theorem 1.3.** It follows from diagram (7.2) and Theorem 7.8 that the map  $Q(\Omega_0\partial_*)$  is surjective. Therefore the  $E^2$ -term of the Eilenberg–Moore spectral sequence is

$$E^2 = \text{Cotor}^{H_*(\Omega_0QS^0)}(H_*(Q_0\mathbb{C}P_+^\infty), \mathbb{F}_2) \cong H_*(Q_0\mathbb{C}P_+^\infty) \setminus \setminus \Omega\omega_*$$

and is concentrated on the line  $E_{0,*}^2$ . Therefore it collapses and we get the short exact sequence

$$\mathbb{F}_2 \rightarrow H_*(\Omega^\infty\mathbb{C}P_{-1}^\infty) \xrightarrow{\Omega\omega_*} H_*(Q\mathbb{C}P_+^\infty) \xrightarrow{\Omega\partial_*} H_*(\Omega QS^0) \rightarrow \mathbb{F}_2 \quad \square$$

### 7.3. Homology of $\Omega^\infty\Sigma\mathbb{C}P_{-1}^\infty$

This part of the calculation is similar to the odd primary case. We consider again the spectral sequence (5.1) with splitting (5.2). Notice that fibration (1.1) splits off the fibration  $S^1 \rightarrow S^1 \rightarrow \mathbb{R}P^\infty$  and hence it has trivial local coefficients. As for odd primes, we need to determine the coalgebra structure on  $H_*(QS^0)/\partial_*$ , so we first prove Theorem 1.1 in the case  $p = 2$ .

**Proof of Theorem 1.1** ( $p = 2$ ). Since  $Q$  is right exact we have  $Q(H_*(QS^0)/\partial_*) = \text{Cok}(Q\partial_*)$ , and from the calculation in the proof of Theorem 7.8 it follows that the image of  $Q\partial_*$  contains all  $Q^I \iota$  where  $I$  has at least one odd entry, and therefore that the composition in Theorem 1.1 is surjective.

To prove injectivity, consider again the dual squaring  $\lambda: H_*(Q_0S^0) \rightarrow H_*(Q_0S^0)$ . It is a map of Hopf algebras, and since  $\lambda Q^{2s} = Q^s \lambda$  and  $\lambda Q^{2s+1} = 0$  we get that

$$\lambda: H_*(QS^0)^{(0)} \rightarrow H_*(QS^0)$$

is an isomorphism. Hence

$$H_*(QS^0) = H_*(QS^0)^{(0)} \oplus \text{Ker}(\lambda),$$

where the first summand is a subalgebra and the second is an ideal. Now the injectivity of the map in the theorem follows from the fact that  $\text{Ker}(\lambda)$  is an ideal and that  $\text{Im}(\partial_*) \subseteq \mathbb{F}_2 \oplus \text{Ker}(\lambda)$ .  $\square$

**Theorem 7.10.**  $H_*(Q_0S^0)/\partial_*$  is dual to a polynomial algebra.

**Proof.** This follows since  $\lambda: H_*(Q_0S^0) \rightarrow H_*(Q_0S^0)$  is surjective.  $\square$

Notice that  $H^*(Q_0S^0)$  itself is polynomial. This is in contrast to the odd primary case, where only the subalgebra  $H^*(Q_0S^0) \setminus \partial^* \subseteq H^*(Q_0S^0)$  is polynomial.

**Proof of Theorem 1.3** ( $p = 2$ ). Completely as for odd primes, Theorem 7.10 makes the spectral sequence collapse, and the collapse gives a short exact sequence of Hopf algebras

$$\mathbb{F}_2 \rightarrow H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty) \setminus \omega_* \rightarrow H_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty) \xrightarrow{\omega_*} H_*(Q\Sigma \mathbb{C}P_+^\infty) \setminus \partial_* \rightarrow \mathbb{F}_2. \tag{7.4}$$

Since  $H_*(Q\Sigma \mathbb{C}P_+^\infty)$  is primitively generated, so is  $H_*(Q\Sigma \mathbb{C}P_+^\infty) \setminus \partial_*$ . Hence the sequence is split if and only if  $P(\omega_*)$  is surjective. We have the diagram

$$\begin{array}{ccccccc} QH_*(\Omega^\infty \mathbb{C}P_{-1}^\infty) & \longrightarrow & QH_*(Q\mathbb{C}P_+^\infty) & \longrightarrow & QH_*(\Omega QS^0) & \longrightarrow & 0 \\ \downarrow \sigma_* & & \downarrow \cong & & \downarrow \cong & & \\ PH_*(\Omega^\infty \Sigma \mathbb{C}P_{-1}^\infty) & \longrightarrow & PH_*(Q\Sigma \mathbb{C}P_+^\infty) & \xrightarrow{P\partial_*} & PH_*(QS^0) & \longrightarrow & 0 \end{array}$$

which we know is exact except possibly at  $PH_*(Q\Sigma \mathbb{C}P_+^\infty)$ . But it follows from the rest of the diagram that it is also exact at  $PH_*(Q\Sigma \mathbb{C}P_+^\infty)$ . Since  $\text{Ker}(P\partial_*) = P(H_*(Q\Sigma \mathbb{C}P_+^\infty) \setminus \partial_*)$  we get that the sequence (7.4) splits.  $\square$

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