Note

Upper signed domination number∗

Huajun Tanga, Yaojun Chenb,*

aDepartment of Logistics, The Hong Kong Polytechnic University, Hung Kom, Kowloon, Hong Kong, PR China
bDepartment of Mathematics, Nanjing University, Nanjing 210093, PR China

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Abstract

Let $G = (V, E)$ be a graph. A signed dominating function on $G$ is a function $f : V \rightarrow \{-1, 1\}$ such that $\sum_{u \in N[v]} f(u) \geq 1$ for each $v \in V$, where $N[v]$ is the closed neighborhood of $v$. The weight of a signed dominating function $f$ is $\sum_{v \in V} f(v)$. A signed dominating function $f$ is minimal if there exists no signed dominating function $g$ such that $g \neq f$ and $g(v) \leq f(v)$ for each $v \in V$. The signed upper domination number of a graph $G$, denoted by $\Gamma_u(G)$, equals the maximum weight of a minimal signed dominating function of $G$. In this paper, we establish an upper bound for $\Gamma_u(G)$ in terms of minimum degree and maximum degree. Our result is a generalization of those for regular graphs and nearly regular graphs obtained in [O. Favaron, Signed domination in regular graphs, Discrete Math. 158 (1996) 287–293] and [C.X. Wang, J.Z. Mao, Some more remarks on domination in cubic graphs, Discrete Math. 237 (2001) 193–197], respectively.

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1. Introduction

All graphs considered in this paper are finite simple graphs. Let $G = (V, E)$ be a graph and $v \in V$. The neighborhood of $v$ is $N(v) = \{u \in V | uv \in E\}$ and the closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. The degree of $v$ in $G$ is $d(v) = |N(v)|$. We call $G$ $k$-regular if $d(v) = k$ for all $v \in V$ and nearly $k$-regular if $d(v) = k - 1$ or $k$ for all $v \in V$. For a subset $S \subseteq V$, $N(S) = \bigcup_{u \in S} N(u)$. We denote by $G[S]$ the subgraph induced by $S$ in $G$ and $d_S(v)$ the number of vertices in $S$ adjacent to $v$. The minimum degree and maximum degree of the vertices of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. When no ambiguity can occur, we often simply write $\delta$ and $\Delta$ instead of $\delta(G)$ and $\Delta(G)$, respectively. For vertex-disjoint subsets $A$ and $B$, we use $e(A, B)$ to denote the number of edges between $A$ and $B$.

A signed dominating function on $G$ is a function $f : V \rightarrow \{-1, 1\}$ such that $\sum_{u \in N[v]} f(u) \geq 1$ for each $v \in V$. The weight of a signed dominating function $f$ is $\omega(f) = \sum_{v \in V} f(v)$. A signed dominating function $f$ is minimal if there exists no signed dominating function $g$ such that $g \neq f$ and $g(v) \leq f(v)$ for each $v \in V$. The upper signed domination number of a graph $G$, denoted by $\Gamma_u(G)$, is defined as $\Gamma_u(G) = \max \{\omega(f) | f$ is a minimal signed dominating function of $G\}$.

In [3] Henning and Slater asked for upper bounds on $\Gamma_u(G)$ for cubic graphs. Favaron gave sharp upper bounds on $\Gamma_u(G)$ for regular graphs.

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* Corresponding author.
E-mail address: yaojunc@nju.edu.cn (Y. Chen).

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Theorem 1 (Favaron [2]). If \( G \) is a \( k \)-regular graph, \( k \geq 1 \), of order \( n \), then \( \Gamma_\delta(G) \leq n(k + 1)/(k + 3) \) if \( k \) is even and \( \Gamma_\delta(G) \leq n(k + 1)^2/(k^2 + 4k - 1) \) if \( k \) is odd.

Wang and Mao established the best possible upper bounds on \( \Gamma_\delta \) for nearly regular graphs.

Theorem 2 (Wang and Mao [4]). If \( G \) is a nearly \((k+1)\)-regular graph of order \( n \), then \( \Gamma_\delta(G) \leq n(k+2)^2/(k^2+6k+4) \) for \( k \) even, and \( \Gamma_\delta(G) \leq n(k^2+3k+4)/(k^2+5k+2) \) for \( k \) odd.

Obviously, \( \Delta - \delta \leq 1 \) for regular graphs and nearly regular graphs and the two theorems above establish upper bounds for \( \Gamma_\delta \) in the case when \( \Delta - \delta \leq 1 \). In general, the minimum and maximum degrees of most graphs do not satisfy the condition \( \Delta - \delta \leq 1 \). Thus one may ask what are the upper bounds for \( \Gamma_\delta \) when \( \Delta - \delta \) is arbitrary large. In this paper, we establish the best possible upper bound for \( \Gamma_\delta \) in terms of minimum and maximum degrees. The main result of this paper is the following.

Theorem 3. If \( G \) is a graph of order \( n \), then \( \Gamma_\delta(G) \leq (\delta \Delta + 4 \Delta - \delta)n/(\delta \Delta + 4 \Delta + \delta) \) for \( \delta \) even, and \( \Gamma_\delta(G) \leq (\delta \Delta + 3 \Delta - \delta + 1)n/(\delta \Delta + 3 \Delta + \delta - 1) \) for \( \delta \) odd. Furthermore, if \( G \) is an Eulerian graph, then \( \Gamma_\delta(G) \leq (\delta \Delta + 2 \Delta - \delta)n/(\delta \Delta + 2 \Delta + \delta) \).

In this paper, we need the following lemmas.

Lemma 1 (Dunbar et al. [1]). A signed dominating function \( f \) on a graph \( G \) is minimal if and only if for every vertex \( v \in V \) with \( f(v) = 1 \), there exists a vertex \( u \in N[v] \) with \( f[u] \in \{1, 2\} \).

Lemma 2 (Favaron [2]). If \( \delta \geq 2 \), then we have

\[
\begin{align*}
(1) & \quad n = m + \sum_{i=0}^l a_i, \\
(2) & \quad e(M, P) = \sum_{i=1}^l i a_i \leq m A.
\end{align*}
\]

If \( \delta = 1 \), then the result is trivial. Thus we may assume \( \delta \geq 2 \).

If \( A_0 = \emptyset \), then by Lemma 2, we have \( n = m + \sum_{i=1}^l a_i \leq m + \sum_{i=1}^l i a_i \leq (\Delta + 1)m \), which implies that \( m \geq n/(\Delta + 1) \), and hence \( \Gamma_\delta(G) = n - 2m \leq (\Delta - 1)n/(\Delta + 1) \). Noting that \( (\Delta - 1)n/(\Delta + 1) < \min\{\delta \Delta + 4 \Delta - \delta)n/(\delta \Delta + 4 \Delta + \delta), (\delta \Delta + 3 \Delta - \delta + 1)n/(\delta \Delta + 3 \Delta + \delta - 1)\} \), we see the conclusion holds. Thus we may assume \( A_0 \neq \emptyset \).

For any \( v \in A_0 \), since \( f[v] = d(v) + 1 \geq \delta \geq 3 \) and \( f \) is minimal, by Lemma 1, \( v \) has at least one neighbor \( u \) such that \( u \notin A_0 \) and \( f[u] = 1 \) or \( 2 \). Let \( Q = \{v \in N(A_0) \mid f[v] = 1 \text{ or } 2\} \). Noting that \( f[v] \geq 3 \) for any \( v \in \bigcup_{i=0}^{l-1} A_i \), we see that \( Q \subseteq \bigcup_{i=1}^l A_i \). Obviously, each \( u \in Q \cap A_i \) has at most \( i + 1 \) neighbors in \( A_0 \). Thus \( Q \cap A_i \) has at most \( (i + 1)|Q \cap A_i| \) neighbors in \( A_0 \). By the arguments above, we have \( A_0 \subseteq \bigcup_{i=k}^l N(Q \cap A_i) \), which implies

\[
a_0 = |A_0| \leq \sum_{i=k}^l |N(Q \cap A_i)| \leq \sum_{i=k}^l (i + 1)a_i.
\]
By Lemma 2(1) and (1), we have
\[ n \leq m + \sum_{i=k}^{\ell} (i + 1)a_i + \sum_{i=1}^{\ell} a_i. \] (2)

If \( k = 1 \), then by (2), we have
\[ n \leq m + \sum_{i=1}^{\ell} (i + 2)a_i \] (3)

and if \( k \geq 2 \), then by (2), we have
\[ n \leq m + \sum_{i=k}^{k-1} a_i + \sum_{i=k}^{\ell} (i + 2)a_i. \] (4)

If \( \bar{\delta} \) is odd, then since \((\bar{\delta} + 3)i/(\bar{\delta} - 1) \geq i + 2\) for \( i \geq (\bar{\delta} - 1)/2 = k \), by (3) and (4), we have \( n \leq m + [(\delta + 3)/(\delta - 1)]\sum_{i=1}^{\ell} i a_i \). By Lemma 2(2), we have \( n \leq m + m A(\delta + 3)/(\delta - 1) \), which implies that \( m \geq n(\delta - 1)/(\delta A + 3A + \delta - 1) \), and hence \( \Gamma_s(G) = n - 2m \leq (\delta A + 3A - \delta + 1)n/(\delta A + 3A + \delta - 1) \).

If \( \bar{\delta} \) is even, then since \((\bar{\delta} + 4)i/\bar{\delta} \geq i + 2\) for \( i \geq \bar{\delta}/2 = k \), by (3) and (4), we have \( n \leq m + [(\delta + 4)/\bar{\delta}]\sum_{i=1}^{\ell} i a_i \). By Lemma 2(2), we have \( n \leq m + m A(\delta + 4)/\bar{\delta} \), which implies that \( m \geq n\bar{\delta}/(\delta A + 4A + \bar{\delta}) \), and hence \( \Gamma_s(G) = n - 2m \leq (\delta A + 4A - \bar{\delta})n/(\delta A + 4A + \bar{\delta}) \).

Furthermore, if \( G \) is an Eulerian graph, that is, every vertex of \( G \) has even degree, then each \( u \in Q \cap A_i \) has at most \( i \) neighbors in \( A_0 \). Thus the inequality (2) can be improved as below:
\[ n \leq m + \sum_{i=k}^{\ell} i a_i + \sum_{i=1}^{\ell} a_i. \]

Using similar proof, we have \( n \leq m + m A(\delta + 2)/\bar{\delta} \), which gives \( \Gamma_s(G) \leq (\delta A + 2A - \bar{\delta})n/(\delta A + 2A + \bar{\delta}) \).

**Remark.** Since Theorem 1 is a special case of Theorem 3, we see that the bounds in Theorem 3 are sharp in the case when \( A = \delta \), and the graph which shows the equality holds was given in [2]. In the following, we will show that the bounds in Theorem 3 are best possible in the case when \( A = \delta \geq 1 \). To see this, we first define two graphs \( K^*_r \) and \( K^*_{r.r,r} \) as follows, where \( r = [(s + 2)/2] \) and \( s \geq 2 \). Let \( K^*_r \) be a graph obtained from complete graph \( K_{2r} \) by deleting a perfect matching if \( s \) is odd and the edges of a hamiltonian cycle if \( s \) is even, and \( K^*_{r.r,r} \) a graph obtained from complete 3-partite graph \( K_{r,r,r} \) by deleting the edges of a hamiltonian cycle if \( s \) is odd and the edges of a
hamiltonian cycle together with any other edge if \( s \) is even. Let \( t \geq s + 1 \). Now, we define \( G \) to be the graph as shown in Fig. 1, where \( V(G) = X \cup Y \cup Z \) with \( |X| = \lfloor s/2 \rfloor \), \( |Y| = t \) and \( |Z| = rt \), \( G[X \cup Y] \) is a complete bipartite graph, \( G[Z] = (t/2)K^*_2r \) if \( t \) is even and \( G[Z] = [(t-3)/2]K^*_2r \cup K^*_r, r, r, r \) if \( t \) is odd, \( d_Z(y) = r \) for any \( y \in Y \) and \( \bigcup_{y \in Y} N_Z(y) = Z \), and there is no edges between \( X \) and \( Z \). Obviously, \( |G| = t + \lfloor (s + 2)/2 \rfloor t + \lfloor s/2 \rfloor \), \( \delta(G) = s \) and \( \Delta(G) = t \). Let \( f \) be a function defined on \( V(G) \) such that \( f(v) = -1 \) for \( v \in X \) and \( f(v) = 1 \) otherwise. It is easy to check \( f \) is a signed dominating function. Since \( f(y) = 2 \) for any \( y \in Y \) and \( Y \) is a dominating set of \( G \), by Lemma 1, \( f \) is minimal. Clearly, \( \omega(f) = |G| - 2|X| = t + \lfloor (s + 2)/2 \rfloor t - \lfloor s/2 \rfloor \). If \( s \) is odd, then it is easy to check that \( \omega(f) = t + (s + 1)t/2 - (s - 1)/2 = (st + 3t - s + 1)/2 = (\delta A + 3A - \delta - 1)n/(\delta A + 3A + \delta - 1) \). If \( s \) is even, then it is not difficult to see that \( \omega(f) = t + (s + 2)t/2 - s/2 = (st + 4t - s)/2 = (\delta A + 4A - \delta)n/(\delta A + 4A + \delta) \).

References