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Note

Generalized Towers of Flag-Transitive Circular Extensions of a Non-classical C_3-Geometry

ANTONIO PASINI

Department of Mathematics, University of Siena, Via del Capitano 15, I-53100 Siena, Italy

AND

SATOSHI YOSHIARA

Division of Mathematical Sciences, Osaka-Kyoiku University, 851 Osaka, Koshimara, Japan

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The classification of generalized towers of flag-transitive circular extensions of the sporadic A_7-geometry is completed by characterizing two flat geometries on 16 points, constructed in terms of the Steiner system $S(24, 8, 5)$, as the flag-transitive circular extensions of the duals of the sporadic A_7-geometry and the Neumaier geometry for A_8, and then by showing the non-existence of flag-transitive circular extensions of these geometries. © 1994 Academic Press, Inc.

1. INTRODUCTION

In this note, we basically follow the standard notation of (incidence) geometries in [3], together with the notation $\mathscr{G}_F$ and $\mathscr{G}_i(F)$ for denoting the residue of a geometry $\mathscr{G}$ at a flag F and the set of elements of $\mathscr{G}_i$ (the set of i-varieties) incident with F, respectively. We also assume that the geometries we consider are residually connected.

A *circular extension* of a geometry $\mathscr{H}$ is a geometry $\mathscr{G} = (\mathscr{G}_0, ..., \mathscr{G}_r; *)$ with the residues $\mathscr{G}_P$ at elements $P \in \mathscr{G}_0$ isomorphic to $\mathscr{H}$, in which the residues $\mathscr{G}_F$ at flags F of type $\{0, i\}$ are isomorphic to a circle geometry for $i = 1$ and generalized digons for any $i = 2, ..., r$. A circular extension of a geometry $\mathscr{H}$ belonging to a diagram X is called a *c.X-geometry*. A sequence $(\mathscr{G}^0, ..., \mathscr{G}^n)$ of geometries is called a *tower* (resp. *generalized*

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tower) of circular extensions of a geometry $\mathscr{G}^0$ if $\mathscr{G}^i$ is a circular extension of $\mathscr{G}^{i-1}$ (resp. $\mathscr{G}^{i-1}$ or its dual) for each $i = 1, ..., n$.

Recently, much work has been done to classify towers of flag-transitive circular extensions of geometries belonging to spherical diagrams or their duals. Specifically, those of C_n-geometries or their duals came to attention, because they contain interesting examples related with the sporadic simple groups McL, HS, Suz, $Co2$, $Co3$, $Co1$, F_{22}, F_{23}, F'_{24}, and M, as well as some infinite families of examples related with classical linear groups [8]. The classification of flag-transitive $c.C_n$-geometries pioneered by [2] has recently been completed by [4] when $n \geqslant 3$, and by [2, 4, 9] when $n = 2$ and the starting C_2-geometries are classical. Towers of flag-transitive circular extensions of C_n-geometries are completely classified by [5] for $n \geqslant 3$ and for classical C_2-geometry modulo some special cases. Some progress is reported [10] on the classification of flag-transitive circular extensions of duals of some buildings of type C_3.

Since flag-transitive geometries of type C_n are not always buildings, a non-classical example naturally occurs in the classification of flag-transitive $c.C_n$-geometries. It was first found by Neumaier [7] and called the *Neumaier geometry* in [4, p. 258]. This is a circular extension of the sporadic A_7-geometry $\mathscr{S}$, which is the unique known example of a finite thick flag-transitive geometry belonging to the diagram C_3 but not a building (see a survey [6] on C_n-geometries). Note that the above classification [4] also implies that $(\mathscr{S}, c(\mathscr{S}))$ is the unique maximal tower of flag-transitive circular extensions of the sporadic A_7-geometry, where $c(\mathscr{S})$ denotes the Neumaier geometry.

In this note, we complete the classification of generalized towers of flag-transitive circular extensions of the duals of the sporadic A_7-geometry. Precisely, we first characterize the flag-transitive circular extensions of the duals of the sporadic A_7-geometry and the Neumaier geometry, and then prove the non-existence of flag-transitive circular extensions of these geometries.

THEOREM 1.1. *There is a unique isomorphism class of flag-transitive circular extensions of the dual of the sporadic A_7-geometry. It is a flat geometry defined on* 16 *points with the full automorphism group* $2^4 : A_7$.

THEOREM 1.2. *There is a unique isomorphism class of flag-transitive circular extensions of the dual of the Neumaier geometry. It is a flat geometry defined on* 16 *points with the full automorphism group* $2^4 : A_8$.

THEOREM 1.3. *For each geometry in the theorems above, there is no flag-transitive circular extension of it.*

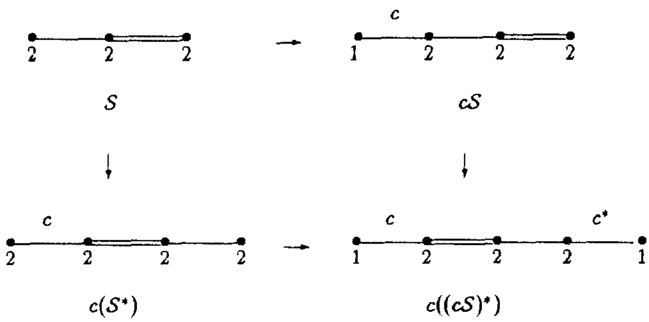

FIGURE 1

These theorems and the uniqueness of the maximal tower of flag-transitive circular extension of the spodradic A_7-geometry imply that $(\mathscr{S}, c\mathscr{S}, c((c\mathscr{S})^*))$ and $(\mathscr{S}, c(\mathscr{S}^*), c((c\mathscr{S})^*))$ are all the maximal generalized towers of flag-transitive circular extensions of the sporadic A_7-geometry, where $c(\mathscr{S}^*)$ and $c((c\mathscr{S})^*)$ are the geometries in Theorems 1.1 and 1.2, respectively (the latter can also be viewed as an extension of the dual $c(\mathscr{S}^*)$). See Fig. 1.

The geometries in Theorems 1.1 and 1.2 above can be constructed as follows in terms of the Steiner system $S(24, 8, 5)$ on the set Ω of 24 letters with the set $\mathcal{O}$ of blocks (usually called *octads*). This construction is an extension of a presentation of the Neumaier geometry implicitly used in [7, Prop. 1]. We fix an octad $\mathcal{O}$, and let $\mathscr{G}_0$, $\mathscr{G}_4$, $\mathscr{G}_1$, and $\mathscr{G}_3$ be the set of 16 letters of $\Omega - O$, the set of 8 letters of O, the set of $\binom{16}{2} = 120$ pairs of letters of $\Omega - O$, and the set of $\binom{8}{2} = 28$ pairs of letters of O, respectively. We call elements of $\mathscr{G}_0$, $\mathscr{G}_1$, $\mathscr{G}_3$, and $\mathscr{G}_4$ *points*, *lines*, *dual lines*, and *dual points*, respectively.

Recall that any subset T of Ω of four letters determines a unique *sextet* $\{T_i \mid i = 1, ..., 6\}$ with $T = T_1$, where a sextet means a 6-tuple of 4-subsets T_i $(i = 1, ..., 6)$ of Ω such that $\bigcup_{i=1}^{6} T_i = \Omega$ and $T_i \cup T_j$ is an octad for any $1 \leqslant i < j \leqslant 6$. We define the set $\mathscr{G}_2$ of *planes* as the set of ordered pairs (T_1, T_2) of a 4-subset T_1 of O and a 4-subset T_2 of $\Omega - O$ in the sextet determined by T_1. Since there are $70 = \binom{8}{4}$ 4-subsets of O, we have $|\mathscr{G}_2| = 70 \cdot 4 = 280$.

Now we define the incidence $*$ on $\bigcup_{i=0}^{4} \mathscr{G}_i$. All elements of $\mathscr{G}_0 \cup \mathscr{G}_1$ are incident with all elements of $\mathscr{G}_3 \cup \mathscr{G}_4$. A point (resp. dual point) P is incident with a line (resp. dual line) l if and only if P is contained in l. For a plane $u = (T_1, T_2) \in \mathscr{G}_2$, u is incident with $X \in \mathscr{G}_0 \cup \mathscr{G}_1$ (resp. $X \in \mathscr{G}_3 \cup \mathscr{G}_4$) if and only if X is a subset of T_2 (resp. T_1).

It is straightforward to verify that the resulting geometry $\mathscr{G} = (\mathscr{G}_0, \mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3, \mathscr{G}_4; *)$ is a circular extension of the dual of the Neumaier geometry, on which the stabilizer ($\cong 2^4 : A_8$) of an octad O in $\mathrm{Aut}(S(24, 8, 5)) \cong M_{24}$ acts flag-transitively. In the proof of Theorem 1.2, we also show that $\mathrm{Aut}(\mathscr{G}) \cong 2^4 : A_8$ is the unique flag-transitive group on $\mathscr{G}$. The residue $\mathscr{H} := \mathscr{G}_Q$ at a dual point Q is a flag-transitive circular extension of the dual of the sporadic A_7-geometry. We have $|\mathscr{H}_i| = 16$, 120, 140, and 7, respectively, for $i = 0$, 1, 2, and 3. In the proof of Theorem 1.1, we show that $\mathrm{Aut}(\mathscr{H}) \cong 2^4 : A_7$ is the unique flag-transitive group on $\mathscr{H}$.

2. Preliminaries

We first summarize some definitions.

A geometry $\mathscr{G} = (\mathscr{G}_1, ..., \mathscr{G}_r; *)$ is called *flat* if all elements of $\mathscr{G}_1$ are incident with all elements of $\mathscr{G}_r$. A *collinearity graph* $\Gamma = \Gamma(\mathscr{G})$ of $\mathscr{G}$ is a graph (with multiple edges) defined on $\mathscr{G}_1$ by declaring that two vertices x and y of $\mathscr{G}_1$ are joined by an edge labelled by an element $l \in \mathscr{G}_2$ whenever x and y are incident with l. Two elements of $\mathscr{G}_1$ are called *collinear* if they are joined by an edge in $\Gamma(\mathscr{G})$. We say that a geometry $\mathscr{G}$ satisfies *the* (*LL*) *property* when the collinearity graph $\Gamma(\mathscr{G})$ of $\mathscr{G}$ has no multiple edges, that is, any two distinct elements of $\mathscr{G}_1$ are incident with at most one element of $\mathscr{G}_2$.

For a flag-transitive geometry $\mathscr{G}$, $\mathrm{Aut}(\mathscr{G})$ means the full automorphism group of $\mathscr{G}$. If it is known that $\mathscr{G}$ has a smallest flag-transitive subgroup of $\mathrm{Aut}(\mathscr{G})$ (which is true for any flag-transitive classical $c.C_n$-geometry [4, Tables]), we denote it by $M(\mathscr{G})$. For a flag-transitive group G on $\mathscr{G}$, G_F means the stabilizer of a flag F in G. We follow the *Atlas* notation [1] to denote isomorphism classes of groups.

Now we note some elementary lemmas, in which Lemma 2.2 is known as the remark following Lemma 2 in [9] or as Lemma 6 in [4].

Lemma 2.1. *For a circular extension $\mathscr{G} = (\mathscr{G}_0, ..., \mathscr{G}_r; *)$ of a geometry $\mathscr{H}$, the following hold.*

(1) *The collinearity graph Γ of $\mathscr{G}$ is connected.*

(2) *If $\mathscr{H}$ is flat, all elements of $\mathscr{G}_0 \cup \mathscr{G}_1$ are incident with all elements of $\mathscr{G}_r$.*

Proof. Claim (1) is easily verified by induction on the rank of $\mathscr{H}$. Take any $\pi \in \mathscr{G}_r$ and any $P \in \mathscr{G}_0$ incident with π. For any line m incident with P, m is incident with π, since $\mathscr{G}_P \cong \mathscr{H}$ is a flat geometry. Thus any element $Q \in \mathscr{G}_0(m)$ is also incident with π. Then it follows from claim (1) above that all elements of $\mathscr{G}_0$ are incident with π. Similarly, we can also show that every element of $\mathscr{G}_1$ is incident with all elements of $\mathscr{G}_r$. ∎

LEMMA 2.2. *Assume that* $\mathscr{H}=(\mathscr{H}_1, ..., \mathscr{H}_r; *)$ *is a flag-transitive geometry in which any minimal flag-transitive automorphism group acts primitively on the set* $\mathscr{H}_1$. *If* $\mathscr{G}=(\mathscr{G}_0, ..., \mathscr{G}_r; *)$ *is a circular extension of* $\mathscr{H}$, *then it satisfies the (LL) property.*

LEMMA 2.3. *Let* $\mathscr{G}$ *be a circular extension of a geometry* $\mathscr{H}=(\mathscr{H}_1, ..., \mathscr{H}_r; *)$, *admitting a flag-transitive group* $G \subseteq \mathrm{Aut}(\mathscr{G})$. *If one of the following conditions is satisfied, then the stabilizer* G_P *of* $P \in \mathscr{G}_0$ *acts faithfully on the residue* $\mathscr{G}_P$ *of* $\mathscr{G}$ *at* P.

(1) *The geometry* $\mathscr{H}$ *is flat, and* $\mathrm{Aut}(\mathscr{H})$ *acts faithfully on the set* $\mathscr{H}_r$.

(2) *The residues of* $\mathscr{H}$ *at flags of cotype* $\{2, i\}$ *are generalized digons for any* $i=3, ..., r$ *except one, and the subgroup of* $\mathrm{Aut}(\mathscr{H})$ *fixing an element* $x \in \mathscr{H}_1$ *acts faithfully on the set of elements of* $\mathscr{H}_1$ *collinear with* x.

Proof. Let K_P be the kernel of the action of the stabilizer G_P of $P \in \mathscr{G}_0$ on the residue $\mathscr{G}_P$ at P. For any element $Q \in \mathscr{G}_0$ collinear with P, the image $K_Q K_P / K_P$ is a subgroup of $\mathrm{Aut}(\mathscr{G}_P)=\mathrm{Aut}(\mathscr{H})$. Assume that condition (1) holds. Since Q is incident with any $\pi \in \mathscr{G}_r$ by Lemma 1(2), K_Q acts trivially on $\mathscr{G}_r(P)=\mathscr{H}_r$. By our assumption, $K_Q K_P / K_P$ is the trivial group, and so we have $K_P = K_Q$. By Lemma 1(1), this implies that K_P fixes all points. Hence, it fixes all varieties. Thus $K_P = 1$.

Assume that condition (2) holds. Let l be an element of $\mathscr{G}_1(P)=\mathscr{H}_1$ incident with Q, and let m be an element of $\mathscr{H}_1$ collinear with l. Then there is an element $u \in \mathscr{G}_2(P)=\mathscr{H}_2$ incident with both l and m. Since $(\mathscr{G}_0(u), \mathscr{G}_1(u); *)$ is a circle geometry by our assumption, l and m can be identified with pairs $\{P, Q\}$ and $\{P, R\}$ of elements of $\mathscr{G}_0(u)$, respectively. Then there is a unique element $n \in \mathscr{G}_1(u)$ incident with Q and R. Since K_Q fixes n and so R, it also fixes m. Thus the image of K_Q in G_P/K_P fixes every element of $\mathscr{G}_1(P)$ collinear with l, and therefore it is trivial by our assumption. Thus we have $K_P = 1$ by Lemma 2.1(1). ▮

3. PROOFS OF THE THEOREMS

Proof of Theorem 1.1. Let $\mathscr{G}=(\mathscr{G}_0, \mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3; *)$ be a circular extension of the dual $\mathscr{H}$ of the sporadic A_7-geometry, admitting a flag-transitive group G, and let (P, l, u, π) be a maximal flag of $\mathscr{G}$. We first remark some facts obtained by recalling a usual presentation of the sporadic A_7-geometry, the dual of $\mathscr{H}$. We may identify $\mathscr{H}_3$ and $\mathscr{H}_2$ with the sets $\{1, ..., 7\}$ of 7 letters and $\binom{7}{3}=35$ triples of 7 letters, respectively. There are 30 different presentations of the projective plane $\mathbf{PG}(3, 2)$ on the 7 letters, on which the alternating group A_7 has two orbits of length 15. The set $\mathscr{H}_1$

corresponds to one orbit. The incidence is given by natural inclusion. Note that the geometry $\mathscr{H}$ is flat and that $\mathrm{Aut}(\mathscr{H}) = M(\mathscr{H}) \cong A_7$ acts faithfully and primitively on both $\mathscr{H}_1$ and $\mathscr{H}_3$.

By Lemma 2.1(2), the geometry $\mathscr{G}$ is flat. Since $\mathrm{Aut}(\mathscr{H}) \cong A_7$ acts primitively on the set $\mathscr{H}_1$, $\mathscr{G}$ satisfies the (LL) property by Lemma 2.2. It follows from Lemma 2.3 (1) that we have $G_P \cong A_7$ for a point $P \in \mathscr{G}_0$. In particular, $|G| = 2^4\,|A_7|$.

Since the residue $\mathscr{G}_{\{P,\pi\}}$ is a classical generalized quadrangle $W(2)$ for the symplectic group $S_4(2)$, it follows from the classification of flag-transitive classical $c.C_2$-geometries (e.g., [4, Th. 4]) that $\mathscr{G}_\pi$ is isomorphic either to an affine polar space related to the polar space for $S_6(2)$ or to a geometry $\mathscr{A}$ on 16 points (the standard quotient of the affine polar space for 2^5A_6). Suppose $\mathscr{G}_\pi$ is an affine polar space. Since affine polar spaces satisfy the intersection property and the (BH) property, $\mathscr{G}_\pi$ is uniquely determined by its collinearity graph ([4, 2.1], especially the comment on p. 256). Thus $\mathscr{G}_2(\pi)$ is uniquely determined by $\mathscr{G}_i = \mathscr{G}_i(\pi)$ $(i = 0, 1)$. Since G acts on $\mathscr{G}_i$ $(i = 0, 1)$, this implies that G also acts on $\mathscr{G}_2(\pi)$. By the transitivity of G on $\mathscr{G}_2$, we have $\mathscr{G}_2 = \mathscr{G}_2(\pi)$. Then every $\pi \in \mathscr{G}_3$ is incident with every $x \in \mathscr{G}_2$, which cannot occur in the sporadic A_7-geometry $\mathscr{G}_P$. Thus we have $\mathscr{G}_\pi \cong \mathscr{A}$, in which the collinearity graph is a complete graph (so it does not uniquely determine $\mathscr{A}$) and $M(\mathscr{A}) \cong 2^4 : A_6$ with the stabilizers of elements of $\mathscr{G}_0(\pi)$ isomorphic to A_6. In particular, $O_2(G_\pi) \cap G_P = 1$ and therefore $O_2(G_\pi) \cong 2^4$ coincides with the kernel K of the action of G on the set $\mathscr{G}_3$ of 7 dual points. Thus G is a non-trivial split extension of $G_P \cong A_7$ by $K \cong 2^4$. Since there is a unique conjugacy class of subgroups isomorphic to A_7 in the group $A_8 \cong GL_4(2)$, the action of G_P on K is uniquely determined. Thus the structure of G is uniquely determined.

Since $\mathscr{G}$ is a flat geometry satisfying the (LL) property, we have $|\mathscr{G}_0| = |\mathscr{G}_0(\pi)| = 16$, $|\mathscr{G}_1| = |\mathscr{G}_0(P)| \cdot |\mathscr{G}_0|/2 = \binom{16}{2}$ and $|\mathscr{G}_3| = |\mathscr{G}_3(P)| = 7$, and the incidence on $\mathscr{G}_0 \cup \mathscr{G}_1 \cup \mathscr{G}_3$ is uniquely determined. Note that this is equivalent to say that G and the parabolic subgroups G_P, G_l, G_π together with their intersections are uniquely determined. (We already have G_P and G_π. The group G_l is determined as $N_G(G_{P,Q}) = C_K(G_{P,Q}) \times G_{P,Q} \cong 2 \times L_3(2)$, where we identify l with $\mathscr{G}_0(l) = \{P, Q\}$ by the (LL) property.) In order to determine the incidence between $\mathscr{G}_2$ and other varieties, we show that G and the parabolic subgroups G_u, $G_{u,x}$ $(x = P, l, \pi)$ are uniquely determined. Then $\mathscr{G}$ can be uniquely reconstructed as a coset geometry from G and its parabolic subgroups.

The parabolics $G_{u,x}$ $(x = P, l, \pi)$ are easy to determine in the parabolic G_x. Observing the dual of the sporadic A_7-geometry $\mathscr{G}_P$ and $G_P \cong A_7$, we see that $G_{P,u} \cong (3 \times A_4).2$ induces S_3 on the set $\mathscr{G}_1(P, u)$, which corresponds to the set $\mathscr{G}_0(u) - \{P\}$ of three points. Thus G_u induces S_4 on $\mathscr{G}_0(u)$ with the kernel $K_u = G'_{P,l,u} \cong A_4$. Since $G_{P,u}$ is a maximal subgroup of G_P

normalizing K_u, it follows from the fact that G is a split extension of G_P by K that we have $N_G(K_u) = C_K(K_u)\, G_{P,u}$. Since $G_u \subset N_G(K_u)$ and $|G : G_u| = |\mathscr{G}_2| = 35\, |\mathscr{G}_0|/4 = 140$, we have $G_u = N_G(K_u)$, which is uniquely determined by G and the parabolics in G_P. Hence G and all parabolics of G are uniquely determined, which implies the uniqueness of $\mathscr{G}$. ∎

Proof of Theorem 1.2. Let $\mathscr{G} = (\mathscr{G}_0, \mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3, \mathscr{G}_4; *)$ be a circular extension of the dual of the Neumaier geometry, and let (P, l, u, m, Q) be a maximal flag of $\mathscr{G}$. Then $\mathrm{Aut}(\mathscr{G}_P) = M(\mathscr{H}_P) \cong A_8$ acts faithfully and primitively both on $\mathscr{G}_1(P)$ and $\mathscr{G}_4(P)$ [7, Sect. 3]. Thus by Lemma 2.2 and 2.3 (1), $\mathscr{G}$ satisfies the (LL) property and the stabilizer G_P acts faithfully on the residue $\mathscr{G}_P$. In particular, $G_P \cong A_8$.

Note that the dual $\mathscr{G}^*$ of $\mathscr{G}$ is also a circular extension of the dual of the unique flag-transitive geometry in Theorem 1.1. Then $\mathrm{Aut}(\mathscr{G}_Q) = M(\mathscr{G}_Q) \cong 2^4 : A_7$ acts faithfully and primitively both on $\mathscr{G}_0(Q)$ and $\mathscr{G}_3(Q)$. Thus by Lemma 2.2 and 2.3 (1), $\mathscr{G}^*$ satisfy the (LL) property and the stabilizer G_Q acts faithfully on the residue $\mathscr{G}_Q$. In particular, $G_Q \cong 2^4 : A_7$.

By applying Lemma 2.1(2) to $\mathscr{G}$ and $\mathscr{G}^*$, all elements of $\mathscr{G}_0 \cup \mathscr{G}_1$ are incident with all elements of $\mathscr{G}_3 \cup \mathscr{G}_4$. In particular, $|\mathscr{G}_0| = |\mathscr{G}_0(Q)| = 16$ and $|\mathscr{G}_4| = |\mathscr{G}_4(P)| = 8$. Since $\mathscr{G}$ and $\mathscr{G}^*$ satisfy the (LL) property, we have $|\mathscr{G}_1| = \binom{16}{2} = 120$ and $|\mathscr{G}_1| = \binom{8}{2} = 28$. Moreover, the incidence on $\mathscr{G}_0 \cup \mathscr{G}_1$ and $\mathscr{G}_3 \cup \mathscr{G}_4$ is uniquely determined. In order to determine $\mathscr{G}_2$ and the remaining incidence, we have to determine G and the stabilizers G_u, $G_{x,u}$ $(x = P, l, m, Q)$.

Let K be the kernel of the action of G on $\mathscr{G}_4$. Then G/K is a subgroup of S_8. Since S_8 does not contain any subgroup of shape $2^n A_7$ $(n > 1)$, we have $O_2(G_Q) = K$. Since $|G| = 8\, |G_Q|$, G is a non-trivial split extension of $G_P \cong A_8$ by $K \cong 2^4$. Since $A_8 \cong GL_4(2)$, the structure of G is uniquely determined. Now, observing $G_P \cong A_8$, the kernel K_u of G_u on the set $\mathscr{G}_0(u) = \mathscr{G}_0(u, Q)$ of four points coincides with $G'_{P,l,u} \cong A_4$. Then $N_G(K_u) = C_K(K_u)\, N_{G_P}(K_u)$, and $G_{P,u}$ is a subgroup of $N_{G_P}(K_u) \cong (A_4 \times A_4)\, 2$ of index 2. Since $|\mathscr{G}_2| = |\mathscr{G}_0| \cdot |\mathscr{G}_2(P)|/4 = 280 = 2|G : N_G(K_u)|$, the stabilizer G_u is uniquely determined as $C_K(K_u)\, G_{P,u}$. The stabilizers $G_{x,u}$ $(x = P, l, m, Q)$ are determined by observing G_x. Thus the geometry $\mathscr{G}$ is uniquely determined. ∎

Proof of Theorem 1.3. Suppose there is a circular extension $\mathscr{G}$ of the geometry $\mathscr{H}$ in Theorem 1.1, admitting a flag-transitive group G, and let (P, l, u, α, π) be a maximal flag of $\mathscr{G}$. Since G_α acts flag-transitively on a geometry $(\mathscr{G}_0(\alpha), \mathscr{G}_1(\alpha), \mathscr{G}_2(\alpha); *)$ belonging to the diagram $c.C$, G_α acts triply transitive on the set $\mathscr{G}_0(\alpha)$. Since the stabilizer of l in $\mathrm{Aut}(\mathscr{H}) \cong 2^4 : A_7$ acts faithfully on the set of elements of $\mathscr{H}_1$ collinear with l, the stabilizer G_P acts faitfully on the residue $\mathscr{G}_P$ by Lemma 2.3 (2). Observing $\mathscr{G}_P$, we may verify that $G_{P,\alpha} \cong 2^2(3 \times A_4)\, 2$ induces S_4 on the set $\mathscr{G}_1(P, \alpha)$ of four elements. Thus G_α induces S_5 on the set $\mathscr{G}_0(\alpha)$ of five points. Note that the kernel K_α of the action of G_α on $\mathscr{G}_0(\alpha)$ coincides with $G'_{P,l,u,\alpha} \cong A_4$ and that $\mathrm{Aut}(A_4)$ does not contain A_5. Thus there is a subgroup $A \cong A_5$ of G_α centralizing K_π with $G_\alpha = (K_\pi \times A)\,2$. Since there is an element of order 3 of $G_{P,l,\alpha} \cong (3 \times A_4)\,2$ inducing a 3-cycle on $\mathscr{G}_0(\alpha)$, we conclude that $A \cap G_{P,l} \neq 1$. Let K be the kernel of the action of G on $\mathscr{G}_4$. Since $\mathscr{G}$ is flat by Lemma 2.1(2), $|\mathscr{G}_4| = |\mathscr{G}_4(P)| = 8$ and so G/K is a subgroup of S_8. By the proof of Theorem 1.1, we know $K \cap G_{P,l} = 1$. Since $A \cong A_5$ and $A \cap G_{P,l} \neq 1$, the image of A in G/K is isomorphic to A. However, since $K_\pi \cap K = 1$, this implies that G/K contains a subgroup isomorphic to $A_4 \times A_5$, which is a contradiction. Hence there is no flag-transitive circular extension of the geometry $\mathscr{H}$ in Theorem 1.1. Since $\mathscr{H}$ is the residue of the geometry $\mathscr{K}$ in Theorem 1.2 at $P \in \mathscr{K}_0$, there is no flag-transitive circular extension of $\mathscr{K}$. ∎

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