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The Modular Hecke Algebra of a Sylow p-Subgroup

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DEDICATED TO SANDY GREEN ON THE OCCASION OF HIS 60TH BIRTHDAY

INTRODUCTION

Modules became a powerful tool in modular representation theory when vertex theory and Green correspondence was developed by Sandy Green.

If a module is induced from a subgroup H of a finite group G—over a suitable ring such that Krull–Schmidt holds—the indecomposable summands have vertices contained in subgroups of the form $H \cap H^\gamma$. However, we do not know exactly which γ's to use, nor do we know precisely which subgroups to pick. In fact, we do not even have useful criteria which allow us to deduce when the induction of an indecomposable module decomposes. Here is a typical question we would like to answer: Let R be a complete discrete valuation ring (including a field) with maximal ideal (π) (which may be 0) such that $R/(\pi)$ is of characteristic p.

(Q) For which indecomposable nonprojective $R[Q]$-modules M, where $Q \in \mathrm{Syl}_p(G)$, does $M\uparrow_Q^G$ have a projective summand?

To answer questions in general about $M\uparrow_H^G$ it is useful to study $E := (M\uparrow_H^G, M\uparrow_H^G)^G$, the endomorphism ring of $M\uparrow_H^G$. In the present note, we give the answer in the case where $M = F$, the trivial $F[G]$-module, F a field of characteristic p, in which case E is called the modular Hecke algebra (perhaps Schur algebra would be better). The key to our result will be a study of a natural homomorphism

$$Z(R[G]) \xrightarrow{\Phi} \mathscr{H}_R(H).$$

The permutation module $O\uparrow_H^G$ is isomorphic to the right ideal in $O[G]$ generated by $\overline{H}$. The modular Hecke algebra $\mathscr{H}_O(H)$ associated with H is the endomorphism algebra of $O\uparrow_H^G$. Its rank equals the number of double

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H-cosets in G and the Schur basis $\{S_\gamma \mid \gamma \in H\backslash G/H\}$ consists of the elements S_γ given by $S_\gamma(\overline{H}) = \overline{H\gamma H}$. For details see Landrock [2, p. 178] and [3].

As we are considering a permutation module we may as well carry our first considerations out over $\mathbb{Z}$.

LEMMA 1.1. *Set* ${}^H\mathbb{Z}[G]^H$ *equal to* $\operatorname{span}_{\mathbb{Z}}\{\overline{H\gamma H} \mid \gamma \in G\}$. *Then*

$$ {}^H\mathbb{Z}[G]^H = \overline{H}\mathbb{Z}[G] \cap \mathbb{Z}[G]\overline{H}. $$

Proof. Recall that $\overline{H\gamma H} = (\overline{H/H \cap H^{\gamma^{-1}}})\gamma \overline{H} = \overline{H}\gamma(\overline{H \cap H^{\gamma}\backslash H})$ from which $\subseteq$ follows.

Next let $\overline{H}a \in \mathbb{Z}[G]\overline{H}$, say $\overline{H}a = \sum_{g \in H\backslash G} \alpha_g \overline{H}g$. Then $\alpha_g = \alpha_{gh}$ for all $h \in H$, as $\overline{H}ah = \overline{H}a$ by assumption. Hence all elements in an H-orbit on $\{\overline{H}g\}$ occur with the same coefficient and $\supseteq$ follows.

There is a natural map $\mathscr{H}_{\mathbb{Z}}(H) \xrightarrow{\Gamma} {}^H\mathbb{Z}[G]^H \subseteq \mathbb{Z}[G]^H$ defined on the Schur basis of $\mathscr{H}_{\mathbb{Z}}(H)$ by

$$ \Gamma(S_\gamma) = \overline{H\gamma H} = S_\gamma(\overline{H}) $$

and extending linearily. By Lemma 1.1, this sets up a $\mathbb{Z}$-isomorphism

$$ \mathscr{H}_{\mathbb{Z}}(H) \simeq {}^H\mathbb{Z}[G]^H. $$

We also have a ring homomorphism

$$ \Phi\colon \mathbb{Z}[G]^H \to \mathscr{H}_{\mathbb{Z}}(H) $$

given by $\Phi(a) = a\overline{H}$. Note that $\Phi \circ \Gamma$ is just multiplication by $|H|$. (For a more general homomorphism, see [2, p. 178]). Of special interest is the restriction of Φ to $Z(\mathbb{Z}[G])$:

Let $\mathscr{K}_1, \ldots, \mathscr{K}_k$ denote the conjugacy classes of G and choose $x_i \in \mathscr{K}_i$. Set

$$ \Phi(\overline{\mathscr{K}_i}) = \sum \kappa_{i\gamma} S_\gamma. $$

For $x \in G$, let $\overline{x} = \operatorname{Tr}^H_{C_H(x)}(x) = \sum_{g \in C_H(x)\backslash H} g^{-1}xg$. Then

$$ \begin{aligned} \overline{x}\,\overline{H} &= \overline{H/C_H(x)}\, x\overline{H} \\ &= |H \cap H^{x^{-1}} : C_H(x)|\, \overline{H/H \cap H^{x^{-1}}}\, x\overline{H} \\ &= |H \cap H^{x^{-1}} : C_H(x)|\, \overline{HxH}. \end{aligned} $$

Thus

$$ \kappa_{i\gamma} = \sum_{H\text{-orbits in } \mathscr{K}_i} |H \cap H^{x^{-1}} : C_H(x)|. $$

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We use the same notation as above and let $Q \in \mathrm{Syl}_p(G)$. Furthermore, we replace $\mathbb{Z}$ by a field F of characteristic p. Thus we have already seen

LEMMA 2.1. *Let $\mathscr{X}_i = \bigcup X_j$, where the intersection is over all Q-orbits. Then*

$$\overline{X_jQ} = \begin{cases} \overline{Q\gamma Q} & \text{if there exists } g_j \in X_j \cap \gamma Q \text{ such that } Q \cap Q^{g_j} = C_Q(g_j) \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\kappa_{i\gamma}$ equals the number of Q-orbits in $\mathscr{X}_i \cap Q\gamma Q$ with a representative g such that $Q \cap Q^g = C_Q(g)$.

THEOREM 2.2. *Assume there exists $x \in \mathscr{X}_i \cap \gamma Q$ such that $D := C_Q(x) = Q \cap Q^x$. Then*

$$\kappa_{i\gamma} \equiv |\mathscr{X}_i \cap xC_Q(D)| \mod p.$$

Otherwise, $\kappa_{i\gamma} = 0 \mod p$.

Proof. If $\kappa_{i\gamma} \not\equiv 0$, there exists x as above, where $QxQ = Q\gamma Q$. Now each Q-orbit in QxQ contains an element of the form $y = xz$, $z \in Q$. Assume $\overline{y}\,\overline{Q} \neq 0$. Then

$$Q \cap Q^y = Q \cap Q^{xz} = D^z \geqslant C_Q(y)$$

so equality, which happens if and only if $\overline{y}\,\overline{Q} \neq 0$, implies

$$D = C_Q(y)^{z^{-1}} = C_Q(zx)$$

and thus $z \in C_Q(D)$.

Conversely, if z_1, $z_2 \in C_Q(D)$ and xz_1, $xz_2 \in \mathscr{X}_i$, we claim they belong to different Q-orbits. Indeed, if $q^{-1}xz_1q = xz_2$ for some $q \in Q$,

$$x^{-1}q^{-1}x = z_2q^{-1}z_1^{-1} \in Q \cap Q^x = D.$$

Thus $q \in D$ and $q^{-1}xz_1q = xz_1 = xz_2$.

COROLLARY. *If i runs through the set $I_D = \{j \mid \mathscr{X}_j$ is p-regular with defect group $D\}$, then*

$$N_D = \{\kappa_{i\gamma}\}_{i \in I_D, \gamma \in Q \backslash G/Q}$$

is Robinson's matrix (see Robinson [4]).

Theorem 2.2 enables us to prove the result promised in the Introduction.

THEOREM 2.3. *Let $x_1,\ldots, x_h$ be representatives of the p-regular conjugacy classes, and denote by $\mathcal{P}_i$ the set of elements in G whose p'-part is conjugate to x_i. Then the following are equivalent*

(i) *$F\uparrow_Q^G$ is projective-free.*

(ii) *For any $\gamma \in G$, any $i = 1,\ldots, h$,*

$$\sum_{j \in J_i} \kappa_{j\gamma} \equiv 0 \mod p,$$

where $j \in J_i$ if and only if $\mathcal{K}_j \subseteq \mathcal{P}_i$.

In other words, $F\uparrow_Q^G$ is projective-free if and only if the number of Q-orbits in $Q\gamma Q$ consisting of elements in $\mathcal{P}_i$ with the property that if x belongs to such an orbit, then $C_Q(x) = Q \cap Q^x$, is a multiple of p. It is easy to see that this number is always a multiple of p if $Q \cap Q^\gamma \neq 1$. This fits well with vertex theory which states that a *necessary* condition for a projective summand is that $Q \cap Q^\gamma = 1$ for some γ. Note also that although our result is phrased in terms of a property of Robinson's matrix, we do not use any of his results. We derive the matrix along different paths.

The proof is based on the following observations: Let $\bar{\beta}_1,\ldots, \bar{\beta}_h$ be the simple modular characters (i.e., the reduction modulo (π) of the irreducible Brauer characters). Set $f_i = \sum_{x \in G} \bar{\beta}_i(x^{-1})\, x$. Then f_i belongs to Reynold's ideal, the intersection of the center of $F[G]$ and the socle, and $f_i F[G]$ is the sum of all right ideals isomorphic to the simple module E_i corresponding to β_i.

LEMMA 2.4. *Let $H \leqslant G$. Then P_{E_i}, the projective cover of E_i, is a direct summand of $F\uparrow_H^G$ if and only $f_i \overline{H} \neq 0$.*

Proof. Consider the homomorphism

$$F[G] \to \overline{H} F[G]$$

given by $a \to \overline{H} a$. Then $f_i \overline{H} \neq 0$ if and only if $f_i F[G]$ is not in the kernel, which in turn happens if and only if P_{E_i} is a direct summand.

Remark. As pointed out in the introduction, $\overline{H}$ belongs to the radical of $F[G]$ if ans only if $\overline{H} F[G]$ is projective-free. Lemma 2.4 is slightly stronger.

COROLLARY. *Let $Z_R = \operatorname{span}_F \{f_i \mid i = 1,\ldots, h\}$, Reynold's ideal. Then $\dim_F \Phi(Z_R)$ equals the number of isomorphism classes of indecomposable projective summands of $F\uparrow_Q^G$.*

Proof. As $\{\Phi(f_i) | \Phi(f_i) \neq 0\}$ is a linearily independent set.

The proof of Theorem 2.3 is now quite trivial: $F\uparrow_Q^G$ is projective-free if and only if $Z_R \overline{Q} = 0$, as we have just seen. As $\overline{\mathscr{P}_i} \in Z_R$, and in fact Z_R is spanned by the $\overline{\mathscr{P}_i}$'s, $i = 1, \ldots, h$, Theorem 2.3 follows.

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