Transposition Hypergroups: Noncommutative Join Spaces

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The study of various kinds of algebraic hypergroups is unified in the theory of transposition hypergroups. The hyperoperation in any hypergroup has two inverses given by \( a/b = \{ x \mid a \in xb \} \) and \( b\setminus a = \{ x \mid a \in bx \} \). A transposition hypergroup is a hypergroup where transposition, \( b\setminus a \cap c/d \neq \emptyset \) implies \( ad \cap bc \neq \emptyset \), holds. The algebra of transposition hypergroups is developed. Closed subhypergroups \( N \) that are reflexive, \( a\setminus N = N/a \), have distinguished structural significance. The quotient space of a transposition hypergroup modulo a reflexive closed subhypergroup forms a transposition hypergroup that is a polygroup. Then generalizations of the isomorphism theorems and the Jordan–Hölder theorem of group theory are obtained.

1. INTRODUCTION

A transposition hypergroup, defined in the next section, is a hypergroup that satisfies a postulated property of transposition. Neither commutativity nor the existence of identity elements is assumed. Many well-known hypergroups such as join spaces, weak cogroups, double coset spaces, polygroups, and canonical hypergroups, as well as ordinary groups are all transposition hypergroups. In addition a new geometrically motivated noncommutative hyperstructure is introduced and shown to be a transposition hypergroup. The main theme of this work is the study of quotient spaces for transposition hypergroups. A treatment of quotient spaces is given that generalizes the isomorphism theorems of group theory through the theorem of Jordan–Hölder. The treatment admits quotients by subhy-

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pergroups that are closed and reflexive, a pair of conditions that is shown to be weaker than the usually assumed pair, invertibility and normality.

2. THE ALGEBRAIC STRUCTURE

Let \( H \) be a set, elements of which will be denoted \( a, b, \ldots \), and subsets of which will be denoted \( A, B, \ldots \). Let \( \mathcal{P}(H)^\ast \) be the family of nonempty subsets of \( H \) and \( \cdot \) a hyperoperation or join operation in \( H \), that is, \( \cdot \) is a function from \( H \times H \) into \( \mathcal{P}(H)^\ast \). If \((a, b) \in H \times H\), its image under \( \cdot \) in \( \mathcal{P}(H)^\ast \) is denoted by \( a \cdot b \) or \( ab \). The join operation is extended to subsets of \( H \) in a natural way, so that \( A \cdot B \) or \( AB \) is given by \( \{ a \in H \mid a \in A \land b \in B \} \). The relational notation \( A \approx B \) (read \( A \) meets \( B \)) is used to assert that the sets \( A \) and \( B \) have an element in common, that is, that \( A \cap B \neq \emptyset \). The notation \( aA \) is used for \( \{ a \in H \mid a \in A \} \) and \( Aa \) for \( \{ a \in H \mid a \in A \} \). Generally, the singleton \( \{ a \} \) is identified with its member \( a \). In fact, \( a \approx A \) or \( A \approx a \) is used as a convenient substitute notation for \( a \in A \).

In \( H \) two hypercompositions right extension / and left extension \( \backslash \) each an inverse to \( \cdot \) are defined by

\[
a/b = \{ x \mid a \in xb \} \quad \text{and} \quad b \backslash a = \{ x \mid a \in bx \}.
\]

Hence \( x \approx a/b \) if and only if \( a \approx xb \), and \( x \approx b \backslash a \) if and only if \( a \approx bx \). For sets each of \( A/B \) and \( B \backslash A \) is defined by \( A/B = \bigcup \{ a/b \mid a \in A, b \in B \} \) and \( B \backslash A = \bigcup \{ b \backslash a \mid a \in A, b \in B \} \). Then \( A \approx B/C \) if and only if \( B \approx AC \), and \( A \approx C \backslash B \) if and only if \( B \approx CA \). Note also that \( A \subseteq B \) and \( C \subseteq D \) imply that \( AC \subseteq BD \), that \( A/C \subseteq B/D \), and that \( C \backslash A \subseteq D \backslash B \).

**Definition.** A hypergroup is a structure \((H, \cdot)\) that satisfies two axioms,

\[
\begin{align*}
(\text{Reproduction}) & \quad aH = Ha = H \text{ for all } a \in H; \\
(\text{Associativity}) & \quad a(bc) = (ab)c \text{ for all } a, b, c \in H.
\end{align*}
\]

A hypergroup is called a transposition hypergroup if it satisfies the axiom,

\[
(\text{Transposition}) \quad b \backslash a \approx c/d \text{ implies } ad \approx bc \text{ for all } a, b, c, d \in H.
\]

Observe the following elementary properties of a hypergroup \( H \). The reproductive condition implies that \( a/b \) and \( b \backslash a \) are each nonempty, and is indeed equivalent to this condition. The associative condition implies associativity for subsets of \( H \), that is, \( A(BC) = (AB)C \). Correspondingly, the transposition axiom yields \( B \backslash A \approx C/D \) implies \( AD \approx BC \) for sets in a transposition hypergroup.
3. TRANSPOSITION HYPERGROUPS

Various kinds of hypergroups that have been introduced and studied can be shown to be transposition hypergroups. Some of them are considered here in Subsections 1–6. A new geometrically motivated noncommutative example is the subject of Subsection 7.

1. Join Spaces. A join space is a commutative \((ab = ba)\) transposition hypergroup. In such a hypergroup, the hyperoperations / and \(\setminus\) obviously coincide. Join spaces play a unifying role in the study of classical geometries. Each of the geometries, descriptive, spherical, and projective, can be formulated as kinds of join spaces. See [14, 15, 9, 1] for the theory of join spaces.

2. Weak Cogroups. A (right-sided) weak cogroup or hypergroup of type \(C\) is a hypergroup that contains a right scalar identity \(e\), meaning that \(ae = a\) for each \(a\), and in which \(ab = ac\) implies \(eb = ec\) for every \(a, b, c\). In a weak cogroup, \(e\) is unique, \(a = ea\), and \(a = ab\) implies \(b = e\). Obviously, the family of all sets of the form \(ea\) is a partition of the weak cogroup. Furthermore, one has that \(ea = eb\) if and only if \(e = eb\) and that \(a \setminus e = a/e = eb\) for any \(b\) such that \(b = a \setminus e\). From here it follows that \(a/b = a(e/b)\) and that \(b/a \subseteq (b \setminus e)a\). Hence it can be seen that a weak cogroup is a transposition hypergroup as follows. Suppose \(b \setminus a = c/d\). Then \(a = b(c/d) = b(c(e/d)) = (bc)(e/d) = bc/d\), from which \(ad = bc\) is immediate.

A cogroup is a weak cogroup in which the cardinalities of \(ac\) and \(bc\) are equal for every \(a, b, c\). The principal example is the family of all left cosets of a subgroup \(H\) of a group under the hyperoperation \(aH \cdot bH = \{xH \mid x \in aHb\}\). See [6, 18, 17, 7] for the study of weak cogroups.

3. Double Coset Spaces. Let \(H\) and \(K\) be subgroups of a group. The family of all double cosets of \(H\) and \(K\) under \(HaK \cdot HbK = \{HxK \mid x \in aKb\}\) forms a hypergroup in which, it is easily seen that, \(HaK/HbK = \{HxK \mid x \in aKb^{-1}H\}\) and \(HbK \setminus HaK = \{HxK \mid x \in Kb^{-1}Ha\}\). Then suppose \(HbK \setminus HaK = HcK/HdK\). Hence in the group, \(HbK^{-1}HaK \approx HcK^{-1}HK\), or equivalently by introducing redundant factors, \(HbK^{-1}HbHcK = HcKd^{-1}HK\). Then by transposing factors in the group, \(HaKHdK \approx HbKHcK\) results, so that back in the hypergroup, \(HaK \cdot HdK = HbK \cdot HcK\). Therefore transposition holds in a double coset space. An investigation of double coset spaces is begun in [19].

4. Polygroups. A polygroup or quasicanonical hypergroup is a hypergroup containing a scalar identity, that is, there exists \(e\) (that is necessarily unique) such that \(ea = ae = a\) for each \(a\). Furthermore, for each \(b\) there exists \(b^{-1}\) such that for each \(a\) each of \(a/b = ab^{-1}\) and \(b \setminus a = b^{-1}a\) holds. For each \(a\) it becomes an elementary consequence that \(a^{-1}\) is
unique and that \((a^{-1})^{-1} = a\). Suppose then that \(b \setminus a \approx c \setminus d\). Then 
\(a \approx b(c \setminus d) = bcd^{-1}\). Thus \(a/d^{-1} = bc\) which yields \(ad = bc\). Therefore a polygroup is a transposition hypergroup. Corollary 3 in Section 6 will characterize a polygroup as a transposition hypergroup with a scalar identity.

A double coset hypergroup where \(H = K\) is a polygroup. Polygroups are studied in [2, 3] where connections with color schemes, relation algebras, and finite permutation groups, as well as with weak cogroups, are brought out. In [8] polygroups appear implicitly as Pasch geometries. Given a Pasch geometry \((A, \Delta, e)\), then \(A\) becomes a polygroup with scalar identity \(e\) and \(a^{-1} = a^\#\) by defining \(a \cdot b = \{(a, b, x^\#) \in \Delta\). Conversely and compatibly, given a polygroup \(H\) with \(e\) the scalar identity, then \((H, \Delta, e)\) becomes a Pasch geometry for \(\Delta = \{(a, b, c) \mid e \in abc\}\). See [12], where the term quasicanonical hypergroup is used, for other aspects of the theory of polygroups.

5. Canonical Hypergroups. A canonical hypergroup is a commutative polygroup. A canonical hypergroup may also be characterized as a join space with a scalar identity. Examples of canonical hypergroups are the spherical and projective join spaces mentioned above. The “additive” structure of a hyperfield is the archetype from which the study of canonical hypergroups began in [13]. In [16], where the term “reversible abelian hypergroup” is used, the canonical hypergroup of conjugacy classes of a group and the canonical hypergroup of irreducible complex characters for a group representation are studied.

6. Groups. Note that a binary operation on a set may be considered as a join operation for which the join of any pair of elements is a singleton set. Hence it is easily seen that groups are polygroups and so transposition hypergroups.

7. Pivot Products. Transposition hypergroups that are idempotent \((aa = a)\) and also satisfy \(ac \subseteq abc\) for every \(a, b, c\) where \(a \neq c\) are considered. A Euclidean line as a descriptive join space, that is, where join is defined for two distinct points as the open segment between them and for a point with itself as itself, is such a hypergroup. Let \(H_1\) and \(H_2\) be any two such hypergroups and let \(H = H_1 \times H_2\). For \(x \in H\) the notation \(x = x_1 \times x_2\) where \(x_1 \in H_1\) and \(x_2 \in H_2\) is used. Join \(\circ\) is defined in \(H\) by

\[x \circ y = x_1 y_1 \times x_2 \cup y_1 \times x_2 \cup y_1 \times x_2 y_2,\]

where the joins \(x_1 y_1\) and \(x_2 y_2\) are in \(H_1\) and \(H_2\), respectively. The hyperstructure \((H, \circ)\) is, as is shown in Section 12, an idempotent transposition hypergroup that satisfies the stronger than above, \(a \circ c \subseteq a \circ b \circ c\) for every \(a, b, c\). It is called the (left) pivot product of \(H_1\) and \(H_2\).
Note that
\[ y \ast x = y_1x_1 \times y_2 \cup x_1 \times y_2 \cup x_1 \times y_2x_2. \]

Thus if \( x_1 \neq y_1 \) and \( H \) is assumed commutative then \( y_1 \times x_2y_2 \subseteq y_1x_1 \times y_2 \)
must hold for any \( x_2 \) and \( y_2 \). Hence \( x_2y_2 = y_2 \) must hold in \( H \). But then
reproduction \( H_2y_2 = H_2 \) yields that \( H_2 = y_2 \). Therefore \( H \) is noncommutative if \( H_1 \) and \( H_2 \) each has at least two members.

Of course, a second hyperoperation \( * \) can be defined on \( H = H_1 \times H_2 \)
giving the right pivot product of \( H_1 \) and \( H_2 \). The hyperoperation \( * \) is the
symmetric counterpart of \( \ast \) given by
\[ x \ast y = x_1 \times x_2y_2 \cup x_1 \times y_2 \cup x_1y_1 \times y_2. \]

Compare \( x \ast y \) and \( y \ast x \).

Lastly, it may be remarked that the process of forming a pivot product
may be repeated with \( H \) as one of the two hypergroups involved.

The study of the algebraic structure of transposition hypergroups is next.
The development given adapts and extends that given for join spaces in
[14] to allow for noncommutativity of join.

4. DUALLITY AND THE ELEMENTARY ALGEBRA

Two statements of the theory of hypergroups are dual statements if each
results from the other by interchanging the order of the join operation \( \cdot \); that is, interchanging any join \( ab \) with the join \( ba \). Observe that each of
the axioms, reproduction and associativity, is self-dual. The extensions / and \( \backslash \) have dual definitions. The transposition axiom, upon reversing the
order of the join operation and necessarily interchanging / and \( \backslash \) to
construct the dual, is seen then to be self-dual. Therefore, a principle of duality holds for the theory of hypergroups and for that of transposition
hypergroups.

Given a theorem, the dual statement, which results from the interchanging
of the order of the join operation \( \cdot \) is also a theorem.

The principle of duality is used throughout.

**Proposition 1.** In any hypergroup,

(a) \((A/B)/C = A/CB\) and (a') \(C \backslash (B \backslash A) = BC \backslash A\);

(b) \((B \backslash A)/C = B \backslash (A/C)\);

(c) \(A \neq \emptyset\) implies \( B \subseteq (A/B) \backslash A\) and (c') \(A \neq \emptyset\) implies \(B \subseteq A/(B \backslash A)\).
Proof. (a) follows from associativity and the sequence of equivalent statements
\[ x \approx (A/B)/C; \quad xC \approx A/B; \quad xCB \approx A; \quad x \approx A/CB. \]
(a’) is the dual of (a).

(b) follows from associativity and the sequence of equivalent statements
\[ x \approx (B\setminus A)/C; \quad xC \approx B\setminus A; \quad Bx \approx A; \quad Bx \approx A/C; \quad x \approx B\setminus (A/C). \]

(c) Let \( x \approx B \). Then \( A/x \subseteq A/B \). The reproductive axiom and \( A \neq \emptyset \) imply \( A/x \neq \emptyset \). Thus \( A/x \approx A/B \). Hence \( A \approx (A/B)x \), so that \( x \approx (A/B)\setminus A \). Therefore (c) holds. (c’) is the dual of (c).

Proposition 2. In a transposition hypergroup,
\begin{enumerate}
  \item[(a)] \( A(B/C) \subseteq AB/C \) and \( (a’)(C\setminus B)A \subseteq C\setminus BA; \)
  \item[(b)] \( A/(B/C) \subseteq AC/B \) and \( (b’)(C\setminus B)\setminus A \subseteq B\setminus CA. \)
\end{enumerate}

Proof. (a) follows from the sequence of implications below in which transposition is used at the second step:
\[ x \approx A(B/C); \quad A\setminus x \approx B/C; \quad xC \approx AB; \quad x \approx AB/C. \]
(a’) is the dual of (a).

(b) follows from transposition and the sequence of implications
\[ x \approx A/(B/C); \quad x(B/C) \approx A; \quad x\setminus A \approx B/C; \quad AC \approx xB; \quad x \approx AC/B. \]
(b’) is the dual of (b).

Propositions 1 and 2 underlie much of the algebra that is to be developed and are used often without citation in what follows.

5. CLOSED SETS

Let \( H \) be a hypergroup. A \textit{subhypergroup} of \( H \) is a subset \( K \) that is itself a hypergroup under the hyperoperation \( \cdot \) restricted to \( K \). Hence it immediately follows that a subset \( K \) of \( H \) is a subhypergroup if and only if \( aK = Ka = K \) for each \( a \in K \). If \( H \) is a transposition hypergroup then a subhypergroup \( K \) is seen to be a transposition hypergroup also, although right extension and left extension in \( K \) must be distinguished from the corresponding extensions in \( H \).
Subsets of $H$ that are closed under both right and left extensions are most important.

**Definition.** A subset $K$ of a hypergroup is a *closed* or linear set if $a, b \in K$ implies $a/b \subseteq K$ and $b \setminus a \subseteq K$.

Note that $K$ is closed if and only if $K/K \subseteq K$ and $K \setminus K \subseteq K$. Each of the concepts, subhypergroup and closed set, is self-dual.

**Proposition 3.** In any hypergroup, if $K$ is closed then $K$ is a subhypergroup.

**Proof.** Let $a \approx K$. That $aK = K$ will be shown. By duality, $Ka = K$ will follow. By Proposition 1,

$$aK \subseteq (K/aK) \setminus K = ((K/K)/a) \setminus K \subseteq (K/a) \setminus K$$

$$\subseteq (K/K) \setminus K \subseteq K \setminus K \subseteq K. $$

To show the reverse inclusion, let $x \approx K$. Then $a \setminus x \subseteq K$, and so $a \setminus x \approx K$. Thus $x \approx aK$ and $K \subseteq aK$ follows. Therefore $aK = K$.

**Proposition 4.** In any hypergroup, if $K$ is closed and $a \approx K$ then

$$a/K = K/a = K = K \setminus a = a \setminus K. $$

**Proof.** Obviously $a/K \subseteq K$ and $K/a \subseteq K$. Let $x \approx K$. Proposition 3 gives $a \approx xK$ and $xa \approx K$. Then $x \approx a/K$ and $x \approx K/a$. Thus $K \subseteq a/K$ and $K \subseteq K/a$. Therefore, $a/K = K/a = K$. That $K \setminus a = a \setminus K = K$ follows by duality.

**Corollary 1.** In any hypergroup, $K$ is closed if and only if $K/K = K \setminus K = K$.

The intersection of any family of closed sets is closed. Hence given a set $A$, the intersection of all closed sets containing $A$ is a closed set containing $A$, in fact, the least closed set containing $A$ or the closed set generated by $A$. The notation $\langle A \rangle$ is used for the closed set generated by $A$ and $\langle A, B \rangle$ is used for $\langle A \cup B \rangle$. For later purposes a special instance of this discussion is recorded.

**Proposition 5.** In any hypergroup, if $M$ and $N$ are closed then $M \cap N$ and $\langle M, N \rangle$ are closed.

Lastly note that in a hypergroup if $N$ is closed, $M \subseteq N$, and $M$ is closed in $N$ then $M$ is closed.
6. SCALAR IDENTITY

Let $H$ be a hypergroup. An element $e$ of $H$ is a scalar identity if $ea = ae = a$ for each $a$. If a scalar identity exists in $H$ then it is unique. If $K$ is a nonempty closed set and a scalar identity $e$ is in $H$, then $e$ is in $K$. For given $a \in K$, then certainly $a \approx ea$, and so $e \approx a/a \subseteq K$.

Given that $H$ has the scalar identity $e$, then $a$ and $b$ are inverse elements if $e \approx ab$ and $e \approx ba$. Scalar identity and inverse elements are each self-dual concepts.

**Proposition 6.** In a transposition hypergroup with a scalar identity $e$, each $a$ has a unique inverse element.

**Proof.** By reproduction, there exist $x$ and $y$ such that $e \approx xa$ and $e \approx ay$. Then $a = x \setminus e$ and $a = e/y$. Hence, $x \setminus e \approx e/y$. By transposition, $y = ey \approx xe = x$. Thus $x = y$ and $x$ is an inverse of $a$. But the above argument applies to any $z$ such that $e \approx za$ or $e \approx az$ and yields $z = x$. Therefore $a$ has a unique inverse.

The inverse of $a$ in a transposition hypergroup with scalar identity $e$ is denoted by $a^{-1}$. Obviously $e^{-1} = e$ and $(a^{-1})^{-1} = a$. For $A$, the set $A^{-1}$ is defined by $A^{-1} = \{a^{-1} | a \in A\}$.

**Corollary 2.** In a transposition hypergroup with a scalar identity $e$, the following are equivalent:

- $e \approx ab$;
- $e \approx ba$;
- $b = a^{-1}$;
- $a = b^{-1}$.

**Proposition 7.** In a transposition hypergroup with a scalar identity $e$,

(a) $a/e = e \setminus a = a$;
(b) $e/a = a \setminus e = a^{-1}$;
(c) $a/b = ab^{-1}$ and (c') $b \setminus a = b^{-1}a$;
(d) $(ab)^{-1} = b^{-1}a^{-1}$.

**Proof.** (a) Here $x = a/e$ if and only if $a \approx xe = x$, that is, $x = a$. The rest follows by duality.

(b) Here $x = e/a$ if and only if $e \approx xa$, which by Corollary 2 is equivalent to $x = a^{-1}$. Duality yields the rest.

(c) By (b), Proposition 2, (a), then (b), and Proposition 2 again.

\[ a/b = a/(e/b^{-1}) \subseteq ab^{-1}/e = ab^{-1} = a(e/b) \subseteq ae/b = a/b, \]

(c') is the dual of (c).

(d) By (b), Proposition 1, (b) again, and then (c),

\[ (ab)^{-1} = e/ab = (e/b)/a = b^{-1}/a = b^{-1}a^{-1}. \]
In a transposition hypergroup with a scalar identity, the results of the proposition hold for sets as well as elements, a fact that was used tacitly in the proof. In particular, $A/B = AB^{-1}$, dually $B \setminus A = B^{-1}A$, and $(AB)^{-1} = B^{-1}A^{-1}$.

From the discussion of polygroups in Section 3.4, the next result is immediate.

**Corollary 3.** A hypergroup is a polygroup if and only if it is a transposition hypergroup with a scalar identity.

It has been observed at the beginning of this section, that a nonempty closed set in any hypergroup with a scalar identity must contain that scalar identity. Therefore by the corollary, a nonempty closed set in a polygroup is itself a polygroup having the same scalar identity as the polygroup. Note also that the scalar identity of a polygroup, considered as a singleton, is a closed set.

### 7. REFLEXIVE CLOSED SETS

In the theory of hypergroups, conditions are imposed on a subhypergroup $K$ of a hypergroup $H$ so that $K$ plays a role in $H$ analogous to the role played by a normal subgroup in a group. The conditions of [5], invertibility (reversibility) and normality (invariance), are more restrictive than is needed when $H$ is a transposition hypergroup. Other conditions, natural in a sense (Proposition 17 below), are set forth here and shown to be weaker conditions.

**Definition.** A subhypergroup $K$ of a hypergroup is **invertible** if $a/b \approx K$ implies $b/a \approx K$, and $b \setminus a \approx K$ implies $a \setminus b \approx K$.

In any hypergroup, a subhypergroup $K$ is known to be closed if $K$ is invertible. To see this, let $x \approx K/K$. Then $K = xK$ and $x \setminus K = K$. Since $K$ is invertible, $K \setminus x \approx K$. Thus, since $K$ is a subhypergroup, $x = KK = K$. Hence $K/K \subseteq K$. Since invertibility is self-dual, then also $K \setminus K \subseteq K$.

**Definition.** A subhypergroup $N$ of a hypergroup is **normal** if $aN \approx Na$.

**Proposition 8.** In any hypergroup, if $N$ is a subhypergroup then $N$ is normal if and only if $N \setminus a = a/N$.

**Proof.** Consider the two pairs (a) and (b) of equivalent statements (i) and (ii):

(a)(i) $x \approx N \setminus y$,  (ii) $y \approx Nx$ and

(b)(i) $x \approx y/N$,  (ii) $y \approx xN$.

Suppose $N$ is normal. Then parts (ii) of (a) and (b) are equivalent. Thus so are parts (i). Therefore, $N \setminus a = a/N$. On the other hand, if $N \setminus a = a/N$ then parts (i) are equivalent, so parts (ii). Thus $N$ is normal.
Both the defining condition for normality and the condition of the proposition hold for sets; that is, if $N$ is normal then $AN = NA$ and $N \setminus A = A/N$.

**Definition.** A subhypergroup $N$ of a hypergroup is reflexive if $a \setminus N = N/\!\!\!/a$.

Note the immediate generalization to sets. If $N$ is reflexive then $A \setminus N = N/A$. Note also that normality and reflexivity are each self-dual notions.

**Proposition 9.** In a transposition hypergroup, a normal closed set is reflexive.

**Proof.** Let $N$ be closed and normal. All is trivial if $N = \emptyset$. Suppose $N$ is nonempty. Then by Proposition 8 to justify the third relation,

$$N/a \subseteq (N/(N/a))\setminus N \subseteq (Na/N)\setminus N = (N \setminus Na)\setminus N \subseteq Na\setminus NN = Na\setminus N = a\setminus (N \setminus N) = a\setminus N.$$ 

Duality yields $a \setminus N \subseteq N/a$. Hence $N$ is reflexive.

Therefore in a transposition hypergroup, normal invertible subhypergroups are closed and reflexive. It is the reflexive closed sets, necessarily subhypergroups (Proposition 3), of a transposition hypergroup that are studied next. First some side remarks are made.

Join spaces provide a setting in which closed sets that are not invertible can be found. For example, in any join geometry of 15 no proper nonempty closed set is invertible. Let $K$ be a nonempty proper closed set in a hypergroup $H$ and let $a \not\in K$. Then $a = Kb$ for some $b$. Then $a/b \approx K$, so that if $K$ were invertible, $b/a = K$. Then $b \approx Ka$, so that $a \approx KKa = Ka$. Hence $a/a \approx K$. Then a contradiction would arise if $H$ were a join geometry, for there $a/a = a$.

Cogroups furnish examples where reflexive closed sets that are not normal can be found. In any weak cogroup the right scalar identity $e$, considered as a singleton, is closed and reflexive. However, $e$ is not necessarily normal since in general $e$ is not a left scalar identity. In fact, in the cogroup of left cosets of a nonnormal subgroup $H$ of a group, the right scalar identity $H$ is not a left scalar identity.

But in a polygroup, a nonempty closed set $K$ is invertible. For such $K$, Proposition 7 implies $K^{-1} = K$. Hence given that $a/b = ab^{-1} \approx K$, then it follows that $b/a = ba^{-1} \approx (ab^{-1})^{-1} \approx K^{-1} = K$. The rest is by duality. Also in a polygroup, normality and reflexivity are the same notion for a subhypergroup $N$ since $aN = Na$ and $a^{-1}N = a\setminus N = N/a = Na^{-1}$ are equivalent conditions. Finally note that the scalar identity of a polygroup is a reflexive closed set.
8. COSETS

Let $H$ be a transposition hypergroup. Let $N$ be a nonempty reflexive closed subset of $H$. The elements of $H$ are partitioned as regards their relation to $N$.

**Definition.** Elements $a$ and $b$ of a transposition hypergroup are said to be equivalent modulo a nonempty reflexive closed set $N$ if $aN = Nb$.

**Proposition 10.** In a transposition hypergroup, equivalence modulo a nonempty reflexive closed set $N$ is an equivalence relation.

**Proof.** Since $N$ is nonempty and reflexive, $aN = Na$. Thus by transposition, $aN = Na$. Equivalence modulo $N$ is a reflexive relation.

For symmetry, suppose that $aN = Nb$. Then $N/aN = N/Nb = Nb/N$, so that

$$bN = b(N \setminus N) = Nb \setminus N = N/aN = (N/N)/a = N/a.$$

By transposition, $bN \approx Na$. Equivalence modulo $N$ is symmetric.

For transitivity, suppose that $aN = Nb$ and $bN = Nc$. Consequently $N \setminus aN = Nc/N$, and therefore transposition gives $aN = aNN \approx NNc = Nc$. Equivalence modulo $N$ is transitive.

Note that the proposition ensures that equivalence modulo $N$ is a self-dual notion.

**Definition.** In a transposition hypergroup, the equivalence class for $a$ modulo a nonempty reflexive closed set $N$ is denoted by $a$ and is called the coset of $N$ determined by $a$.

For sets the coset notation $A_N$ is defined by $A_N = \{aN \mid a \in A\}$.

**Proposition 11.** In a transposition hypergroup, if $N$ is a nonempty reflexive closed set then $aN = Na/N = N/(N/a) = N \setminus aN = (a \setminus N) \setminus N$.

**Proof.** First observe that $x = aN$ is equivalent to $xN = Na$, and so to $x \approx Na/N$. Hence $aN = Na/N$. Next note that

$$N/(N/a) \subseteq Na/N \subseteq (N/(Na/N)) \setminus N \subseteq (NN/Na) \setminus N = (N/Na) \setminus N = N/(N/Na) = N/Na/N \subseteq NN/(N/a) = N/(N/a).$$

Hence $Na/N = N/(N/a)$. Duality gives the rest.

**Corollary 4.** In a transposition hypergroup, if $N$ is a nonempty reflexive closed set then $a \approx N$ if and only if $a_N = N$. 
Proof. Let \( a \approx N \). Then \( a_N = Na/N = N/N = N \). The converse is trivial since \( a \approx a_N \).

The result of the proposition generalizes to sets, that is,
\[
A_N = NA/N = N/(N/A) = N \setminus AN = (A \setminus N) \setminus N.
\]

**Proposition 12.** In a transposition hypergroup, if \( N \) is a nonempty reflexive closed set then the following are equivalent statements:

1. \( ab \approx N \);
2. \( ba \approx N \);
3. \( a_N = N/b \);
4. \( b_N = N/a \).

Proof. (a) is equivalent to \( a \approx N/b \). Dually, (b) is equivalent to \( a \approx b \setminus N \). Since the two latter statements are equivalent for \( N \) reflexive, so are (a) and (b). Next (a) is shown to be equivalent to (c). Then by symmetry, (b) is equivalent to (d) and the proof will be complete.

(a) yields \( b \approx a \setminus N = N/a \). Thus \( N/b \subseteq N/(N/a) = a_N \) by the previous proposition. Therefore,
\[
a_N = (N/b)_N = N(N/b)/N \subseteq (NN/b)/N = (N/b)/N
\]
\[
= (b \setminus N)/N = b \setminus (N/N) = b \setminus N = N/b.
\]

Hence (a) implies (c). Conversely, (c) implies \( a \approx N/b \), and so (a). Therefore, (a) and (c) are equivalent.

Cosets and the defining sets for reflexivity are the same.

**Corollary 5.** In a transposition hypergroup, if \( N \) is a nonempty reflexive closed set then \( N/a = a \setminus N \) is a coset of \( N \). Moreover, any coset of \( N \) has this form.

Proof. Given \( a \), there exists \( b \) such that \( ab \approx N \). Hence \( N/a = b_N \) and \( a_N = N/b \).

The join of a pair of cosets and the union of the cosets determined by the members of the join of the representatives of the pair are related.

**Proposition 13.** In a transposition hypergroup, if \( N \) is a nonempty reflexive closed set then \( a_N b_N \subseteq (ab)_N \).

Proof. By Proposition 11,
\[
a_N b_N = (N \setminus aN)(N/(N/b)) \subseteq (N \setminus aN)N/(N/b)
\]
\[
\subseteq (N \setminus aNN)/(N/b) = (N \setminus aN)/(N/b)
\]
\[
= (Na/N)/(N/b) = Na/(N/b)N
\]
\[
= Na/(b \setminus N)N \subseteq Na/(b \setminus NN) = Na/(b \setminus N)
\]
\[
= Na/(N/b) \subseteq Nab/N = (ab)_N.
\]
The proposition holds for sets, that is, \( A_N B_N \subseteq (AB)_N \). The containment relation of the proposition can be proper. See [15, Exercise 4, p. 371] for an example where \( a_N b_N \neq (ab)_N \).

Lastly an identity property for cosets is derived.

**Proposition 14.** In a transposition hypergroup, if \( N \) is a nonempty reflexive closed set then the following statements are equivalent:

(a) \( n \approx N \);  
(b) \((na)_N = a_N \);  
(c) \((an)_N = a_N \).

**Proof.** Suppose (a). Then \((na)_N = N na / N = Na / N = a_N \), hence (b).

Suppose (b). Since \( N \) is nonempty, \( a \) may be chosen in \( N \). Then Corollary 4 yields \((na)_N = N \), and so \( na \approx N \). Thus Proposition 4 yields \((n / a)_N = N \) and (a) holds. Therefore (a) and (b) are equivalent. The rest follows by duality.

The proposition has a generalization to sets in the form \((MA)_N = (AM)_N = A_N \) for a nonempty subset \( M \) of \( N \).

Note that for polygroups the results of this section appear in a simpler form since the expression for a coset \( a_N = Na / N = aN / N = aNN^{-1} = aNN = aN \). Thus, for example, Proposition 13 takes on a more familiar form, \( aNbN = abNN = abN \).

### 9. Quotient Spaces

Let \( H \) be a hypergroup and \( \theta \) an equivalence relation in \( H \). Let \( a_\theta \) denote the equivalence class of \( a \) with respect to \( \theta \) and \( A_\theta = \cup \{ a_\theta \mid a \in A \} \). Let \( H : \theta \), read \( H \) modulo \( \theta \), denote the family of equivalence classes, that is, \( H : \theta = \{ a_\theta \mid a \in H \} \). A hyperoperation \( \circ \) is defined in \( H : \theta \) by

\[
a_\theta \circ b_\theta = \{ x_\theta \mid x \in a_\theta b_\theta \}.
\]

The equivalence \( \theta \) is known as a regular or type 2 equivalence if \( a_\theta b_\theta \subseteq (ab)_\theta \). If \( \theta \) is regular then it readily follows that \( a_\theta \circ b_\theta = \{ x_\theta \mid x \in ab \} \). It is well known for \( \theta \) regular that \( (H : \theta, \circ) \) is a hypergroup (see [10, 4]). The hypergroup \( H : \theta \) is known as the factor or quotient hypergroup of \( H \) modulo \( \theta \). In \( H : \theta \), the right and left extensions are denoted respectively by \( \gamma \) and \( \delta \). If \( \theta \) is a regular equivalence in \( H \), an element \( n \) of \( H \) is called a scalar identity for \( \theta \) if \( (na)_\theta = (an)_\theta = a_\theta \). Regular equivalence is a self-dual notion and so is being a scalar identity for a regular equivalence.

**Proposition 15.** If \( H \) is a transposition hypergroup and \( \theta \) is a regular equivalence in \( H \) then \( H : \theta \) is a transposition hypergroup. Furthermore, if there exists a scalar identity \( n \) for \( \theta \), then \( H : \theta \) is a polygroup whose scalar identity is \( n_\theta \).
Proof. Suppose that $H$ is a transposition hypergroup. It needs to be shown that transposition holds in $H: \theta$. Suppose $b_{\theta} \setminus a_{\theta} = c_{\theta} \setminus d_{\theta}$. Then there exists $x_{\theta}$ such that $a_{\theta} = b_{\theta} \circ x_{\theta}$ and $c_{\theta} = x_{\theta} \circ d_{\theta}$. Hence in $H$, since $\theta$ is regular, $a_{\theta} \approx b x$ and $c_{\theta} \approx x d$. Eliminating $x$ gives $b \setminus a_{\theta} \approx c_{\theta} \setminus d_{\theta}$. Transposition in $H$ then gives $a_{\theta} d \approx b c_{\theta}$. Then $a_{\theta} d_{\theta} \approx b_{\theta} c_{\theta}$, so that in $H: \theta$, as wanted $a_{\theta} \circ d_{\theta} \approx b_{\theta} \circ c_{\theta}$. Therefore, since $H: \theta$ is known to be a hypergroup, the first assertion of the proposition holds.

Suppose that $n$ is a scalar identity for $\theta$. Then

$$n_{\theta} \circ a_{\theta} = \{x_{\theta} \mid x \in n_{\theta} a_{\theta}\} \subseteq \{x_{\theta} \mid x \in (n a)_{\theta}\} = \{x_{\theta} \mid x \in a_{\theta}\} = a_{\theta}.$$ 

Thus it is immediate that $n_{\theta} \circ a_{\theta} = a_{\theta}$. By duality, $a_{\theta} \circ n_{\theta} = a_{\theta}$. Hence, $n_{\theta}$ is a scalar identity in $H: \theta$. Therefore Corollary 3 yields $H: \theta$ is a polygroup.

Let $H$ be a transposition hypergroup and $N$ a nonempty reflexive closed set. For equivalence modulo $N$, the family of equivalence classes (cosets of $N$) is denoted by $H: N$.

**Proposition 16.** If $H$ is a transposition hypergroup and $N$ is a nonempty reflexive closed set, then $(H: N, \circ)$ is a polygroup in which $N$ is the scalar identity and $a_{N^{-1}} = N/a$.

**Proof.** By Proposition 13, equivalence modulo $N$ is a regular equivalence. By Proposition 14, each member of $N$ is a scalar identity for equivalence modulo $N$. Therefore the previous proposition and Corollary 4 give $H: N$ is a polygroup whose scalar identity is $N$. Lastly, Corollary 5 gives $N/a = b_{N}$ for some $b$. By Proposition 12 then $a b \approx N$, so that $N \in a_{N} \circ b_{N}$. Corollary 2 then yields $b_{N} = a_{N^{-1}}$. Thus, $a_{N^{-1}} = N/a$.

**Definition.** For a transposition hypergroup $H$ and a nonempty reflexive closed set $N$, the space $H: N$ is called the factor or quotient hypergroup of $H$ modulo $N$.

Propositions 13 and 14 assert that equivalence modulo a nonempty reflexive closed set $N$ is a regular equivalence with $N$ its set of scalar identities. The next result characterizes equivalences modulo a nonempty reflexive closed set in a transposition hypergroup as the only regular equivalences having a nonempty set of scalar identities.

**Proposition 17.** In a transposition hypergroup, if $\theta$ is a regular equivalence relation that has a nonempty set $N$ of scalar identities, then $N$ is a reflexive closed set and $\theta$ is the relation equivalence modulo $N$. 
Proof. It is first shown that \( N_\theta = N \). Let \( x \approx N_\theta \). Then \( x \approx N_\theta \) for some scalar identity \( n \) for \( \theta \). Then \( (xa)_\theta \subseteq (n_\theta a_\theta)_\theta \subseteq ((na)_\theta)_\theta = (a_\theta)_\theta = a_\theta \). Hence \( (xa)_\theta = a_\theta \). By duality, \( (ax)_\theta = a_\theta \). Thus \( x \approx N \). Therefore, \( N_\theta \subseteq N \) and \( N_\theta = N \) follows.

Next \( N \) is shown to be closed. Let \( x \approx N/N \). Then \( N \approx (xN)_\theta = x_\theta \). Thus \( x \approx N_\theta = N \). Hence \( N/N \subseteq N \). By duality, \( N \setminus N \subseteq N \). Therefore, \( N \) is closed.

Now \( N \) is shown to be reflexive. Let \( x \approx a \setminus N \). There exists \( b \) such that \( x \approx N/b \). Then \( a \setminus N \approx N/b \), so that \( aN \approx Nb \). Hence, \( a_\theta = (aN)_\theta = (Nb)_\theta = b_\theta \), so that \( a_\theta = b_\theta \). Then \( N \approx xb \subseteq x_\theta b_\theta = x_\theta a_\theta \subseteq (xa)_\theta \). Thus \( xa \approx N_\theta = N \), so that \( x \approx N/a \). Hence \( a \setminus N \subseteq N/a \). By duality, \( N/a \subseteq a \setminus N \). Therefore \( a \setminus N = N/a \) and \( N \) is reflexive.

Finally \( \theta \) is shown to be the relation equivalence modulo \( N \) by showing \( a_\theta = a_N \). Let \( x \approx a_\theta \). Choose \( b \) such that \( ab \approx N \). Then \( N \approx a_\theta b_\theta = x_\theta b_\theta \subseteq (xb)_\theta \). Hence \( xb \approx N_\theta = N \). Then \( x \approx N/b = a_N \) by Proposition 12. Thus \( a_\theta \subseteq a_N \). Let \( x \approx a_\theta \). Then \( xN \approx Na \) and so \( x_\theta = (xN)_\theta = (Na)_\theta = a_\theta \). Hence \( x_\theta = a_\theta \) and \( x \approx a_\theta \). Thus \( a_\theta \subseteq a_\theta \). Therefore \( a_\theta = a_N \).

10. THE ISOMORPHISM THEOREMS

In this section the three isomorphism theorems of group theory are derived in the context of transposition hypergroups. Let \( H \) and \( H' \) be hypergroups. Given a function \( \phi : H \to H' \), the subsets \( \phi^{-1}(\phi(a)) \) for \( a \) in \( H \) form a partition of \( H \). Through abuse of notation let the corresponding equivalence relation in \( H \) also be denoted by \( \phi \). Hence the equivalence class \( a_\phi = \phi^{-1}(\phi(a)) \).

The function \( \phi \) is known as a homomorphism if \( \phi(ab) \subseteq \phi(a)\phi(b) \). The defining condition for a homomorphism is equivalent to \( ab \subseteq \phi^{-1}(\phi(a)\phi(b)) \). From here it follows that \( (ab)_\phi \subseteq \phi^{-1}(\phi(a)\phi(b)) \) and that \( a_\phi b_\phi \subseteq \phi^{-1}(\phi(a)\phi(b)) \). The homomorphism \( \phi \) is of type 2 if \( (ab)_\phi = \phi^{-1}(\phi(a)\phi(b)) \) (see [10, 4]). Note then for \( \phi \) a type 2 homomorphism that \( a_\phi b_\phi \subseteq (ab)_\phi \), so that the equivalence relation \( \phi \) is of type 2, that is, regular. Next note that if \( \phi \) is onto \( H' \), then \( \phi \) is a homomorphism of type 2 if and only if \( \phi(ab) = \phi(a)\phi(b) \). The function \( \phi \) is an isomorphism if \( \phi \) is one-to-one, onto \( H' \), and \( \phi(ab) = \phi(a)\phi(b) \). In this case, \( H \) is isomorphic to \( H' \), symbolized \( H \cong H' \).

The first isomorphism theorem comes next.

Proposition 18. Let \( H \) be a transposition hypergroup and \( H' \) a polygroup in which \( e \) is the scalar identity. Let \( \phi : H \to H' \) be a homomorphism of type 2 onto \( H' \). Let the kernel of \( \phi \) be given by \( \ker \phi = \phi^{-1}(e) \). Then \( \ker \phi \) is closed and reflexive in \( H \) and \( H : \ker \phi \cong H' \).
Proof. The equivalence relation \( \phi \) is regular. The set of scalar identities for \( \phi \) is shown to be \( \ker \phi \). Let \( n = \ker \phi \). Then

\[
(na)_{\phi} = \phi^{-1}(\phi(n)\phi(a)) = \phi^{-1}(\epsilon \phi(a)) = \phi^{-1}(\phi(a)) = a_{\phi}.
\]

By duality, \((an)_{\phi} = a_{\phi}\). Thus \( n \) is a scalar identity for the regular equivalence \( \phi \). On the other hand, let \( n \) be a scalar identity for \( \phi \) and \( \phi(a) \) an arbitrary member of \( H' \). Then \((na)_{\phi} = a_{\phi}\) gives \( \phi(n)\phi(a) = \phi(na) = \phi(a) \).

By duality, \( \phi(a)\phi(n) = \phi(a) \). Hence \( \phi(n) = \epsilon \) and \( n = \ker \phi \). Therefore \( \ker \phi \) is the set of scalar identities for the relation \( \phi \).

Proposition 17 then yields that \( \ker \phi \) is closed and reflexive and that \( a_{\ker \phi} = a_{\phi} \). Let \( \psi : H : \ker \phi \to H' \) be given by \( \psi(x_{\phi}) = \phi(x) \). Then \( \psi \) is well defined, one-to-one, onto \( H' \), and

\[
\psi(a_{\phi} \circ b_{\phi}) = \{\psi(x_{\phi}) | x = ab\} = \{\phi(x) | x = ab\} = \phi(ab) = \phi(a)\phi(b) = \psi(a_{\phi})\psi(b_{\phi}).
\]

Therefore \( \psi \) is an isomorphism.

To prepare for the second isomorphism theorem, two propositions dealing with a pair of closed sets are proven.

**Proposition 19.** In any hypergroup, if \( M \) and \( N \) are closed and \( N \) is reflexive, then \( M \cap N \) is reflexive in \( M \).

Proof. Let \( m = M \). Suppose \( x = m \setminus (M \cap N) \). Certainly \( x = M \), so that \( xm \subseteq M \). But also \( x = M \setminus N \), so that \( xm \subseteq N \). Consequently, \( xm = M \setminus N \). Thus \( x = (M \cap N)/m \) and \( m \setminus (M \cap N) \subseteq (M \cap N)/m \).

Dually, \( (M \cap N)/m \subseteq m \setminus (M \cap N) \). Therefore, \( m \setminus (M \cap N) = (M \cap N)/m \) and so \( M \cap N \) is reflexive in \( M \).

**Proposition 20.** In a transposition hypergroup, suppose \( M \) and \( N \) are closed sets such that \( M = N \). If \( N \) is reflexive then \( N/M = M_N = \langle M, N \rangle \).

Proof. By hypothesis, \( M \cap N \neq \emptyset \). Then Proposition 4 yields

\[
M = (M \cap N)/M \subseteq N/M \quad \text{and} \quad N = N/(M \cap N) \subseteq N/M. \quad (1)
\]

Observe by Corollary 5 that \( N/M \) is a union of cosets of \( N \). Then the first result of (1) gives

\[
M_N \subseteq (N/M)_N = N/M \subseteq N/(N/M) = M_N.
\]

Thus the first equality of the proposition holds.
Obviously, $N/M \subseteq \langle M, N \rangle$. Since (1) holds, to prove the rest of the proposition, it suffices to show that $N/M$ is closed. Note that

\[
\begin{align*}
(N/M)/(N/M) &= (M \setminus N)/(N/M) = M \setminus (N/(N/M)) = M \setminus M_N \\
&= M \setminus (N/M) = M \setminus (M \setminus N) = M_M \setminus M_N \\
&= M \setminus N = N/M.
\end{align*}
\]

Since $(N/M)/(N/M) = (M \setminus N)/(M \setminus N)$, dually $(N/M)/(N/M) = M \setminus N = N/M$. Therefore, $N/M$ is closed.

Now comes the second isomorphism theorem.

**Proposition 21.** In a transposition hypergroup, suppose $M$ is a closed set, $N$ is a reflexive closed set, and $M \subseteq N$. Then $M \approx N \approx M \cap N$.

**Proof.** Both $\langle M, N \rangle$ and $M$ are transposition hypergroups. Proposition 19 and $M \subseteq N$ yield $N$ and $M \cap N$ are nonempty reflexive closed sets in $\langle M, N \rangle$ and $M$, respectively. By the last proposition, $a_N$ for $a$ in $M$ is arbitrary in $\langle M, N \rangle : N$. Let $\phi : \langle M, N \rangle : N \rightarrow M : M \cap N$ be given by $\phi(a_N) = a_M \cap N$ for $a$ in $M$.

To show that $\phi$ is well defined, suppose $a_N = b_N$ for $b$ also in $M$. In $M$ there exists $c$ such that $ac \approx M \cap N$. Proposition 12 is employed repeatedly. Then $a_{M \cap N} = (M \cap N)/c$. Since $ac \approx N$, then $a_N = N/c$. Thus $b_N = N/c$, so that $bc \approx N$. But $bc \subseteq M$. Hence $bc \approx M \cap N$, so that $b_{M \cap N} = (M \cap N)/c$. Therefore $a_{M \cap N} = b_{M \cap N}$ and $\phi$ is well defined.

Clearly $\phi$ is onto $M : M \cap N$. To show $\phi$ is one-to-one, suppose $a_{M \cap N} = b_{M \cap N}$. Then $a(M \cap N) \approx (M \cap N)b$, from which $aN \approx Nb$ follows. Therefore $a_N = b_N$ and $\phi$ is one-to-one.

Finally since equivalence modulo $N$ and modulo $M \cap N$ are regular

\[
\phi(a_N \circ b_N) = \{x \in M \cap N \mid ab\} = a_{M \cap N} \circ b_{M \cap N} = \phi(a_N) \circ \phi(b_N).
\]

Therefore $\phi$ is an isomorphism and the proposition is proven.

At last the third isomorphism theorem comes.

**Proposition 22.** In a transposition hypergroup $H$, suppose that $M$ and $N$ are nonempty reflexive closed sets and $M \subseteq N$. Then $H : N \approx (H : M) : (M : N)$.

**Proof.** Let $\phi : H : M \rightarrow H : N$ be given by $\phi(x_M) = x_N$. Then $\phi$ is well defined since $a_M = b_M$ implies $aM \approx Mb$. Then $M \subseteq N$ gives $aN \approx Nb$, so
that \( a_N = b_N \). Obviously, \( \phi \) is onto \( H : N \). Moreover,
\[
\phi(a_M \circ b_M) = \{ \phi(x_M) \mid x \preceq ab \} = \{ x_N \mid x \preceq ab \} = a_N \circ b_N = \phi(a_M) \circ \phi(b_M).
\]
Hence \( \phi \) is a homomorphism of type 2. Note by Proposition 16 that \( H : N \) is a polygroup with scalar identity \( N \). Then \( \ker \phi = \phi^{-1}(N) = \{ x_M \mid x \preceq N \} = N : M \). Therefore Proposition 18 gives that \( N : M \) is closed and reflexive in \( H : M \) and that \((H : M) : (N : M) \equiv H : N\).

11. THE JORDAN–HÖLDER THEOREM

In this section a Jordan–Hölder theorem is derived for transposition hypergroups. The methods used are adapted from those used in group theory by Schreier and Zassenhaus. A comparison of the treatment here with that given for hypergroups in \[11\] and also in \[5\] may be of interest.

**Proposition 23.** In a transposition hypergroup, if \( M \) and \( N \) are reflexive closed sets and \( M \preceq N \) then \( \langle M, N \rangle \) is reflexive.

**Proof.** Suppose \( x \preceq a \langle M, N \rangle \). Then \( ax \preceq \langle M, N \rangle \). Since \( M \) is reflexive, Proposition 20 gives \( ax \preceq M / N \), and so \( axN \preceq M \). Then the reflexivity of \( M \) and Proposition 12 give \( xNa \preceq M \). Then Proposition 20 then yields \( xa \preceq \langle M, N \rangle \). Hence \( x \preceq \langle M, N \rangle / a \), so that \( a \setminus \langle M, N \rangle \subseteq \langle M, N \rangle / a \). By duality, \( \langle M, N \rangle / a \subseteq a \setminus \langle M, N \rangle \). Therefore \( \langle M, N \rangle \) is reflexive.

An interesting question arises as to whether \( M \cap N \) is reflexive under the conditions of the proposition.

Next a variant of Dedekind modularity is proven.

**Proposition 24.** In a transposition hypergroup, suppose \( L, M, \) and \( N \) are closed sets, \( L \preceq M \), and \( M \subseteq N \). Then \( \langle L, M \rangle \cap N = \langle L \cap N, M \rangle \).

**Proof.** First note that clearly \( \langle L \cap N, M \rangle \subseteq \langle L, M \rangle \) and \( \langle L \cap N, M \rangle \subseteq N \), so that \( \langle L \cap N, M \rangle \subseteq \langle L, M \rangle \cap N \). Next, Proposition 20 gives \( \langle L, M \rangle \cap N = \langle L / M \rangle \cap N \). Suppose \( x \equiv \langle L / M \rangle \cap N \). Then \( x \equiv L / M \), so that \( xM \equiv L \). But also \( x \equiv N \), so that \( xM \subseteq NN = N \). Thus \( xM \equiv L \cap N \). Therefore \( x \equiv (L \cap N) / M \subseteq \langle L \cap N, M \rangle \), and consequently \( \langle L, M \rangle \cap N \subseteq \langle L \cap N, M \rangle \). The proof is complete.

Now two rather technical results are needed.
Proposition 25. In a transposition hypergroup, suppose \( L, M, \) and \( N \) are closed sets and \( M \subseteq N \). Suppose \( L \) is reflexive, \( M \) is reflexive in \( N \), and \( \emptyset \neq L \cap N \subseteq M \). Then \( \langle L, M \rangle \) is reflexive in \( \langle L, N \rangle \) and \( \langle L, N \rangle \): \( \langle L, M \rangle \equiv N : M \).

Proof. Obviously \( \langle L, M \rangle \subseteq \langle L, N \rangle \) holds. Let \( a \equiv \langle L, N \rangle \). Suppose \( x \equiv a \setminus \langle L, M \rangle \). Then \( x \equiv \langle L, N \rangle \) and \( ax \equiv \langle L, M \rangle \). Next since \( L \) is reflexive and \( L \subseteq N \), then \( \langle L, N \rangle = N \). Therefore, there exist \( b \) and \( y \) in \( N \) such that \( a \equiv b \) and \( x \equiv y \). Then \( ax \subseteq b \leq y \subseteq \langle L, M \rangle \). Hence, \( by \equiv \langle L, N \rangle \subseteq \langle L, M \rangle \). Since clearly \( L \equiv M \), then \( \langle L, M \rangle = L / M \). Thus \( by \equiv L / M \), and so \( byM \equiv L \). But \( byM \subseteq N \equiv N \), so that \( byM \equiv L / N \subseteq M \). Then \( b \equiv M / M \equiv M \). Since \( M \) is reflexive in \( N \), Proposition 12 gives \( yb \equiv M \). But \( y \equiv x \), and so \( yb \subseteq x \cdot ax \subseteq \langle x \rangle \). Hence \( \langle x \rangle \equiv M \), and so \( xa \equiv M \subseteq \langle L, M \rangle \). Thus \( x \equiv \langle L, M \rangle \). Similarly, \( \langle L, M \rangle / a \subseteq \langle L, M \rangle / a \). Dually, \( \langle L, M \rangle / a \subseteq \langle L, M \rangle / a \). Therefore \( \langle L, M \rangle \) is reflexive in \( \langle L, N \rangle \).

Finally, by Proposition 21 followed by the last proposition,
\[
\langle L, N \rangle : \langle L, M \rangle = \langle \langle L, M \rangle, N \rangle : \langle L, M \rangle = N : \langle L, M \rangle \cap N \\
= N : \langle L \cap N, M \rangle = N : M.
\]

Proposition 26. In a transposition hypergroup, suppose \( K, L, M, \) and \( N \) are closed sets where \( K \subseteq L \), where \( M \subseteq N \), and where \( K \equiv M \). Suppose \( K \) is reflexive in \( L \) and \( M \) is reflexive in \( N \). Then \( \langle K, L \cap M \rangle \) is reflexive in \( \langle K, L \rangle \cap \langle M, L \rangle \), symmetrically \( \langle M, K \cap N \rangle \) is reflexive in \( \langle M, L \rangle \cap \langle N, M \rangle \), and
\[
\langle K, L \cap M \rangle : \langle K, L \rangle = \langle M, L \cap N \rangle : \langle M, K \cap N \rangle. \tag{1}
\]

Proof. Proposition 19 for the hypergroup \( L \) gives \( K \cap N = K \cap (L \cap N) \) is reflexive in \( L \cap N \). Similarly, \( L \cap M \) is reflexive in \( L \cap N \). Then Proposition 23 for the transposition hypergroup \( L \cap N \) gives \( \langle K, L \cap N \rangle \) is reflexive in \( L \cap N \). Then in the transposition hypergroup \( L \), since \( K \) is reflexive, since \( \langle K, N \cap L \rangle \) is reflexive in \( L \cap N \). Since
\[
K \cap (L \cap N) = K \cap N \subseteq \langle K, N \cap L \rangle,
\]
the previous proposition applies and yields \( \langle K, \langle K, N \cap L \rangle \rangle \) is reflexive in \( \langle K, L \cap N \rangle \) and
\[
\langle K, L \cap N \rangle : \langle K, L \rangle \equiv L \cap N : \langle K, N \cap L \rangle. \tag{2}
\]
Similarly, \( \langle M, K \cap N \rangle \) is reflexive in \( \langle M, L \cap N \rangle \) and
\[
\langle M, L \cap N \rangle : \langle M, K \cap N \rangle \equiv L \cap N : \langle K, N \cap L \rangle. \tag{3}
\]
Therefore, \( (1) \) follows from \( (2) \) and \( (3) \).

The Jordan–Hölder theorem is the final result.
PROPOSITION 27. In a transposition hypergroup, suppose $M$ and $N$ are nonempty closed sets and $M \subseteq N$. Suppose

$$M = K_0 \subseteq \cdots \subseteq K_k = N \quad \text{and} \quad M = L_0 \subseteq \cdots \subseteq L_l = N,$$

where, for $i = 1, \ldots, k$ and $j = 1, \ldots, l$, each $K_{i-1}$ and each $L_{j-1}$ is a maximal proper reflexive closed set in $K_i$ and $L_j$, respectively. Then $k = l$ and there exists a one-to-one correspondence between the families of factor spaces

$$\{K_i : K_{i-1} \mid i = 1, \ldots, k\} \quad \text{and} \quad \{L_j : L_{j-1} \mid j = 1, \ldots, l\}$$

such that correspondents are isomorphic.

Proof. Let

$$K_{i,j} = \langle K_{i-1}, K_i \cap L_j \rangle \quad \text{for } i = 1, \ldots, k \text{ and } j = 0, \ldots, l$$

and

$$L_{j,i} = \langle L_{j-1}, K_i \cap L_j \rangle \quad \text{for } i = 0, \ldots, k \text{ and } j = 1, \ldots, l.$$ 

Then

$$K_0 = K_{0,0} \subseteq \cdots \subseteq K_{1,1} = K_1 = K_{2,0} \subseteq \cdots \subseteq K_{k,l} = K_k$$

and

$$L_0 = L_{0,0} \subseteq \cdots \subseteq L_{1,1} = L_1 = L_{2,0} \subseteq \cdots \subseteq L_{k,k} = L_k.$$ 

Furthermore by the last proposition, $K_{i,i-1}$ is reflexive in $K_{i,j}$, symmetrically $L_{j,i-1}$ is reflexive in $L_{j,i}$, and

$$K_{i,j} : K_{i,j-1} \equiv L_{j,i} : L_{j,i-1} \quad (1)$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, l$. By the maximality of $K_{i-1}$ in $K_i$, there is exactly one nontrivial factor space $K_{i,j} : K_{i,j-1}$ for each $i$ and the number of such spaces is $k$. Similarly there is exactly one nontrivial factor space $L_{j,i} : L_{j,i-1}$ for each $j$ and the number of such spaces is $l$. Therefore, $k = l$ and the rest readily follow from (1).

12. PIVOT PRODUCT

In this section $(H, \circ)$ is taken to be the pivot product, as defined in Section 3.7, of the idempotent transposition hypergroups $H_1$ and $H_2$, each of which satisfy

$$ac \subseteq abc \quad \text{for every } a, b, c \text{ where } a \neq c. \quad (1)$$
In $H$ let $\not\in$ and $\backslash$ denote the right and left extensions, respectively, and recall that

$$x \circ y = x_1y_1 \times x_2 \cup y_1 \times x_2 \cup y_1 \times x_2y_2.$$  

(2)

Observe that $x = a \not\in b$ if and only if

$$a_1 \times a_2 = a \equiv x \circ b = x_1b_1 \times x_2 \cup b_1 \times x_2 \cup b_1 \times x_2b_2.$$  

Thus

$$a \not\in b = \begin{cases} a_1/b_1 \times a_2, & \text{if } a_1 \neq b_1; \\ H_1 \times (a_2 \cup a_2/b_2), & \text{if } a_1 = b_1. \end{cases}$$  

(3)

Similarly

$$b \backslash a = \begin{cases} a_1 \times b_2 \backslash a_2, & \text{if } a_2 \neq b_2; \\ (a_1 \cup b_1 \backslash a_1) \times H_2, & \text{if } a_2 = b_2. \end{cases}$$  

(4)

It is now shown that $H$ is an idempotent transposition hypergroup that satisfies

$$a \circ c \subseteq a \circ b \circ c \quad \text{for every } a, b, c.$$  

(5)

Reproduction. Since (3) and (4) yield $a \not\in b$ and $b \backslash a$ are both nonempty, reproduction holds.

Associativity. By (2),

$$a \circ (b \circ c) = a \circ (b_1c_1 \times b_2) \cup a \circ (c_1 \times b_2) \cup a \circ (c_1 \times b_2c_2)$$

$$= a_1b_1c_1 \times a_2 \cup b_1c_1 \times a_2 \cup b_1 \times c_1 \times a_2b_2$$

$$\cup a_1c_1 \times a_2 \cup a_1 \times c_1 \times a_2 \cup a_1 \times c_2$$

$$\cup a_1c_1 \times a_2 \cup a_1 \times c_2 \cup a_2 \times c_1 \times a_2b_2$$

$$= (a_1b_1c_1 \cup b_2c_1 \cup a_1c_1 \cup c_1) \times a_2 \cup b_1 \times c_1 \times a_2b_2$$

$$\cup c_1 \times (a_2b_2 \cup a_2b_2c_2).$$

If $a_1 \neq c_1$ then (1) for $H_1$ yields $a_1c_1 \subseteq a_1b_2c_1$, and if $a_1 = c_1$ then idempotency for $H_1$ yields $a_1c_1 = c_1c_1 = c_1$. In any case the above simplifies to

$$a \circ (b \circ c) = (a_1b_1c_1 \cup b_1c_1 \cup c_1) \times a_2 \cup b_1 \times c_1 \times a_2b_2$$

$$\cup c_1 \times (a_2b_2 \cup a_2b_2c_2).$$  

(6)

Similarly,

$$(a \circ b) \circ c = (a_1b_1c_1 \cup b_1c_1) \times a_2 \cup b_1 \times c_1 \times a_2b_2$$

$$\cup c_1 \times (a_2b_2 \cup a_2b_2c_2).$$
which simplifies to
\[(a \circ b) \circ c = (a_1 b_1 c_1 \cup b_1 c_1) \times a_2 \cup b_1 c_2 \times a_2 b_2 \]
\[\cup c_1 \times (a_2 \cup a_2 b_2 \cup a_2 b_2 c_2), \quad (7)\]

A comparison of (6) and (7) gives associativity.

Transposition. Suppose that \(b \not\subset a \approx c \not\subset d\). Four cases are considered.

Case 1. \(a_2 \neq b_2 \) and \(c_1 \neq d_1\).
Then (3) and (4) yield \(a_1 b_1 c_1 \cup b_1 c_1 = c_1 \cup d_1 \times c_2\). Thus \(c_1 = a_1 d_1\) and \(a_2 = b_2 c_2\). Then (2) gives \(c_1 \times a_2 \approx a_1 d_1 \times a_2 \subseteq a \circ d\) and \(c_1 \times a_2 \approx c_1 \times b_2 c_2 \subseteq b \circ c\). Hence \(a \circ d \approx b \circ c\).

Case 2. \(a_2 \neq b_2 \) and \(c_1 = d_1\).
Then (3) and (4) yield \(a_1 b_1 c_1 \cup b_1 c_1 = H_2 \times (c_2 \cup c_2 / d_2)\). Since transposition holds in \(H_2\) then \(a_2 \cup a_2 d_2 \approx b_2 c_2\). Since \(c_1 = d_1\) then \(d_1 \times (a_2 \cup a_2 d_2) = c_1 \times b_2 c_2\). But (2) yields \(d_1 \times (a_2 \cup a_2 d_2) \subseteq a \circ d\) and \(c_1 \times b_2 c_2 \subseteq b \circ c\). Hence \(a \circ d \approx b \circ c\).

Case 3. \(a_2 = b_2 \) and \(c_1 \neq d_1\).
An argument symmetric to the previous one gives \(a \circ d \approx b \circ c\).

Case 4. \(a_2 = b_2 \) and \(c_1 = d_1\).
Then (2) gives \(a \circ d \approx c_1 \times a_2 = c_2 \times b_2 = b \circ c\). Thus \(a \circ d \approx b \circ c\).
Therefore transposition holds.

Idempotency. By (2) and the idempotency of \(H_1\) and of \(H_2\) clearly

\[a \circ a = a_1 a_1 \times a_2 \cup a_1 \times a_2 \cup a_1 \times a_2 a_2 = a_1 \times a_2 = a.\]

Idempotency is established.

Condition (5). Since \(a \circ b \circ c\) is given by the formula for \(a \circ (b \circ c)\) that precedes (6) and also by the formula for \((a \circ b) \circ c\) that precedes (7), a comparison of these formulas with \(a \circ c = a_1 c_1 \times a_2 \cup c_1 \times a_2 \cup c_1 \times a_2 c_2\) yields (5).

It having been established that a pivot product is an idempotent transposition hypergroup satisfying (5), a final remark is made. Pivot products are extremely simple hypergroups in the sense that they contain no proper nonempty closed sets. For given \(a \in H\), by (3) and (4)

\[a \not\subset a = H_1 \times a_2 / a_2 \quad \text{and} \quad a \not\subset a = a_1 \setminus a_1 \times H_2.\]

Hence for any \(x \in H\), by (2)

\[x = x_1 \times x_2 = (a_1 \times x_2) \circ (x_1 \times a_2) \subseteq (a \not\subset a) \circ (a \not\subset a),\]
so that \(\langle a \rangle = H\).
REFERENCES