Matrices and Set Intersections*

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ABSTRACT

The incidence matrix of a (v,k,λ)-design is a $(0,1)$-matrix A of order v that satisfies the matrix equation $AA^T=(k-\lambda)I+\lambda J$, where A^T denotes the transpose of the matrix A, I is the identity matrix of order v, J is the matrix of 1's of order v, and v,k,λ are integers such that $0<\lambda<k<v-1$. This matrix equation along with various modifications and generalizations has been extensively studied over many years. The theory presents an intriguing joining together of combinatorics, number theory, and matrix theory. We survey a portion of the recent literature. We discuss such varied topics as integral solutions, completion theorems, and λ-designs. We also discuss related topics such as Hadamard matrices and finite projective planes. Throughout the discussion we mention a number of basic problems that remain unsolved.

1. INTRODUCTION

Let $X=\{x_1,\dots,x_n\}$ be an n-set (a set of n elements), and let $X_1,\dots,X_m$ be m not necessarily distinct subsets of X. We set $a_{ij}=1$ if $x_j\in X_i$, and we set $a_{ij}=0$ if $x_j\notin X_i$. The resulting $(0,1)$-matrix

$$A=[a_{ij}] \qquad (i=1,\dots,m; \quad j=1,\dots,n) \tag{1.1}$$

of size m by n is the *incidence matrix* for the subsets $X_1,\dots,X_m$ of X. Row i of A displays the subset X_i, and column j of A displays the occurrences of the element x_j among the subsets. Thus A gives us a complete description of the subsets and the occurrences of the elements within the subsets.

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We now form the matrix equation

$$AA^T = B, \tag{1.2}$$

where A^T denotes the transpose of the matrix A. The matrix B is symmetric of order m, and B has in its (i, j) position the cardinality of the set intersection $X_i \cap X_j$.

The matrix equation (1.2) is of fundamental importance. But it is difficult to deal with this matrix equation in its full generality. Nevertheless, two fundamental problems emerge that are already evident in the writings of Boole [1, 12].

PROBLEM 1. Let $B = [b_{ij}]$ be a symmetric matrix of order m with nonnegative integral elements. Determine necessary and sufficient conditions on B in order that there exist subsets $X_1, \ldots, X_m$ of some n-set X such that

$$|X_i \cap X_j| = b_{ij} \qquad (i, j = 1, \ldots, m). \tag{1.3}$$

If such subsets exist, then we say that B is *realizable*. The problem of realizability is an exceedingly difficult one and is largely unsolved.

The second fundamental problem is the following.

PROBLEM 2. Let B be realizable. Then determine the minimal value of n for which B is realizable.

This minimal integer n is called the *content* of B. Even in very special situations the content of B is often unknown.

2. SYMMETRIC BLOCK DESIGNS

We now look at a very special case of the fundamental matrix equation (1.2), but one that is of considerable significance in its own right. We begin with the following definition. The subsets $X_1, \ldots, X_v$ of a v-set $X = \{x_1, \ldots, x_v\}$ are called a (v, k, λ)*-design* (*symmetric block design*) provided they satisfy the following three postulates:

Each X_i is a k-subset of X. (2.1)

Each $X_i \cap X_j$ for $i \neq j$ is a λ-subset of X. (2.2)

The integers v, k, λ satisfy $0 < \lambda < k < v - 1$. (2.3)

The above postulates imply that the incidence matrix A of a (v, k, λ)-design is a $(0,1)$-matrix of order v that satisfies the matrix equation

$$AA^T=(k-\lambda)I+\lambda J, \tag{2.4}$$

where I is the identity matrix of order v, and J is the matrix of 1's of order v.

Conversely, if $0<\lambda<k<v-1$ and if A is a $(0,1)$-matrix of order v that satisfies the matrix equation (2.4), then we are assured of the existence of a (v, k, λ)-design.

Symmetric block designs have been extensively studied, and summaries of their basic properties are available in [7, 15]. One proves without difficulty that the incidence matrix of a (v, k, λ)-design is normal, namely,

$$AA^T=A^TA. \tag{2.5}$$

It also follows that the parameters v, k, λ must satisfy the relationship

$$k-\lambda=k^2-\lambda v. \tag{2.6}$$

Thus it is already clear that the parameters v, k, λ are far from arbitrary, because they must satisfy the relations (2.3) and (2.6). The only other known necessary conditions on the parameters are those given by the following Bruck-Ryser-Chowla theorem on the existence of (v, k, λ)-designs [7, 15].

THEOREM 2.1. *Suppose that v, k, λ are integers for which there exists a (v, k, λ)-design. If v is even, then the integer $k-\lambda$ is equal to a square. If v is odd, then the Diophantine equation*

$$x^2=(k-\lambda)y^2+(-1)^{(v-1)/2}\lambda z^2 \tag{2.7}$$

has a solution in integers x, y, z not all zero.

The case of v even in Theorem 2.1 follows at once by applying determinants to the matrix equation (2.4). Thus in this case we obtain

$$[\det(A)]^2=k^2(k-\lambda)^{v-1}, \tag{2.8}$$

and the conclusion follows. But the case of v odd is harder and requires the study of matrix congruences over the field of rational numbers Q.

PROBLEM 3. Determine the precise range of values of v, k, λ for which (v, k, λ)-designs exist.

Certain special parameter sets v, k, λ are of great importance in their own right. A *finite projective plane* of *order* n is a (v, k, λ)-design on the parameters

$$v=n^2+n+1, \quad k=n+1, \quad \lambda=1 \qquad (n\geq 2). \tag{2.9}$$

The first undecided order for the existence of a finite projective plane is $n=10$. The construction of a plane of order 10 is equivalent to finding a $(0,1)$-matrix A of order 111 that satisfies the matrix equation

$$AA^T=10I+J. \tag{2.10}$$

This problem is perhaps the most famous unsolved problem that is of a purely finite character.

Another important class of (v, k, λ)-designs is associated with the parameters $v=4t-1$, $k=2t-1$, $\lambda=t-1$, $t\geq 2$. These designs are called *Hadamard designs* and are easily seen to be equivalent to Hadamard matrices of order $4t$. We recall that a *Hadamard matrix* of order n is a $(1,-1)$-matrix of order n that satisfies the matrix equation

$$HH^T=nI. \tag{2.11}$$

It is elementary to verify that a Hadamard matrix must have orders $n=1,2$, or else $n\equiv 0 \pmod 4$.

PROBLEM 4. Show that Hadamard matrices exist for all orders $n\equiv 0$ (mod 4).

We next mention a theorem of Ryser [7, 8, 14, 15] that shows that under certain conditions the existence of (v, k, λ)-designs reduces to a classical problem involving matrices with integral elements.

THEOREM 2.2. *Let A be a matrix of order v with integral elements such that*

$$AA^T=(k-\lambda)I+\lambda J, \tag{2.12}$$

where $0<\lambda<k<v-1$ and $k-\lambda=k^2-\lambda v$. We agree to write A so that all of the column sums c_i of A satisfy $c_i\geq 0$. Suppose further that (k, λ) is

square-free and that $k-\lambda$ is odd. Then A is a $(0,1)$-matrix and hence the incidence matrix of a (v, k, λ)-design.

We remark that Hadamard matrices are easily constructed for orders that are a power of 2 and that the direct product of Hadamard matrices is Hadamard. Hence it suffices to deal with Problem 4 on the existence of Hadamard matrices of order n for the case $n=4t$, where t is odd. But then the corresponding Hadamard designs are covered by Theorem 2.2, so that Problem 4 may actually be regarded as a difficult unsettled question dealing with integral matrices.

We next mention a recent deep theorem of Verheiden [21] that concerns completions of Hadamard matrices and extends the earlier investigations of Hall [9].

THEOREM 2.3. *Let X be a $(1,-1)$-matrix of size $n-r$ by n with $n\equiv 0 \pmod 4$. Suppose that X satisfies the matrix equation*

$$XX^T = nI_{n-r}, \tag{2.13}$$

where I_{n-r} is the identity matrix of order $n-r$. Then if $r\leq 7$, there exists a Hadamard matrix of order n with X as its first $n-r$ rows.

The result is best possible in the sense that the following matrix of size 4 by 12 cannot be completed to a Hadamard matrix [9]:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix}. \tag{2.14}$$

Completion theorems for (v, k, λ)-designs have also been extensively studied [10, 11, 22]. The following theorem of Verheiden [22] establishes the existence of highly restricted rational solutions of the matrix equation (2.4).

THEOREM 2.4. *Suppose that v, k, λ satisfy the known necessary conditions for the existence of a (v, k, λ)-design. Then there exists a rational and normal matrix A of order v such that*

$$AA^T = (k-\lambda)I + \lambda J, \tag{2.15}$$

where $2^e A$ is integral for e sufficiently large.

3. VARIANTS OF SYMMETRIC BLOCK DESIGNS

In recent years there have been important investigations that deal with variants of (v, k, λ)-designs. We begin with the following theorem of Ryser [16] and Woodall [23].

THEOREM 3.1. *Let A be a $(0,1)$-matrix of order $n>1$ that satisfies the matrix equation*

$$AA^T = \operatorname{diag}[k_1-\lambda,\ldots,k_n-\lambda] + \lambda J. \tag{3.1}$$

Suppose that not all of the k_i are equal and that $0<\lambda<k_i$. Then A has exactly two distinct column sums c_1 and c_2, and these numbers satisfy

$$c_1 + c_2 = n+1. \tag{3.2}$$

The configurations associated with the incidence matrix A of this theorem are variants of symmetric block designs called *λ-designs* on n elements. The λ-designs with $\lambda=1$ have an especially simple structure. The de Bruijn–Erdös theorem [6] asserts that for each $n>3$ there exists a unique λ-design with $\lambda=1$:

$$X_1 = \{2,3,\ldots,n\},$$

$$X_2 = \{1,2\}, \qquad X_3 = \{1,3\},\ldots, \qquad X_n = \{1,n\}.$$

This is in sharp contrast to the state of affairs for a finite projective plane.

One may judiciously modify symmetric block designs in an elementary manner and thereby construct λ-designs [2, 23]. All of the λ-designs constructed by this procedure (including the λ-designs with $\lambda=1$) are called λ-designs of *type* 1.

PROBLEM 5. Show that all λ-designs are of type 1.

The combined efforts of Bridges and Kramer [2, 3, 13] have verified this conjecture for $\lambda \leqslant 9$, and a theorem of Singhi and Shrikhande [20] solves this problem for the case of λ equal to a prime number. But the possibility of the existence of exotic λ-designs remains open.

The following theorem on λ-designs has been established by Woodall [23].

THEOREM 3.2. *The number of λ-designs for each fixed value of $\lambda > 1$ is finite.*

The corresponding result for (v, k, λ)-designs is far from resolved.

PROBLEM 6. Show that the number of (v, k, λ)-designs for each fixed value of $\lambda > 1$ is finite.

Many other variants of (v, k, λ)-designs are now being studied. We mention in particular the concept of a *multiplicative design* [17]. Such a design is characterized by a $(0,1)$-matrix A of order n that satisfies a matrix equation of the form

$$AA^T = \mathrm{diag}[k_1 - \lambda_1, \ldots, k_n - \lambda_n] + \left[\sqrt{\lambda_i}\sqrt{\lambda_j}\right]. \tag{3.3}$$

In these designs the symmetric matrix λJ of the (v, k, λ)-design and the λ-design is replaced by the more general situation of a symmetric matrix of rank 1. These designs are under intensive investigation by Bridges and Mena [4, 5] and have many remarkable properties.

Now let Z be a matrix of order n, and suppose that the elements of Z consist of only two elements x and y, which are elements of a field F. We call Z an (x, y)-*matrix* over F. The following theorem deals with this concept [19].

THEOREM 3.3. *Let Z be a nonsingular (x, y)-matrix of order $n > 1$ over F, and suppose that Z satisfies the matrix equation*

$$ZEZ^T = D + \lambda J, \tag{3.4}$$

where the diagonal matrices

$$D = \mathrm{diag}[d_1, \ldots, d_n], \qquad E = \mathrm{diag}[e_1, \ldots, e_n] \tag{3.5}$$

are nonsingular over F. Then if c_i denotes the sum of column i of Z, it follows that

$$\lambda(1+\lambda w)[c_i - (x+y)]^2 - t(1+\lambda w)(x+y)[c_i - (x+y)] + t^2\left(xyw + \frac{1}{e_i}\right) = 0, \tag{3.6}$$

where

$$1+\lambda w\neq 0, \tag{3.7}$$

$$w=\frac{1}{d_1}+\cdots+\frac{1}{d_n}, \tag{3.8}$$

$$t=\lambda(n-1)-(e_1+\cdots+e_n)xy. \tag{3.9}$$

We remark that this theorem has a number of applications. In particular, it implies that the column sums of a λ-design satisfy (3.2).

The following corollary is immediate [18].

COROLLARY 3.4. *Let Z be a $(0,1)$-matrix of order n over the rational field Q, and suppose that Z satisfies the matrix equation*

$$ZEZ^T=D, \tag{3.10}$$

where D and E are diagonal matrices of order n over Q, and D is nonsingular. Then Z is a permutation matrix.

An interesting analogous result holds for $(1,-1)$-matrices over Q.

COROLLARY 3.5. *Let Z be a $(1,-1)$-matrix of order n over the rational field Q, and suppose that Z satisfies the matrix equation*

$$ZEZ^T=D, \tag{3.11}$$

where D and E are diagonal matrices of order n over Q, and D is nonsingular. Then Z is a Hadamard matrix.

We remark that these corollaries are not difficult to establish from first principles.

The preceding discussion suggests the following two problems.

PROBLEM 7. Extend Theorem 3.3 so that the matrix Z is an (x,y,z)-matrix over F, and, in particular, find such an extension for Z a $(0,1,-1)$-matrix over the rational field Q.

PROBLEM 8. Classify the $(0,1,-1)$-matrices Z of order n over the rational field Q that satisfy the matrix equation

$$ZEZ^T=D, \tag{3.12}$$

where D and E are diagonal matrices of order n over Q, and D is nonsingular.

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