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Global Homeomorphism of Vector-Valued Functions*

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1. INTRODUCTION

In 1959, Palais [1] proved that the necessary and sufficient conditions for a function $f : R^n \to R^n$ to be a diffeomorphism of R^n onto itself are

(1) $\det J_f(x) \neq 0$ and

(2) $\lim_{\|x\| \to \infty} \|f(x)\| = \infty$,

where R^n denotes the Euclidean n-space, $J_f(x)$ denotes the Jacobian matrix of f, and $\| x \|^2 = \sum_{i=1}^{n} x_i^2$. This powerful theorem has played a fundamental role in many recent research works in nonlinear network theory [2–9]. On many occasions, however, Palais' theorem is used only to show that a function possesses a continuous global inverse f^{-1}, and the differentiability of the inverse map is not really essential. Since there exist many globally one-to-one functions which fail to satisfy condition (1) of Palais' theorem, our objective in this paper is to derive weaker global inversion theorems which do not require the Jacobian to be nonzero everywhere. Since we will be concerned exclusively with functions from R^n into R^n, the following classical theorem due to Brouwer shows that the class of globally one-to-one and continuous functions from R^n into R^n is identical to the class of globally homeomorphic functions from R^n into R^n:

THEOREM 1.1 (Invariance of Domain [10]). *If A is open in R^n and $f : A \to R^n$ is one-to-one and continuous, then $f(A)$ is open and f is a homeomorphism.*

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In view of this theorem, we will be concerned with theorems on global homeomorphisms in this paper. Unless otherwise stated, all functions are assumed to be from the Euclidean n-space R^n into R^n and are of class C^1.

Our main result in Section 2 consists of a weaker form of Palais' Theorem for global homeomorphic onto functions which allows the Jacobian to vanish on a set of isolated points. In Section 3, we study the properties of a class of global homeomorphic functions which arise frequently in nonlinear network theory; namely, the class of "increasing functions." Several theorems will be presented which guarantee that a vector-valued function is increasing. Unlike the results in Section 2, most of the theorems in this section are valid not only for vector-valued functions from R^n into R^n, but also from an open convex subset $K \subset R^n$ into R^n. The hypothesis in these theorems clearly reveals that the class of increasing functions is a natural generalization of the class of functions which are expressible as the gradient of a strictly convex scalar-potential function [11-12]. The main result in this section consists of a sufficient condition which replaces the requirement that the Jacobian matrix be positive definite by an "almost-positive definite" requirement to be defined in Section 3. Finally, in Section 4 we consider a class of "quasi-increasing" functions which need not be increasing or decreasing, but are nevertheless globally homeomorphic.

2. Sufficient Conditions for Global-Homeomorphic onto Functions

A mapping $f : X \to Y$ is called a "local homeomorphism" if for each $x \in X$, a neighborhood of x is mapped homeomorphically by f onto a neighborhood of $f(x)$. In order to prove our main theorem on global homeomorphism, it is convenient to introduce the notion of a covering map [1, 13]:

Definition 2.1. Let X and Y be a connected and locally connected topological space. If f is a mapping of X onto Y with the property that each $y \in Y$ has a neighborhood V such that each component of $f^{-1}(V)$ is mapped homeomorphically onto V by f, then f is called a "covering map" and (X, f) is called a "covering space" of the space Y. In this case, the cardinal number n of the set $f^{-1}(y)$ is the same for all $y \in Y$. If n is a finite integer, then f is called a finite covering, or more specifically, an n-covering.

It is well known that every homeomorphic onto function $f : X \to Y$ is a covering map and every covering map is a local homeomorphism [13]. However, the converse is not true: A local homeomorphism need not be a covering map and a covering map need not be a homeomorphic onto function. Hence, an n-covering map lies somewhere in between a local homeomorphism and a global homeomorphism. A 1-covering map, of course, must necessarily be a global homeomorphic onto function. The following standard results on covering maps will be needed in the proof of our global inversion theorem and are reproduced here for handy reference:

LEMMA 2.1 [1]. *Let $f: X \to Y$, $X = Y = R^n$, be a local homeomorphism. A necessary and sufficient condition that f be a finite covering is that*

$$\lim_{\|x\| \to \infty} \|f(x)\| = \infty.$$

Furthermore, if f is a finite covering, then the set

$$Y_m = \{y \in Y : f^{-1}(y) \text{ contains at least } m \text{ distinct points}\}$$

is either empty or equal to Y for each positive integer m.

LEMMA 2.2 [13]. *Let $f: X \to Y$, $X = Y = R^n$, be a covering map. If A is any component of $f^{-1}(Y)$, then A is open and f restricted to A is a 1-covering of Y, i.e., $f: A \to Y$ is bijective.*

We are now ready to state the main result in this section.

THEOREM 2.1. *Let $f: X \to Y$, $X = Y = R^n$, $n \neq 2$, be a C^1 map. Let $S = \{x \in R^n : \det J_f(x) = 0\}$ and $T = \{x \in R^n : x \notin S\}$. Then the following conditions are sufficient for f to be a homeomorphism of R^n onto R^n:*

(1) $\det J_f(x) > 0$ *for all $x \in T$ and S is at most a set of isolated points.*

(2) $\lim_{\|x\| \to \infty} \|f(x)\| = \infty$.

Proof. Consider first the case $n = 1$. Suppose $f: R^1 \to R^1$ is not one-to-one. Then there exist two distinct points x_1 and x_2 such that $f(x_1) = f(x_2)$. Let $x_1 < x_2$. Condition (1) implies that there exists an interval (a, b) with $x_1 \leqslant a < b \leqslant x_2$ such that $f'(x) > 0$ on (a, b). Since f is C^1, we can write

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} f'(x)\, dx \geqslant \int_a^b f'(x)\, dx > 0,$$

which is a contradiction. Hence, f is one-to-one. By Theorem 1.1, f is a homeomorphism on R^1. Moreover, in view of Condition (2) and Lemma 2.1, f is a homeomorphism of R^1 onto R^1.

It remains to consider $n \geqslant 3$. Let p be a point in S. There exists an open neighborhood U_p about p in R^n such that $U_p \cap S = \{p\}$: Since $\det J_f(x) > 0$ on U_p except at the isolated point p where $\det J_f(x) = 0$, it follows from a recent result by Church and Hemmingsen [14] that f is a local homeomorphism on U_p, for each $p \in S$. Since $\det J_f(x) > 0$ for each $x \in T$, f is a local homeomorphism on R^n for all $n \geqslant 3$. In view of Lemma 2.1, we know f is a finite covering. Let A be any component of $f^{-1}(Y)$. Then it follows from

Lemma 2.2 that A is open in R^n and f restricted to A is a bijective map onto R^n. By Theorem 1.1, f restricted to A is a homeomorphism of A onto Y. Hence, if we can show that $A = X$, then we would have completed the proof of this theorem.

Suppose A is a proper subset of X. Since A is open in X, A is an open proper subset of X. Let $b \in X$ be a boundary point of A and let M_b be an open connected neighborhood of $f(b)$. Since f is a finite covering map on X, $f^{-1}(M_b)$ has a finite and nonzero number of components. Let N_b be a component of $f^{-1}(M_b)$ that contains the point b. Let $N_b^* = A \cap f^{-1}(M_b)$. Since f is continuous, f^{-1} is open. Hence, both N_b and N_b^* are open and connected. Also note that f maps both N_b and N_b^* topologically onto M_b. Since N_b is an open set that contains b, the set $N_b \cap A$ is not empty. It follows that $N_b \cap N_b^*$ is not empty, for otherwise there will be at least one point x_1 in $N_b \cap A$ and a point x_2 in N_b^* such that $f(x_1) = f(x_2) \in M_b$ and f restricted to A will not be one-to-one on A, which is a contradiction. Since both N_b and N_b^* are connected, we have $N_b = N_b^*$. Hence, b is in N_b^* and, therefore, is in A. This implies that A cannot be an open proper subset of X. That is, A is closed in X. We have A is both open and closed in X, and A is nonempty. Therefore, we can conclude that $A = X$. This completes the proof of Theorem 2.1.

We remark that Condition 1 of Theorem 2.1 is not necessary for f to be a homeomorphism on R^n. However, the following lemma shows that Condition 2 of Theorem 2.1 is also a necessary condition:

LEMMA 2.3. *Let $f : X \to Y$, $X = Y = R^n$. If f is a homeomorphic function of R^n onto R^n, then*

$$\lim_{\|x\| \to \infty} \|f(x)\| = \infty.$$

Proof. We first note that if f is a homeomorphic onto function, then f^{-1} is also a homeomorphic onto function from Y to X. Let y be an arbitrary point in Y and $V \subset Y$ be an open connected set containing y. Let $U = f^{-1}(V)$. Since f^{-1} is continuous and V is connected, it follows that U is connected. Since f^{-1} is a homeomorphism on Y, f^{-1} is open on Y, hence U is open in X. It is well known that the restriction of a continuous function to an open connected set is continuous. Hence, f and f^{-1} are continuous mappings on U and V, respectively. In order to show that f maps U onto V homeomorphically, we only need to show that f maps U univalently onto V. But this is trivial because (1) f is one-to-one on X, so f is one-to-one on U, (2) in view of our definition for U, corresponding to each point v in V, there is a u in U such that $f(u) = v$, where $u = f^{-1}(v)$. Since y is an arbitrary point in Y, we have shown that f has the property that for each y in Y, there is a neighborhood V about y such that f maps $f^{-1}(V)$ onto V homeomorphically. This implies that f is a covering map. Moreover, f is a 1-covering map since f is a homeomorphism on X. By Lemma 2.1, we have the conclusion of this lemma.

An immediate consequence of Theorem 2.1 is the following global implicit function theorem:

COROLLARY 2.1. *Let* $f: X \times Z \to Y$, i.e., $f(x, z) = y$ *where* $x \in X = R^n$, $z \in Z = R^m$, $y \in Y = R^n$, $n \geqslant 1$ *and* $n + m \neq 2$. *Suppose* f *satisfies the following two conditions:*

(1) $\det \partial f/\partial x \geqslant 0$ *for all* x *and* z *and* $\det \partial f/\partial x = 0$ *on at most a set of isolated points in* $X \times Z$.

(2) $\lim_{\|x\| \to \infty} \|f(x, z)\| = \infty$ *for all* z.

Then there exists a unique continuous function g *such that* $x = g(y, z)$ *for all* (y, z) *in* $Y \times Z$.

Proof. The following proof is virtually identical to that given by Kuh and Hajj [7] for their version of the global inversion theorem which is based upon Palais' theorem. Let us define the following vectors:

$$\hat{x} = \begin{bmatrix} x \\ z \end{bmatrix}, \qquad \hat{f}(\hat{x}) = \hat{f}(x, z) = \begin{bmatrix} f(x, z) \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \end{bmatrix} = \hat{y}.$$

Then

$$J_{\hat{f}}(\hat{x}) = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial z} \\ & \\ 0 & 1 \end{bmatrix}$$

and $\det J_{\hat{f}}(\hat{x}) = \det \partial f/\partial x$. Now conditions (1) and (2) imply the following:

(1)′ $\det J_{\hat{f}}(\hat{x}) \geqslant 0$ for all $\hat{x}$ and $\det J_{\hat{f}}(\hat{x}) = 0$ on at most a set of isolated points in $X \times Z$.

(2)′ $\lim_{\|\hat{x}\| \to \infty} \|\hat{f}(\hat{x})\| = \infty$.

It follows from Theorem 2.1 that $\hat{f}$ is a homeomorphism of $R^n \times R^m$ onto itself, and hence we have a unique, continuous function $\hat{g} = \hat{f}^{-1}$ such that $\hat{g}(\hat{y}) = \hat{x}$; i.e.,

$$\begin{bmatrix} x \\ z \end{bmatrix} = \hat{x} = \hat{g}(\hat{y}) = \begin{bmatrix} g(y, z) \\ g^*(y, z) \end{bmatrix}$$

for all $\hat{y} = (y, z)$ in $R^n \times R^m$. Consequently, we obtain the first n components as $x = g(y, z)$ for all x and z, and g is a unique, continuous function for all (y, z) in $R^n \times R^m$. This completes the proof.

By requiring the function f in Theorem 2.1 to be of class C^n, $n \geqslant 3$, it is possible to allow $\det J_f(x) = 0$ on a somewhat larger set which we define next [15]:

DEFINITION 2.2. Let U be a nonempty subset of R^n. Then U is said to be of dimension zero if and only if U is a totally disconnected set. The empty set and only the empty set has dimension -1.

THEOREM 2.2. *Let $f: X \to Y$, $X = Y = R^n$, $n \neq 2$, be a C^n map. Let $S = \{x \in R^n : \det J_f(x) = 0\}$ and $T = \{x \in R^n : x \in S\}$. Then the following conditions are sufficient for f to be a homeomorphism of R^n onto R^n:*

(1) $\det J_f(x) > 0$ *for all $x \in T$ and S is a set of dimension* 0 *or* -1.

(2) $\lim_{\|x\| \to \infty} \|f(x)\| = \infty$.

Proof. We first show the theorem is true for the case $n = 1$. From Condition (1), S is a set of dimension 0 or -1, hence, for any two distinct points a and b in R^1, say $a < b$, there is some point c, $a < c < b$, such that $f'(c) > 0$. Since f is C^1, we have f' is continuous. Hence, there is a neighborhood N_c about the point c scuh that for any x in N_c, we have $f'(x) > 0$.

Suppose f is not one-to-one. Then there exists at least two distinct points x_1 and x_2 such that $f(x_1) = f(x_2)$. From the preceding paragraph, we know that there is an interval (d_1, d_2), $x_1 \leqslant d_1 < d_2 \leqslant x_2$ such that $f'(x) > 0$ for all x in (d_1, d_2). But

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} f'(x)\, dx \geqslant \int_{d_1}^{d_2} f'(x)\, dx > 0.$$

This is absurd. Hence, f is one-to-one on X. Theorem 1.1 says that f is a homeomorphism. In view of Condition (2) and Lemma 2.1, f is a homeomorphism of R^1 onto itself. For the case $n \geqslant 3$, it suffices to prove that f is a local homeomorphism on R^n because the remaining proof then follows exactly that of Theorem 2.1. In this case, the fact that f is a local homeomorphism follows from another recent result by Church [16–17], provided that $n \geqslant 3$. This completes the proof of Theorem 2.2.

We remark that the first part of Condition (1) of Theorems 2.1 and 2.2 can obviously be replaced by $\det J_f(x) < 0$. We also remark that the converse of Theorem 2.2 is true for the case $n = 1$, i.e., Conditions (1) and (2) are both necessary and sufficient for the function f in Theorem 2.2 to be a homeomorphism of R^1 onto R^1. For otherwise, S must have dimension > 0 and hence must contain a nonempty open interval (a, b). This would imply

$$f(x) = f(a) + \int_a^x f'(y)\, dy = f(a), \qquad \forall x \in (a, b),$$

which is absurd. For the case $n \geqslant 3$, the converse of Theorem 2.2, and for that matter, Theorem 2.1, is obviously not true since Condition (1) of both theorems is not necessary. There are many homeomorphic onto mappings with their Jacobian vanishing on an $(n-1)$-dimensional set. The following simple example is a case in point:

Let $f : R^3 \to R^3$ be defined by

$$\begin{aligned} y_1 &= f_1(x) = x_1{}^3, \\ y_2 &= f_2(x) = x_2{}^3, \\ y_3 &= f_3(x) = x_1 + x_2 + x_3 \,. \end{aligned}$$

This function f has a global inverse on R^3; namely,

$$x_1 = y_1^{1/3}, \qquad x_2 = y_2^{1/3},$$

$$x_3 = y_3 - y_1^{1/3} - y_2^{1/3}.$$

Hence, $f : R^3 \to R^3$ is a homeomorphic onto mapping. But $\det J_f(x) = 9x_1{}^2x_2{}^2$ vanishes on two 2-dimensional hyperplanes: one defined by $x_1 = 0$ and the other defined by $x_2 = 0$. However, the following theorem gives a partial converse to both Theorems 2.1 and 2.2:

THEOREM 2.3. *Let $f : X \to Y$, $X = Y = R^n$ be a C^n map. If f is a homeomorphism of R^n onto itself, then*

(1) *either* $\det J_f(x) \geqslant 0$ *or* $\det J_f(x) \leqslant 0$, *and there does not exist an n-dimensional open set $N \subset X$ such that* $\det J_f(x) = 0 \ \forall x \in N$.

(2) $\lim_{\|x\| \to \infty} \|f(x)\| = \infty$.

Proof. It suffices to prove only Property (1) because Property (2) follows immediately from Lemma 2.3. Since f is a homeomorphism of R^n onto itself, it is one-to-one and open on X. This implies that f is a light and open map [18]. Moreover, since f is C^n, it follows from Corollary 1.7 in Church [17] that either $\det J_f(x) \geqslant 0$ or $\det J_f(x) \leqslant 0$.

Suppose there is an n-dimensional open set $N \subset X$ such that $\det J_f(x) = 0$ for all x in N.

Let $\hat{x}$ be a point in N and let r be small enough so that $B(\hat{x}, r)$, an open ball centered at $\hat{x}$ with radius r, is contained in N. Then $(\hat{x} + rz)$ is in N for all z in $B(0, 1) \subset R^n$. Hence, $\det J_f(\hat{x} + rz) = 0$ whenever $\| z \| < 1$.

Let us define a mapping $\hat{f} : B(0, 1) \subset R^n \to R^n$ by $\hat{f}(z) \equiv f(\hat{x} + rz)$ for z in $B(0, 1)$. $\hat{f}$ is C^n since f is and $\det J_{\hat{f}}(z) \equiv 0$ on $B(0, 1)$. It had been shown in [6] that if an n-dimensional vector-valued function h is a homeomorphism on an n-dimensional open ball B, then $\det J_h(x) \not\equiv 0$ on B. Hence $\hat{f}$ cannot be a homeomorphism on $B(0, 1) \subset R^n$. In view of Theorem 1.1, $\hat{f}$ is not one-to-one on $B(0, 1)$. That is, there exists at least a pair of distinct points z_1 and z_2 in $B(0, 1)$ such that $\hat{f}(z_1) = \hat{f}(z_2)$. This implies that $f(\hat{x} + rz_1) = f(\hat{x} + rz_2)$, or that f is not one-to-one on X. This is a contradiction. Hence, our conclusion follows.

We remark that although Theorem 2.3 is only a partial converse to Theorems 2.1 and 2.2, it is as far as one can go because there exist many homeomorphic onto functions whose Jacobians vanish on an $(n - 1)$-dimensional set.

As a consequence of Theorem 2.3, the following partial converse to the global implicit function theorem in Corollary 2.1 can be given:

COROLLARY 2.2. *Let* $f : X \times Z \to Y$, i.e., $f(x, z) = y$, *where* $x \in X = R^n$, $z \in Z = R^m$ *and* $y \in Y = R^n$, *be a* C^0 *map. If there exists a unique* C^0 *mapping* $g : Y \times Z \to X$, i.e., $g(y, z) = x$, *then*

1. $\lim_{\|x\| \to \infty} \|f(x, z)\| = \infty$ *for all* z *in* Z.
2. $\lim_{\|y\| \to \infty} \|g(y, z)\| = \infty$ *for all* z *in* Z.

If, in addition, f *is* C^n, *then either* $\det(\partial f/\partial x) \geqslant 0$ *or* $\det(\partial f/\partial x) \leqslant 0$ *for all* (x, z). *Moreover, there does not exist an* $(n + m)$*-dimensional open set* $N \subset X \times Z$ *such that* $\det[\partial f(x, z)/\partial x] = 0$ *for all* (x, z) *in* N.

Proof. Let

$$\hat{x} = \begin{bmatrix} x \\ z \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} y \\ z \end{bmatrix}, \quad \hat{f} = \begin{bmatrix} f \\ z \end{bmatrix}, \quad \hat{g} = \begin{bmatrix} g \\ z \end{bmatrix},$$

$$\hat{X} = X \times Z = R^{n+m} \quad \text{and} \quad \hat{Y} = Y \times Z = R^{n+m}.$$

Then,

$$\hat{g}(\hat{y}) = \begin{bmatrix} g(y, z) \\ z \end{bmatrix} = \begin{bmatrix} x \\ z \end{bmatrix} = \hat{x}, \qquad \hat{f}(\hat{x}) = \begin{bmatrix} f(x, z) \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \end{bmatrix} = \hat{y}.$$

Hence, we have $\hat{g} : \hat{Y} \to \hat{X}$ and $\hat{f} : \hat{X} \to \hat{Y}$ such that $\hat{f} \circ \hat{g}(\hat{y}) = \hat{y}$ for all $\hat{y}$ in R^{n+m} and $\hat{g} \circ \hat{f}(\hat{x}) = \hat{x}$ for all $\hat{x}$ in R^{n+m}. This implies that $\hat{g} = \hat{f}^{-1}$ and $\hat{f} = \hat{g}^{-1}$ on R^{n+m}. Hence, $\hat{g}$ and $\hat{f}$ are both one-to-one and onto. Since they are C^0 maps, Theorem 1.1 indicates that both $\hat{f}$ and $\hat{g}$ are homeomorphisms on R^{n+m}. Hence, they are homeomorphism of R^{n+m} onto itself. By Lemma 2.3, we have

$$\lim_{\|\hat{x}\| \to \infty} \|\hat{f}(\hat{x})\| = \infty \tag{2.1}$$

and

$$\lim_{\|\hat{y}\| \to \infty} \|\hat{g}(\hat{y})\| = \infty. \tag{2.2}$$

But (2.1) implies that

$$\lim_{(\|x\|+\|z\|)\to\infty} \|f(x, z)\| + \|z\| = \infty. \tag{2.3}$$

Let z be an arbitrary point in Z with $\|z\| < \infty$. Then (2.3) implies that

$$\lim_{\|x\|\to\infty} \|f(x, z)\| = \infty. \tag{2.4}$$

Hence, (2.4) is true for all z in Z. Similarly, we can show that

$$\lim_{\|y\|\to\infty} \|g(y, z)\| = \infty.$$

If f is C^n, then $\hat{f}$ is C^n also. Since

$$\det J_{\hat{f}}(\hat{x}) = \det \begin{bmatrix} \dfrac{\partial f(x, z)}{\partial x} & \dfrac{\partial f(x, z)}{\partial z} \\ 0 & I \end{bmatrix} = \det \frac{\partial f(x, z)}{\partial x},$$

our conclusion follows from Theorem 2.3.

On first thought, one might be tempted to surmise that Theorems 2.1 and 2.2 might also hold for the case $n = 2$. Unfortunately, the following counterexample shows that this is not possible.

COUNTEREXAMPLE. Let $f : R^2 \to R^2$ be defined by

$$f_1(x) = x_1^2 - x_2^2,$$
$$f_2(x) = 2x_1x_2.$$

It can be easily verified that det $J_f(x) = 0$ only at the origin. Hence, S is isolated and of dimension 0. Moreover, $\lim_{\|x\|\to\infty} \|f(x)\| = \infty$. However, the function f is not one-to-one for the two points (1, 1) and $(-1, -1)$ both map to the same point (0, 2).

The fact that Theorems 2.1 and 2.2 are not valid for $n = 2$ and yet are true for $n \geqslant 3$ is somewhat surprising. This unexpected result has its origin in a recent result by Church and Hemmingsen [14] to the effect that the branch points of a mapping from R^n to R^n for $n \geqslant 3$ form a perfect set.

Both Theorems 2.1 and 2.2 require a norm condition on the function f. Our next theorem replaces this with a condition on the inverse map.

THEOREM 2.4. *Let $f : X \to Y$, $X = Y = R^n$, be a one-to-one and continuous map. Suppose that given any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x_2 - x_1\| < \epsilon$ for every $\|f(x_2) - f(x_1)\| < \delta$. Then f is a homeomorphism of R^n onto itself.*

Proof. f is one-to-one on X implies f^{-1} exists on $f(X)$. Moreover, f^{-1} is uniformly continuous on $f(X)$ because given any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|f^{-1}(y_2) - f^{-1}(y_1)\| = \|x_2 - x_1\| < \epsilon$ whenever $\|y_2 - y_1\| = \|f(x_2) - f(x_1)\| < \delta$, where we have let $y_i = f(x_i)$, $i = 1, 2$. Let $\{x_k\}$ be a sequence in X such that $\|x_k\| \to \infty$ as $k \to \infty$ and consider the sequence $\{y_k = f(x_k)\}$. Suppose the sequence $\{y_k\}$ is bounded, i.e., $\|y_k\| = \|f(x_k)\| \leqslant B < \infty$ for all k. Then there exists a Cauchy subsequence $\{y_{k_i}\}$ which converges to a point y_0, where $\|y_0\| \leqslant B$. From a standard result in analysis [19], we know that the image of a Cauchy sequence under a uniformly continuous mapping is Cauchy. Hence, the subsequence $\{x_{k_i} = f^{-1}(y_{k_i})\}$ is also a Cauchy sequence, and $\lim_{k_i\to\infty} \|x_{k_i}\| \neq \infty$, which is a contradiction. Therefore, the sequence $\{y_k = f(x_k)\}$ cannot be bounded. This implies that $\lim_{\|x\|\to\infty} \|f(x)\| = \infty$. Moreover, since f is one-to-one and continuous, it follows again from Theorem 1.1 and Lemma 2.1 that f is a homeomorphism of R^n onto R^n. This completes the proof.

Remark. Let $f : R^n \to R^n$. If f^{-1} exists on $f(R^n)$ and is uniformly continuous, then we have

$$\lim_{\|x\|\to\infty} \|f(x)\| = \infty.$$

3. Properties and Characterizations of Increasing Functions

Our objective in this section is to characterize an important class of homeomorphic vector-valued functions in Euclidean n-space which are frequently encountered in the physical sciences [20, 21]. These functions are usually called "monotone operators" and have been treated extensively in more general spaces by Minty [22] and Browder [23]. Since the term monotone operator or monotone function has been used in the literature under various different definitions [18, 22–26] which, unfortunately, are not consistent with one another, we have adopted the following concise and unambiguous terminologies:

Definition 3.1. Let $f : X \to Y$, $X \subset R^n$, $Y = R^n$. Let the following inner product be denoted by

$$\langle f(x_1) - f(x_2), x_1 - x_2 \rangle \equiv \alpha(x_1, x_2).$$

Then f is said to be:

(a) increasing on X, or simply an *increasing function* if and only if

$$\alpha(x_1, x_2) > 0, \qquad \forall x_1, x_2 \in X \qquad \text{and} \qquad x_1 \neq x_2;$$

(b) nondecreasing on X, or simply a *nondecreasing function* if and only if

$$\alpha(x_1, x_2) \geqslant 0, \qquad \forall x_1, x_2 \in X;$$

(c) decreasing on X, or simply a *decreasing function* if and only if

$$\alpha(x_1, x_2) < 0, \qquad \forall x_1, x_2 \in X \quad \text{and} \quad x_1 \neq x_2;$$

(d) nonincreasing on X, or simply a *nonincreasing function* if and only if

$$\alpha(x_1, x_2) \leqslant 0, \qquad \forall x_1, x_2 \in X.$$

Since each property in this section concerning an increasing (nondecreasing) function has an obvious corresponding property concerning a decreasing (nonincreasing) function, all results will be stated only in terms of increasing or nondecreasing functions.

LEMMA 3.1. *Let $f : U \to R^n$, where U is an open convex subset of R^n.*

(a) *If f is increasing on U, then f is one-to-one on U.*

(b) *If f is continuous and increasing on U, then f is a homeomorphism on U and its inverse function $f^{-1} : f(U) \to U$ is also increasing on $f(U)$.*

Proof. To prove (a), suppose the contrary. Then there exists two distinct points x and y in U such that $f(x) = f(y)$. If we let

$$g(\lambda) \equiv (y - x)^t f(x + \lambda(y - x)),$$

then $g(1) = g(0)$. But f is increasing on U implies that $g(1) - g(0) > 0$, hence the contradiction. To prove (b), we use Theorem 1.1 to assert that f is a homeomorphism on U. To show that f^{-1} is increasing on $f(U)$, let y_1 and y_2 be two arbitrary but distinct points in $f(U)$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then

$$\langle f^{-1}(y_1) - f^{-1}(y_2), y_1 - y_2 \rangle = \langle x_1 - x_2, f(x_1) - f(x_2) \rangle > 0.$$

This completes the proof of Lemma 3.1.

DEFINITION 3.2. Let $f : X \to Y$, $X \subset R^n$, $Y = R^n$. Then f is said to be a state function on X, or simply a state function, if, and only if, the Jacobian matrix $J_f(x)$ is symmetric for all $x \in X$.

DEFINITION 3.3. Let $f : U \to R^1$, where U is an open convex subset of R^n. Then f is said to be

(a) strictly convex on U, or simply a *strictly convex function* if, and only if,

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y)$$
$$\forall x, y \neq x \in U \qquad \text{and} \qquad 0 < \lambda < 1.$$

(b) convex on U, or simply a *convex function* if and only if the inequality sign in (a) is replaced by "$\leqslant$".

The following relationship between increasing and strictly convex functions had been proved in [6] and is reproduced here for handy reference:

THEOREM 3.1. *Let U be an open convex subset of R^n and $\varphi : U \to R^1$ be a C^1 map. Let $f = \nabla\varphi$ on U. Then f is increasing on U if and only if φ is strictly convex on U.*

It is well known that a scalar function $\varphi : U \to R^1$ defined on an open convex set is strictly convex if the Hessian matrix $H_\varphi(x)$ of φ [which is also the Jacobian matrix $J_f(x)$ of $f = \nabla\varphi$] is positive definite [26]. Our next objective is to weaken this hypothesis by allowing the Hessian to vanish on a set of isolated points.

LEMMA 3.2. *Let U be an open convex subset of R^1 and let $\varphi : U \to R^1$ be a C^2 scalar function. Let $S = \{x \in U : \varphi''(x) = 0\}$ and $T = \{x \in U : x \notin S\}$. Then φ is strictly convex on U if and only if $\varphi''(x) > 0$ for all x in T and S is at most a set of isolated points.*

Proof. Suppose $\varphi''(x) > 0$ for all x in T and S is a set of isolated points. Then for any two distinct points u and v in U with $u > v$, we have

$$\varphi'(u) - \varphi'(v) = \int_v^u \varphi''(x)\, dx > 0. \tag{3.1}$$

Let $w = (1-\lambda)p + \lambda q$ where $\lambda \in (0, 1)$, p and q are any two distinct points in U. It follows from (3.1) that

$$\varphi(w) - \varphi(p) = \int_p^w \varphi'(x)\, dx < \varphi'(w)(w - p), \tag{3.2}$$

$$\varphi(q) - q(w) = \int_w^q \varphi'(x)\, dx > \varphi'(w)(q - w). \tag{3.3}$$

Multiplying (3.2) by $(1-\lambda)$ and (3.3) by λ and adding, we obtain

$$\varphi((1-\lambda)p + \lambda q) < (1-\lambda)\varphi(p) + \lambda\varphi(q) \tag{3.4}$$

for all $p \neq q$ in U. Hence φ is strictly convex on U.

To prove the converse, let φ be strictly convex on U and suppose there exists a nonempty interval $(a, b) \subset U$ such that $\varphi''(x) \leqslant 0$ for all x in (a, b). If we let $w = (1 - \lambda) p + \lambda q$ where $\lambda \in (0, 1)$, p and q are any two distinct points in (a, b), then

$$\varphi(w) - \varphi(p) = \int_p^w \varphi'(x)\, dx \geqslant \varphi'(w)\,(w - p), \tag{3.5}$$

$$\varphi(q) - \varphi(w) = \int_w^q \varphi'(x)\, dx \leqslant \varphi'(w)\,(q - w). \tag{3.6}$$

Multiplying (3.5) by $(1 - \lambda)$ and (3.6) by λ and adding, we obtain

$$\varphi((1 - \lambda) p + \lambda q) \geqslant (1 - \lambda)\, \varphi(p) + \lambda \varphi(q). \tag{3.7}$$

This implies φ is not strictly convex on U, which is a contradiction. Hence, there cannot exist a nonempty open interval in U on which $\varphi'' \leqslant 0$. Therefore, $\varphi''(x) \leqslant 0$ can occur at most on a set of isolated points. Suppose there is a point C such that $\varphi''(C) < 0$. Since φ'' is continuous, there is an interval N about c such that $\varphi''(x) < 0$ for all x in N. This is a contradiction. Hence, $\varphi''(x) \geqslant 0$ for all x and $\varphi''(x) = 0$ can occur on at most a set of isolated points.

THEOREM 3.2. *Let U be an open convex subset of R^n and let $\varphi : U \to R^1$ be a C^2 scalar function. Then φ is strictly convex on U if and only if the quadratic form*

$$Q(\lambda, x, z) \equiv (z - x)^t\, H_\varphi(x + \lambda(z - x))\,(z - x) \tag{3.8}$$

is positive for all x and z in U, and for all $\lambda \in (0, 1)$ except for at most a set of isolated points on which $Q(\lambda, x, z) = 0$.

Proof. The strict convexity of φ on U is equivalent to the strict convexity of φ on each straight line segment in U. This is equivalent to the strict convexity of the function

$$g(\lambda) = \varphi(x + \lambda(z - x)), \qquad \lambda \in (0, 1),$$

where x and z are any two distinct points on U. By Lemma 3.2, $g(\lambda)$ is strictly convex on $(0, 1)$ if, and only if, $g''(\lambda) > 0$ except for at most a set of isolated points on which $g''(\lambda) = 0$. But

$$g''(\lambda) = (z - x)^t\, H_\varphi(x + \lambda(z - x))\,(z - x) = Q(\lambda, x, z).$$

Hence, g is strictly convex if, and only if $Q(\lambda, x, z)$ is positive for all $\lambda \in (0, 1)$ except for at most a set of isolated points on which $Q(\lambda, x, z) = 0$. This completes the proof of Theorem 3.2.

The result of Theorem 3.2 motivates the following definitions:

DEFINITION 3.4. An $n \times n$ real matrix A which need not be symmetric is said to be positive definite (positive semidefinite) if, and only if, $y^t A y > 0$ ($y^t A y \geqslant 0$) for all $n \times 1$ real vectors $y \neq 0$.

DEFINITION 3.5. Let U be a subset of R^n. An $n \times n$ matrix $A(x)$ is said to be "almost positive definite" on U if and only if, $A(x)$ is positive definite for all x in U except for at most a set of isolated points on which $A(x)$ is positive semidefinite.

COROLLARY 3.1. *Let U be an open convex subset of R^n and let $\varphi : U \to R^1$ be a C^2 map. If the Hessian matrix H_φ of φ is almost positive definite on U, then φ is strictly convex on U.*

Proof. Let x and z be any two distinct points in U, then by hypothesis, we have for any $y \neq 0$ in R^n,

$$y^t H_\varphi(x + \lambda(z - x))\, y \geqslant 0, \qquad \lambda \in (0, 1), \tag{3.9}$$

where the equality sign may hold only on a set of isolated points in $(0, 1)$ that λ takes on. Now if we let $y = z - x$ in (3.9), and use the notation in (3.8), we obtain $Q(\lambda, x, z) > 0$ for all x and $z \neq x$ in U and for all $\lambda \in (0, 1)$, except possibly for a set of isolated points on which $Q(\lambda, x, z) = 0$. Hence, by Theorem 3.2, φ is strictly convex on U. This completes the proof of Corollary 3.1.

It is well known that if f is a C^1 state function defined on an open rectangular subset $U \subset R^n$, then there exists a C^2 scalar potential function $\varphi : U \to R^1$ such that $f = \nabla\varphi$ [26, 27], where an open rectangular subset of R^n is defined to be the set $\{x : a_i < x_i < b_i,\ i = 1, 2, ..., n\}$. The following theorem is a utilization of state functions:

THEOREM 3.3. *Let U be an open rectangular subset of R^n and let $f : U \to R^n$ be a C^1 state function. Then f is increasing on U if and only if the quadratic form*

$$P(\lambda, u, v) \equiv (v - u)^t J_f(u + \lambda(v - u))\,(v - u) \tag{3.10}$$

is positive for all u and $v \neq u$ in U, and for all $\lambda \in (0, 1)$ except for at most a set of isolated points on which $P(\lambda, u, v) = 0$.

Proof. Since f is a C^1 state function, there exists a C^2 potential function $\varphi : U \to R^1$ such that $f = \nabla\varphi$ on U. The conclusion follows immediately from Theorems 3.1 and 3.2. This completes the proof of Theorem 3.3.

COROLLARY 3.2. *Let U be an open rectangular subset of R^n and let $f : U \to R^n$ be a C^1 state function. Then f is increasing on U if $J_f(x)$ is almost positive definite on U.*

Proof. Since f is a C^1 state function, there exists a C^2 potential function $\varphi : U \to R^1$ such that $f = \nabla\varphi$ on U [26–27]. By Corollary 3.1, φ is strictly convex on U and hence f is increasing on U by Theorem 3.1. This completes the proof of Corollary 3.2.

An example of a state function which satisfies the hypothesis of Corollary 3.2 but whose Jacobian matrix is not positive definite is given by

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_1 + \frac{1}{3}x_1^3 + x_2 \\ x_1 + x_2 + \frac{1}{3}x_2^3 \end{bmatrix}.$$

A simple calculation shows that $\det J_f(x) = 0$ at $x = 0$ and $J_f(x)$ is almost positive definite on R^2. Moreover, f is increasing because

$$\begin{aligned} \langle f(u) - f(v), u - v\rangle = {} & (u_1 - v_1 + u_2 - v_2)^2 \\ & + \tfrac{1}{3}(u_1 - v_1)^2 (u_1^2 + u_1 v_1 + v_1^2) \\ & + \tfrac{1}{3}(u_2 - v_2)^2 (u_2^2 + u_2 v_2 + v_2^2) > 0 \end{aligned}$$

whenever $u \neq v$.

However, the converse to Corollary 3.2 is not true and consequently, the converse to Corollary 3.1 is also not true. The following trivial example will bear this out:

Let $\varphi : R^2 \to R^1$ and $f : R^2 \to R^2$ be defined by

$$\varphi(x) = \tfrac{1}{4}x_1^4 + \tfrac{1}{2}x_2^2 + \tfrac{1}{4}x_2^4,$$
$$f_1(x) = x_1^3, \qquad f_2(x) = x_2 + x_2^3.$$

It is obvious that $f = \nabla\varphi$. Trivially, one can show that f is increasing on R^2, hence φ is strictly convex on R^2. But, a straightforward calculation will show that $J_f(x) = H_\varphi(x)$ is positive definite for all x in R^2 except on the line $x_1 = 0$ and on this line, $J_f(x) = H_\varphi(x)$ is positive semidefinite.

If f is not a state function, it will not be possible to generalize the preceding characterization of an increasing function f in terms of the strict convexity of some scalar function φ because only state functions can be expressed as the gradient of a potential function. However, the following theorem shows that a natural generalization of the preceding characterization to nonstate functions can be achieved through their Jacobian matrices.

THEOREM 3.4. *Let U be an open convex subset of R^n and let $f : U \to R^n$ be a C^1 function. If $J_f(x)$ is almost positive definite on U, then f is increasing on U.*

Proof. Let u and v be any two distinct points in U. Then the points on the straight line $v + \lambda(u - v)$ are in U for $\lambda \in [0, 1]$. Let

$$g(\lambda) = (u - v)^t f(v + \lambda(u - v)). \tag{3.11}$$

Since g is a C^1 function on $[0, 1]$ and $J_f(x)$ is almost positive definite on U, we have

$$g(1) - g(0) = \int_0^1 \frac{dg(\lambda)}{d\lambda}\, d\lambda = \int_0^1 (u - v)^t\, J_f(v + \lambda(u - v))\, (u - v)\, d\lambda > 0. \tag{3.12}$$

But

$$g(1) - g(0) = (u - v)^t f(u) - (u - v)^t f(v) = (u - v)^t\, (f(u) - f(v)). \tag{3.13}$$

It follows, therefore, from (3.12) and (3.13) that f is increasing on U. This completes the proof of Theorem 3.4.

An example of a nonstate function which satisfies the hypothesis of Theorem 3.4 is given by

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_1 + \frac{1}{3}x_1^3 + 2x_2 \\ x_2 + \frac{1}{3}x_2^3 \end{bmatrix}.$$

Let the nonsymmetric Jacobian matrix $J_f(x)$ be resolved into a symmetric part $A(x)$ and a skew-symmetric part $B(x)$

$$J_f(x) = \begin{bmatrix} 1 + x_1^2 & 2 \\ 0 & 1 + x_2^2 \end{bmatrix} = \begin{bmatrix} 1 + x_1^2 & 1 \\ 1 & 1 + x_2^2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$= A(x) + B(x).$$

Clearly, $A(x)$ is almost positive definite, and hence so is $J_f(x)$. To verify that f is indeed increasing, we found

$$\langle f(u) - f(v), u - v \rangle = (u_1 - v_1 + u_2 - v_2)^2 + \tfrac{1}{3}\,[(u_1 - v_1)^2\,(u_1^2 + u_1v_1 + v_1^2)$$
$$+ (u_2 - v_2)^2\,(u_2^2 + u_2v_2 + v_2^2)] > 0 \qquad \text{for all } u \neq v.$$

Although the converse of Theorem 3.4 is not true, the following theorem shows that not much improvement is possible:

THEOREM 3.5. *Let U be an open convex subset of R^n and let $f : U \to R^n$ be a C^1 function. If $J_f(x)$ is positive semidefinite for all x in U, then f is nondecreasing on U.*

Proof. The proof of Theorem 3.5 follows *mutatis mutandis* from the proof of Theorem 3.4 and is, therefore, omitted.

LEMMA 3.3. *Let U be an open convex subset of R^n and $f: U \to R^n$ be a C^1 map. If f is nondecreasing on U, then $J_f(x)$ is positive semidefinite on U.*

Proof. Suppose there exists a point x_0 in U and a pair of distinct vectors x_1 and x_2 in R^n such that the quadratic form

$$Q(x_0, x_1, x_2) \equiv (x_2 - x_1)^t J_f(x_0) (x_2 - x_1)$$

is negative. Then there exists a neighborhood about x_0 which contains an open ball $B(x_0, r) \subset U$ such that for every z in $B(x_0, r)$, we have $Q(z, x_1, x_2) < 0$. Since $B(x_0, r)$ is an n-dimensional ball, for each y in R^n, there exists a vector $\bar{y}$ in $B(x_0, r)$ and $\alpha > 0$ such that $y = \alpha(\bar{y} - x_0)$. Hence, we have two vectors $\bar{x}_1$ and $\bar{x}_2$ in $B(x_0, r)$ such that $x_1 = a_1(\bar{x}_1 - \bar{x}_0)$ and $x_2 = a_2(\bar{x}_2 - \bar{x}_0)$, where $a_1 > 0$, $a_2 > 0$. Let $a = \max\{a_1, a_2\}$. Then we can find $\hat{x}_1$ and $\hat{x}_2$ in $B(x_0, r)$ such that $x_1 = a(\hat{x}_1 - x_0)$ and $x_2 = a(\hat{x}_2 - x_0)$. Thus

$$x_2 - x_1 = a(\hat{x}_2 - \hat{x}_1). \tag{3.14}$$

Since $\hat{x}_1$ and $\hat{x}_2$ are in $B(x_0, r)$, the points $[\hat{x}_1 + \lambda(\hat{x}_2 - \hat{x}_1)]$ with λ in $[0, 1]$ are in $B(x_0, r)$. Hence, for λ in $[0, 1]$, we have

$$Q(\hat{x}_1 + \lambda(\hat{x}_2 - \hat{x}_1), \hat{x}_1, \hat{x}_2) = \frac{1}{a^2} Q(\hat{x}_1 + \lambda(\hat{x}_2 - \hat{x}_1), x_1, x_2) < 0. \tag{3.15}$$

Consider the following scalar function:

$$g(\lambda) \equiv (\hat{x}_2 - \hat{x}_1)^t f(\hat{x}_1 + \lambda(\hat{x}_2 - \hat{x}_1)), \qquad \lambda \in [0, 1].$$

Then, $g(1) - g(0) = (\hat{x}_2 - \hat{x}_1)^t [f(\hat{x}_2) - f(\hat{x}_1)] \geqslant 0$ because f is nondecreasing on U. But $g(1) - g(0) = g'(\lambda^*)$ for some λ^* in $(0, 1)$. Hence, we have

$$\begin{aligned} g(1) - g(0) = g'(\lambda^*) &= (\hat{x}_2 - \hat{x}_1)^t J_f(\hat{x}_1 + \lambda^*(\hat{x}_2 - \hat{x}_1)) (\hat{x}_2 - \hat{x}_1) \\ &= Q(\hat{x}_1 + \lambda^*(\hat{x}_2 - \hat{x}_1), \hat{x}_1, \hat{x}_2) < 0, \end{aligned} \tag{3.16}$$

which is a contradiction. Hence, there does not exist a point x_0 in U and a pair of vectors x_1 and x_2 in R^n such that $Q(x_0, x_1, x_2) < 0$. Therefore, $Q(x_0, x_1, x_2) \geqslant 0$ for all x_0 in U and x_1, x_2 in R^n. This implies that $J_f(x)$ is a positive semidefinite matrix for all x in U.

The preceding results can be summarized as follows:

Let U be an open convex subset of R^n and let $f: U \to R^n$ be a C^1 function.

Consider the following properties:

(A) $J_f(x)$ is positive definite on U.

(B) $J_f(x)$ is almost positive definite on U.

(C) f is increasing on U.

(D) f is nondecreasing on U.

(E) $J_f(x)$ is positive semidefinite on U.

Then we have the following results:

$$(A) \Rightarrow (B) \Rightarrow (C) \Rightarrow (D) \Leftrightarrow (E).$$

4. Properties and Characterization of Quasi-Increasing Functions

Our objective in this section is to study the properties of a class of homeomorphic functions, called "quasi-increasing functions," which includes the class of increasing functions as a proper subset. Our main motivation is to derive weaker global inversion theorems which allow the Jacobian of an appropriately transformed homeomorphic function to vanish on hyperplanes, rather than on a set of isolated points.

Definition 4.1. An $n \times n$ matrix A is said to be a class-E matrix if, and only if, each row and each column of A have one and only one nonzero element which is either 1 or -1.

Notice that premultiplication (postmultiplication) of a matrix J by a class-E matrix A is equivalent to a permutation of the rows (columns) of J with certain rows (columns) of J multiplied by -1. In this respect, class-E matrices act like elementary matrices [28]. The following two important properties of class-E matrices can be shown trivially:

1. If E_a and E_b are class-E matrices, then E_bE_a is a class-E matrix. Conversely, if E_c is a class-E matrix, there are two class-E matrices, E_a and E_b such that $E_bE_a = E_c$.

2. Every class-E matrix is an orthogonal matrix.

Definition 4.2. A function $f: R^n \to R^n$ is said to be quasi-increasing if, and only if, there exist two class-E matrices E_a and E_b such that the transformed function $\hat{f}: R^n \to R^n$ defined by $\hat{f}(x) = E_a f \circ (E_b x)$ is an increasing function.

Lemma 4.1. *Every quasi-increasing function f is a homeomorphic function.*

Proof. By definition, the transformed function $\hat{f}(x) = E_a f \circ (E_b x)$ is increasing and hence by Lemma 3.1(b), is homeomorphic in R^n. Since class-E

matrices are nonsingular, the function $f \circ (E_b x) = E_a^{-1}\hat{f}(x)$ is also homeomorphic in R^n. If we let $z = E_b x$, then $x = E_b^{-1} z$ is well defined and hence $f(z)$ is homeomorphic in R^n. This completes the proof.

The following lemma is an important observation for the development of this section:

LEMMA 4.2. *Let E_a and E_b be two class-E matrices of order n and $f: U \subset R^n \to R^n$ be a C^1 map on U. Then $E_a J_f(x) E_b$ is positive definite (semi-definite) at a point $x \in U$ if, and only if, $E_c J_f(x)$ is positive definite (semi-definite) at x, where $E_c = E_b E_a$.*

Proof. Since the proofs of the positive definite part and the positive semi-definite part are similar, it suffices to prove the positive definite part only.

Since E_b is nonsingular, so is $E_b{}^t$. Hence, for every vector $z \neq 0$ in R^n, there is a vector $s \neq 0$ in R^n such that $z = E_b{}^t s$. So, we have the following:

$$Q(z) \equiv z^t E_a J_f(x) E_b z = s^t E_b E_a J_f(x) E_b E_b{}^t s = s^t E_c J_f(x) s \equiv P(s).$$

This implies that $Q(z) > 0$ for all $z \neq 0$ in R^n if, and only if, $P(s) > 0$ for all $s \neq 0$ in R^n. Hence, our conclusion follows.

In view of Lemma 4.2, we can conclude that if $f: U \subset R^n \to R^n$ and is a C^1 map on U, then $E_a J_f(x) E_b$ is almost positive definite on U if and only if $E_c J_f(x)$ is almost positive definite on U, where $E_c = E_b E_a$, and E_a, E_b are class-E matrices.

THEOREM 4.1. *Let $f: X \to Y$, $X = Y = R^n$, be a C^1 map. Suppose there exists a class-E matrix E_c such that $E_c J_f(x)$ is almost positive definite on R^n, then f is quasi-increasing.*

Proof. Let E_a and E_b be two class-E matrices such that $E_c = E_b E_a$. For each x in X and $y = f(x)$, let $x = E_b u$ and $w = E_a y$. Then

$$w = E_a f \circ (E_b u) \equiv \hat{f}(u)$$

and $\hat{f}: U \to W$, where $U = E_b^{-1} X = R^n$ and $W = E_a Y = R^n$, is a C^1 map. Moreover,

$$J_{\hat{f}}(u) = E_a J_f \circ (E_b u) E_b = E_a J_f(x) E_b. \tag{4.1}$$

It follows from the hypothesis and Lemma 4.2 that $J_{\hat{f}}(u)$ is almost positive definite on R^n. By Theorem 3.4, $\hat{f}$ is an increasing function on R^n. Hence, f is quasi-increasing on R^n and the theorem is proved.

An example of a function $f: R^n \to R^n$ which is neither increasing nor decreasing but which satisfies the hypothesis of Theorem 4.1 is given by

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_2{}^3 \\ -x_1 - x_1{}^3 - x_2 \end{bmatrix}.$$

f is neither increasing or decreasing on R^2 because

$$\langle f(a) - f(b), a - b\rangle = -1 \quad \text{when} \quad a_1 = b_1, \quad a_2 = 1, \quad b_2 = 0$$

and

$$\langle f(a) - f(b), a - b\rangle = 1 \quad \text{when} \quad a_2 = b_2, \quad a_1 = 1, \quad b_1 = 0.$$

Let

$$E_c = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad E_a = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad E_b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$E_c = E_b E_a \quad \text{and} \quad E_c J_f(x) = \begin{bmatrix} 1 + 3x_1^2 & 1 \\ 1 & 1 + 3x_2^2 \end{bmatrix} = A(x).$$

A simple computation will show that $A(x)$ is positive definite except at the point $x_1 = x_2 = 0$ and at this point, $A(x)$ is positive semidefinite. Hence, $A(x)$ is almost positive definite on R^2 and so f is quasi-increasing. Indeed, the function

$$\hat{f}(u) \equiv E_a J_f(E_b u) = \begin{bmatrix} u_1 + u_1^3 - u_2 \\ -u_1 + u_2 + u_2^3 \end{bmatrix}$$

is increasing because

$$\begin{aligned} &\langle \hat{f}(a) - \hat{f}(b), a - b\rangle \\ &\quad = [(a_1 - b_1) - (a_2 - b_2)]^2 + (a_1 - b_1)^2 (a_1 + a_1 b_1 + b_1^2) \\ &\qquad + (a_2 - b_2)^2 (a_2 + a_2 b_2 + b_2^2) > 0 \end{aligned}$$

whenever $a \neq b$.

Our next theorem shows that a function $f: R^n \to R^n$ can be quasi-increasing even if the Jacobian of $\hat{f}$ is allowed to vanish on $(n - 1)$-dimensional hyperplanes.

THEOREM 4.2. *Let $f: X \to Y$, $X = Y = R^n$, be a C^1 map. Suppose there exists a class-E matrix E_c such that $E_c J_f(x)$ is positive definite for all x in R^n except for at most a finite number of hyperplanes of dimension less than or equal to $(n - 1)$ defined as*

$$\sum_{j=1}^{n} p_{ij} x_j = q_j \qquad i = 1, 2,..., m,$$

where $x = (x_1, x_2, ..., x_n)$; p_{ij} and q_j, $j = 1, 2,..., n$, $i = 1, 2,..., m$, are constants. Let

$$S_i = \left\{x : \sum_{j=1}^{n} p_{ij} x_j = q_j\right\}, \qquad i = 1, 2,..., m.$$

If for any two distinct vectors α *and* β *in each* S_i, $i = 1, 2, \ldots, m$, *we have*

$$\langle E_c[f(\alpha) - f(\beta)], \alpha - \beta \rangle > 0, \tag{4.2}$$

then f *is quasi-increasing.*

Proof. Let E_a and E_b be two class-E matrices such that $E_c = E_b E_a$. For each x in X and $y = f(x)$, let $x = E_b u$ and $w = E_b y$. Then

$$U = E_b^{-1} X = R^n, \qquad W = E_a Y = R^n$$

and

$$w = E_a y = E_a f(x) = E_a f \circ (E_b u) \equiv \hat{f}(u).$$

That is, $\hat{f} : U \to W$, $U = W = R^n$, is a C^1 map. Let r and s be any two distinct points in R^n. Let

$$g(\lambda) = (r - s)^t \hat{f}(s + \lambda(r - s)), \qquad \lambda \in [0, 1].$$

Then g is a C^1 map on $[0, 1]$. Hence, we have

$$\begin{aligned} g(1) - g(0) &= \int_0^1 g'(\lambda)\, d\lambda = \int_0^1 (r - s)^t J_{\hat{f}}(s + \lambda(r - s))\, (r - s)\, d\lambda \\ &= \int_0^1 (r - s)^t E_a J_f \circ [E_b(s + \lambda(r - s))]\, E_b(r - s)\, d\lambda \\ &= \int_0^1 (r - s)^t E_b{}^t E_b E_a J_f \circ [E_b(s + \lambda(r - s))]\, E_b(r - s)\, d\lambda \\ &= \int_0^1 (x_r - x_s)^t E_c J_f(x_s + \lambda(x_r - x_s))\, (x_r - x_s)\, d\lambda, \end{aligned} \tag{4.3}$$

where $x_r = F_b r$ and $x_s = E_b s$.

Since $\Gamma \equiv \{x : x = x_s + \lambda(x_r - x_s), \lambda \in [0, 1]\}$ is a straight line connecting the points x_r and $x_s \neq x_r$, only the following three cases can occur:

(a) Γ does not intersect any S_k, $k = 1, 2, \ldots, m$;

(b) Γ intersects some S_k, $k = 1, 2, \ldots, m$;

(c) Γ is contained in some S_k, say $S_{l_1}, S_{l_2}, \ldots, S_{l_p}$, where

$$1 \leqslant l_1, l_2, \ldots, l_p \leqslant m,$$

i.e.,

$$\Gamma \subset S_{l_1} \cap S_{l_2} \cdots \cap S_{l_p}.$$

Notice that a straight line can intersect each hyperplane at most at one point unless it is contained in that hyperplane. If case (a) or (b) occurs, then, except for a finite number of λ in (0, 1) for which the straight line

$$\Gamma = \{x : x = x_s + \lambda(x_r - x_s), \lambda \in [0, 1]\}$$

intersects with some hyperplane S_k, the matrix $E_c J_f(x_s + \lambda(x_r - x_s))$ is positive definite. Hence, (4.3) would imply that

$$g(1) - g(0) > 0,$$

that is,

$$(r - s)^t \hat{f}(r) - (r - s)^t \hat{f}(s) = \langle \hat{f}(r) - \hat{f}(s), r - s\rangle > 0. \tag{4.4}$$

If case (c) occurs, then the straight line Γ connecting the points x_r and x_s is contained in $S_{l_1} \cap S_{l_2} \cap \cdots \cap S_{l_p}$. By assumption,

$$\langle E_c[f(x_r) - f(x_s)], x_r - x_s\rangle > 0.$$

This implies that

$$\begin{aligned}\langle E_b E_a[f \circ (E_b r) - f \circ (E_b s)], E_b r - E_b s\rangle &= \langle E_a f \circ (E_b r) - E_a f \circ (E_b s), r - s\rangle \\ &= \langle \hat{f}(r) - \hat{f}(s), r - s\rangle > 0.\end{aligned}$$

Hence, for any two distinct points r and s in R^n, we have

$$\langle \hat{f}(r) - \hat{f}(s), r - s\rangle > 0.$$

That is, $\hat{f}$ is increasing on R^n and so f is quasi-increasing. This completes the proof of Theorem 4.2.

To illustrate the utility of Theorem 4.2, we will present next an example which applies Theorem 4.2 to show that a function which is neither increasing nor decreasing can nevertheless be quasi-increasing even though it fails to satisfy the hypothesis of Theorem 4.1.

EXAMPLE 4.1. Let $f : R^2 \to R^2$ where

$$f(x) = \begin{bmatrix} x_1 + x_2 + x_2^3 \\ -x_1 - x_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}.$$

f is neither increasing nor decreasing because $\langle f(u) - f(v), u - v\rangle = -1$ when $u = (0, 1)$ and $v = (0, 0)$ and $\langle f(u) - f(v), u - v\rangle = 1$ when $u = (1, 1)$ and $v = (2, 1)$. Moreover, the Jacobian of f vanishes on the line $x_2 = 0$. However, if we choose

$$E_c = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad E_a = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad E_b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

then it is easily verified that $E_c = E_b E_a$ and $E_c J_f(x)$ is positive definite except on the line $x_2 = 0$, and on this line, $\langle E_c[f(u) - f(v)], (u - v)\rangle > 0$. Hence, by Theorem 4.2, f is quasi-increasing and homeomorphic on R^2. Indeed, the inverse function is given by

$$f^{-1} = \begin{bmatrix} y_2 - (y_1 + y_2)^{1/3} \\ (y_1 + y_2)^{1/3} \end{bmatrix}.$$

By choosing $E_c = E_a = E_b = I$, the identity matrix, Theorem 4.2 becomes a sufficient condition for f to be an increasing function while allowing the Jacobian of $J_f(x)$ to vanish on a larger set than a set of isolated points. The next example illustrates this point.

EXAMPLE 4.2. Let $f : R^2 \to R^2$ where

$$f(x) = \begin{bmatrix} \frac{1}{3}x_1^3 + x_2 \\ - x_1 + x_2 + \frac{1}{3}x_2^3 \end{bmatrix}.$$

It is easily verified that $J_f(x)$ is positive definite except on the line $x_1 = 0$, and on this line, $\langle f(u) - f(v), u - v\rangle > 0$. Hence, by Theorem 4.2, f is an increasing function.

5. Concluding Remarks

Several new theorems have been presented for a function $f : R^n \to R^n$ to be homeomorphic on R^n. Unlike the Palais theorem [1], these global inversion theorems share a common feature in that the Jacobian of f is allowed to vanish on at least a set of isolated points. This feature is also preserved in the global implicit function theorems which follow quite naturally from the inversion theorems. A version of partial converse to the global inverse (implicit) function theorem is also presented.

The class of increasing functions is seen to be a natural generalization of the class of state functions associated with strict convex potential functions. A further generalization leads to the definition of a quasi-increasing function which behaves in many respects like an increasing function.

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