A note on the interlacing of zeros and orthogonality

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Received 16 September 2008; accepted 11 November 2008
Available online 18 November 2008

Communicated by Paul Nevai

Abstract

Let \( \{t_n\}_{n=0}^\infty \) be a sequence of monic polynomials with \( \deg(t_n) = n \) such that, for each \( n \in \mathbb{N} \), the zeros of \( t_n \) are real and simple and \( t_n \) and \( t_{n+1} \) have no common zeros. We discuss the connection between the orthogonality of the sequence, the positivity of a certain ratio, and the interlacing of the zeros of \( t_n \) and \( t_{n+1} \) for \( n \geq 1, n \in \mathbb{N} \).

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Keywords: Orthogonality of a monic polynomial sequence; Interlacing of zeros

For any positive Borel measure \( \mu \), if \( \{p_n\}_{n=0}^\infty \) is the uniquely determined sequence of monic orthogonal polynomials with respect to \( \mu \), it is a well-known classical result that the \( n \) zeros of \( p_n \) are real and distinct and lie in the convex hull of \( \text{supp}(\mu) \). Further, if \( x_{1,n} < x_{2,n} < \cdots < x_{n,n} \) are the zeros of \( p_n \) and \( x_{1,n+1} < x_{2,n+1} < \cdots < x_{n+1,n+1} \) are the zeros of \( p_{n+1} \), then

\[
x_{1,n+1} < x_{1,n} < x_{2,n+1} < x_{2,n} < \cdots < x_{n+1,n+1} < x_{n,n} < x_{n+1,n+1},
\]
a property usually called the interlacing of the zeros.

In 2007, Marcellán (see [1] for related work) posed the following question:

If \( \{p_n\}_{n=0}^\infty \) and \( \{q_n\}_{n=0}^\infty \) are two monic orthogonal polynomial sequences corresponding to positive Borel measures \( \mu_1 \) and \( \mu_2 \) respectively, under what circumstances is \( \{r_n\}_{n=0}^\infty \) an orthogonal sequence where \( r_n = a p_n + b q_n \), with \( a \) and \( b \) non-zero constants?

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doi:10.1016/j.jat.2008.11.008
In addressing this question, it has been suggested that if the zeros of \( r_n \) and \( r_{n+1} \) are interlacing for each \( n \in \mathbb{N}, n \geq 1 \), then \( \{r_n\}_{n=0}^{\infty} \) is orthogonal with respect to some positive Borel measure. This seems plausible in light of Wendroff’s theorem (cf. [4]) which proves that if \( \{x_{k,n}\}_{k=1}^{n} \) and \( \{x_{k,n+1}\}_{k=1}^{n+1} \) are two sets of real distinct points satisfying (1), then \( p_n(x) = \prod_{k=1}^{n}(x - x_{k,n}) \) and \( p_{n+1}(x) = \prod_{k=1}^{n+1}(x - x_{k,n+1}) \) can be embedded in an orthogonal sequence.

The purpose of this note is to discuss the connection between three different properties of a sequence of monic polynomials \( \{t_n\}_{n=0}^{\infty} \), namely, the orthogonality of the sequence, the positivity of a certain ratio, and the interlacing of the zeros of \( t_n \) and \( t_{n+1} \) for all \( n \in \mathbb{N}, n \geq 1 \). We give an example to illustrate that the interlacing of zeros of monic polynomials of adjacent degree in a sequence is a far weaker property than the orthogonality of such a sequence. For related work on connections between orthogonal polynomials, their zeros, and their recurrence coefficients, see [3].

We begin by reformulating Theorem 6.2 in [2].

**Lemma.** Let \( \{t_n\}_{n=0}^{\infty} \) be a sequence of monic polynomials with \( \deg(t_n) = n \) and suppose that the zeros of \( t_n \) are real and simple and that \( t_n \) and \( t_{n+1} \) have no common zeros for any \( n \in \mathbb{N} \). Then \( \{t_n\}_{n=0}^{\infty} \) satisfies a three term recurrence relation of the form

\[
t_{n+1}(x) = (x - b_n)t_n(x) - \lambda_n t_{n-1}(x)
\]

if and only if

\[
\left( \frac{t_{n+1}}{t_{n-1}} \right) (t_{i,n}) = \left( \frac{t_{n+1}}{t_{n-1}} \right) (t_{j,n}) \quad \text{for each } i, j = 1, 2, \ldots, n,
\]

where \( \{t_{k,n}\}_{k=1}^{n} \) is the set of zeros of \( t_n \). Note that we take \( t_0(x) \equiv 1 \) and \( t_{-1}(x) \equiv 0 \).

We observe that if (2) holds then

\[
b_n = \sum_{i=1}^{n+1} t_{i,n+1} - \sum_{i=1}^{n} t_{i,n},
\]

and the common value of the ratios in (3) is \(-\lambda_n\).

We can now state and prove our result.

**Theorem.** Let \( \{t_n\}_{n=0}^{\infty} \) be a sequence of real monic polynomials, \( \deg(t_n) = n \), satisfying the property that the zeros of \( t_n \) are real and simple and \( t_n \) and \( t_{n+1} \) have no common zeros for any \( n \in \mathbb{N} \). Assume, in addition, that (3) holds for every \( n \in \mathbb{N}, n \geq 1 \). Then the following are equivalent:

(i) the sequence \( \{t_n\}_{n=0}^{\infty} \) is orthogonal with respect to a positive Borel measure;

(ii) \( \lambda_n \) is positive for each \( n \in \mathbb{N} \), where \(-\lambda_n\) is the common value of the ratios in (3);

(iii) the zeros of \( t_n \) and \( t_{n+1} \) are interlacing for each \( n \in \mathbb{N} \).

**Proof.** If (i) holds, it is a classical result that \( \{t_n\}_{n=0}^{\infty} \) satisfies a three term recurrence relation of the form (2) with \( \lambda_n > 0 \). It follows immediately from (3) that (ii) holds and, since the interlacing property (iii) is a well-known consequence of orthogonality, we see that (i) implies (iii). Now, if (ii) holds, it follows from Favard’s Theorem that (i), and therefore (iii), holds. Finally, if \( t_{1,n} < t_{2,n} < \cdots < t_{n,n} \) are the zeros of \( t_n \) for each \( n \in \mathbb{N} \), we see that if (iii) holds, we have

\[
t_{1,n+1} < t_{1,n} < t_{2,n+1} < \cdots < t_{n,n+1} < t_{n,n} < t_{n+1,n+1}
\]
for each \( n \geq 1, n \in \mathbb{N} \). Evaluating (2) at the smallest zero \( t_{1,n+1} \) of \( t_{n+1}(x) \), we see that
\[
0 = (t_{1,n+1} - b_n)t_n(t_{1,n+1}) - \lambda_n t_{n-1}(t_{1,n+1}),
\]
and therefore,
\[
\lambda_n = (t_{1,n+1} - b_n) \left( \frac{t_n}{t_{n-1}} \right) (t_{1,n+1}).
\]

From (4) and (5) we observe that
\[
t_{1,n+1} - b_n = (t_{1,n} - t_{2,n+1}) + \cdots + (t_{n,n} - t_{n+1,n+1}) < 0,
\]
while \( t_n \) and \( t_{n-1} \) must have a different sign at the smallest zero of \( t_{n+1} \) since neither can have a smaller zero and their degrees differ by exactly one. Therefore \( \lambda_n > 0 \) and this concludes the proof. \( \square \)

**Remark.** Let \( \{t_n\}_{n=0}^\infty \) be a sequence of monic polynomials, where the zeros of \( t_n \) and \( t_{n+1} \) are interlacing for all \( n \geq 1, n \in \mathbb{N} \), that includes, for example,
\[
t_3(x) = (x - 1)(x - 4)(x - 8), \quad t_2(x) = (x - 2)(x - 5), \quad t_1(x) = (x - 3).
\]

Then
\[
\left( \frac{t_3}{t_1} \right) (2) = -12,
\]
while
\[
\left( \frac{t_3}{t_1} \right) (5) = -6,
\]
so the sequence \( \{t_n\}_{n=0}^\infty \) cannot be orthogonal with respect to any positive Borel measure. The point is, in an orthogonal sequence, the three term recurrence relation completely (and uniquely) determines \( t_1 \) if \( t_3 \) and \( t_2 \) are given.

**Acknowledgments**

The author acknowledges constructive and helpful comments by Alan Beardon and the referees. The research was supported by the National Research Foundation Grant Number 61095.

**References**