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The Structure of the Iwasawa Module Associated with a $\mathbf{Z}_p^r$-Extension of a p-adic Local Field of Characteristic 0

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Let K be a finite extension of $\mathbf{Q}_p$. Let $K_{\infty,r}$ be a Galois extension of K such that $\mathscr{G}_r := \mathrm{Gal}(K_{\infty,r}/K) \cong \mathbf{Z}_p^r$ for some integer $r \geqslant 1$. Let $K_{\infty,r}^{ab,p}$ be the maximal abelian pro-p extension of $K_{\infty,r}$, $\mathbf{M}_r = \mathrm{Gal}(K_{\infty,r}^{ab,p}/K_{\infty,r})$, and $\Lambda_r = \varprojlim \mathbf{Z}_p[[\mathscr{G}_r/p^n\mathscr{G}_r]]$. When $r=1$, Iwasawa has determined the Λ_r-module structure of $\mathbf{M}_r$. In this article we determine the rank and depth of the Λ_r-module $\mathbf{M}_r$ for any integer $r \geqslant 1$.

INTRODUCTION

Let K be a finite extension of $\mathbf{Q}_p$. Let $K_{\infty,r}$ be a Galois extension of K such that $\mathscr{G}_r := \mathrm{Gal}(K_{\infty,r}/K) \simeq \mathbf{Z}_p^r$ for some integer $r \geqslant 1$. Let $K_{\infty,r}^{ab,p}$ be the maximal abelian pro-p extension of $K_{\infty,r}$. Let $\mathbf{M}_r = \mathrm{Gal}(K_{\infty,r}^{ab,r}/K_{\infty,r})$ and $\Lambda_r = \varprojlim \mathbf{Z}_p[\mathscr{G}_r/p^n\mathscr{G}_r]$. If we choose topological generators for $\mathscr{G}_r$, then Λ_r can be identified with $\mathbf{Z}_p[|X_1, ..., X_r|]$. Note that $\mathbf{M}_r$ is naturally a Λ_r module which we call the Iwasawa module. When $r=1$, Iwasawa has determined the Λ_r-module structure of $\mathbf{M}_r$. In this note we determine the rank and depth of the Λ_r-module $\mathbf{M}_r$ for any integer $r \geqslant 1$. For the definition of depth of a module see [S].

For related results, we mention the works of J. P. Wintenberger [W] and Nguyen-Quang-Do [N]. J. P. Wintenberger [W] has considered Galois extensions of K (K as above) whose Galois group is isomorphic to $\mathbf{Z}_p^r \times G$, where G is a finite group of order prime to p. In particular, he has shown that the corresponding Λ_ϕ-module $\mathbf{M}_\phi$ is a free module of rank d for any irreducible character ϕ of G other than trivial and cyclotomic character (see [W, Theorem 4.1]).

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1. Structure of the Iwasawa Module for $r = 1$

Here we recall the result of Iwasawa (see [I, p. 316–320]). Let K be as in the introduction and $K_{\infty,1}$ be a $\mathbf{Z}_p$-extension of K. Then $\mathbf{M}_1 = \mathrm{Gal}(K_{\infty,1}^{ab,p}/K_{\infty,1})$ be the corresponding $\Lambda_1 \simeq \mathbf{Z}_p[|X|]$ module.

Let $d = [K : \mathbf{Q}_p]$, $\bar{K}$ be the algebraic closure of K, $\mathbf{W}_n =$ set of p^nth roots of unity in $\bar{K}$ and $\mathbf{W}_\infty = \bigcup_{n \geqslant 0} \mathbf{W}_n$. For any field $L \subset \bar{K}$ set $\mathbf{W}_L = \mathbf{W}_\infty \cap L$.

Theorem (Iwasawa). *The $\Lambda_1 \simeq \mathbf{Z}_p\,[|X|]$-module structure of $\mathbf{M}_1$ is as follows:*

(1) *Suppose $K_{\infty,1} \neq K(\mathbf{W}_\infty)$, i.e., $\mathbf{W}_{K_{\infty,1}} = \mathbf{W}_\infty \cap K_{\infty,1} \neq \mathbf{W}_\infty$, then*

$$\mathbf{M}_1 \subseteq \Lambda_1^d \quad \text{and} \quad \Lambda_1^d/\mathbf{M}_1 \simeq \mathbf{W}_{K_{\infty,1}}$$

(2) *Suppose $K_{\infty,1} = K(\mathbf{W}_\infty)$, i.e., $\mathbf{W}_{K_{\infty,1}} = \mathbf{W}_\infty \cap K_{\infty,1} = \mathbf{W}_\infty$, then*

$$\mathbf{M}_1 \simeq T(\mathbf{W}_\infty) \oplus \Lambda_1^d.$$

where $T(\mathbf{W}_\infty) = \varprojlim \mathbf{W}_n$ is the Tate module of $\mathbf{W}_\infty$.

Proof. See [I, Theorem 25]. ∎

Definition. If N is a finitely generated module over a domain A with quotient field L, then dimension of the vector space $N \otimes_A L$ over L is called the *rank* of N.

Corollary. *Notations as above*

(1) *If $\mathbf{W}_K = (e)$ then $\mathbf{M}_1$ is a free Λ_1-module of rank d.*

(2) *If $\mathbf{W}_K \neq (e)$ then $\mathbf{M}_1$ is not free Λ_1-module but* $\mathrm{depth}_{\Lambda_1}(\mathbf{M}_1) = 1$ *and* $\mathrm{rk}_{\Lambda_1}(\mathbf{M}_1) = d$, *where* $\mathrm{depth}_{\Lambda_1}(\mathbf{M}_1) :=$ *depth of* $\mathbf{M}_1$ *and* $\mathrm{rk}_{\Lambda_1}(\mathbf{M}_1) :=$ *rank of* $\mathbf{M}_1$ *as a Λ_1-module.*

Proof. (1) follows from (1) of the theorem above.

(2) (a) If $\mathbf{W}_K \neq (e)$ and $K_{\infty,1} \neq K(\mathbf{W}_\infty)$ then by (1) of the theorem above we have an exact sequence of Λ_1-modules

$$0 \to \mathbf{M}_1 \to \Lambda_1^d \to \mathbf{W}_{K_{\infty,1}} \to 0.$$

From this sequence it follows that $\mathrm{depth}_{\Lambda_1}(\mathbf{M}_1) = 1$ and $\mathrm{rk}_{\Lambda_1}(\mathbf{M}_1) = \mathrm{rk}_{\Lambda_1}(\Lambda_1^d) = d$.

(b) If $K_{\infty,1} = K(\mathbf{W}_\infty)$ then by (2) of the theorem above we have

$$\mathbf{M}_1 \simeq T(\mathbf{W}_\infty) \oplus \Lambda_1^d$$

hence it is clear that $\mathrm{depth}_{\Lambda_1}(\mathbf{M}_1) = 1$ and $\mathrm{rk}_{\Lambda_1}(\mathbf{M}_1) = d$. ∎

2. Structure of the Λ_r Module $\mathbf{M}_r$ for $r \geqslant 2$

To study the Λ_r-module structure of $\mathbf{M}_r$, we use induction on r; for this we need the following:

LEMMA 1. *Let $\mathcal{G}_r$, $K_{\infty,r}$, $K_{\infty,r}^{ab,p}$, Λ_r, and $\mathbf{M}_r$ be as in the introduction with $r \geqslant 2$. Let H be a subgroup of $\mathcal{G}_r$ such that $\mathcal{G}_r/H \simeq \mathbf{Z}_p^{r-1}$. Let $K_{\infty,r-1} = K_{\infty,r}^H$ be the fixed field of H. Then $\mathrm{Gal}(K_{\infty,r-1}/K) = \mathcal{G}_{r-1} = \mathcal{G}_r/H$. If v is a topological generator of H then $\Lambda_r/(v-1) \simeq \Lambda_{r-1}$ and $v-1$ is injective on $\mathbf{M}_r$ and there is an exact sequence of Λ_{r-1}-modules*

$$0 \to \mathbf{M}_r/(v-1)\,\mathbf{M}_r \to \mathbf{M}_{r-1} \to \mathbf{Z}_p \to 0,$$

where $\mathbf{M}_{r-1} = \mathrm{Gal}(K_{\infty,r-1}^{ab,p}/K_{\infty,r-1})$ and $\mathbf{Z}_p$ is the Λ_{r-1}-module with trivial $\mathcal{G}_{r-1}$ action.

Proof. See [W, Lemma 5.2]. ∎

Remark. By Lemma 1 and the fact about compact Λ_r-modules (see [G, p. 87]) it follows that $\mathbf{M}_r$ is a finitely generated Λ_r-module.

THEOREM 1. *Let $\mathbf{M}_r$ and Λ_r be as above then*

$$\mathrm{rk}_{\Lambda_r}(\mathbf{M}_r) = d,$$

where $d = [K : \mathbf{Q}_p]$.

Proof. Proof is by induction on r. If $r = 1$ then the theorem follows from the explicit structure theorem of the Iwasawa module given by the theorem of Iwasawa quoted above. So we assume $r \geqslant 2$ and the result holds for $r-1$. Now with the notations of Lemma 1 we have an exact sequence of Λ_{r-1}-modules

$$0 \to \mathbf{M}_r/(v-1)\,\mathbf{M}_r \to \mathbf{M}_{r-1} \to \mathbf{Z}_p \to 0.$$

This exact sequence together with the assumption that $r \geqslant 2$ and induction gives $\mathrm{rk}_{\Lambda_{r-1}}(\mathbf{M}_r/(v-1)\,\mathbf{M}_r) = \mathrm{rk}_{\Lambda_{r-1}}(\mathbf{M}_{r-1})$. Since there are infinitely many subgroups of the type H in $\mathcal{G}_r$ the theorem follows from induction and Lemma 2 below. ∎

LEMMA 2. *Let $\mathbf{M}$ be a finitely generated Λ_r-module such that there are infinitely many height 1 prime ideals P of Λ_r with $\Lambda_r/P \simeq \Lambda_{r-1}$ and $P = (t)$ with t injective on $\mathbf{M}$ and $\mathbf{M}/(t)\,\mathbf{M}$ is a Λ_{r-1}-module of fixed rank say d. Then rank of $\mathbf{M}$ as Λ_r-module is d.*

Proof. Let

$$V_{\mathbf{M}}^{d}=\{P\in \mathrm{Spec}(\Lambda_r)\mid \mathrm{rk}_{\Lambda r/P}(\mathbf{M}/P\mathbf{M})\geqslant d\}.$$

Then upper semi continuity of the rank function

$$\mathrm{rk}_{\mathbf{M}}:\mathrm{Spec}(\Lambda_r)\to \mathbf{Z}$$

defined by **M** (See [H, p. 288]) $V_{\mathbf{M}}^{d}$ is a closed subset of $\mathrm{Spec}(\Lambda_r)$. By our assumption on **M** we must have $V_{\mathbf{M}}^{d}=\mathrm{Spec}(\Lambda_r)$. Now the lemma follows from our assumption on **M**. ▌

We next prove some results about depth of Λ_r-module $\mathbf{M}_r$. For the definition of depth and homological-dimension see [S].

THEOREM 2. *Let* $\mathbf{M}_r$ *and* Λ_r *be as above. Then we have the following*:

(a) *If* $r=1$ *and if* $\mathbf{W}_{K_{\infty,1}}=(e)$ *then* $\mathbf{M}_1$ *is a free* Λ_1*-module, otherwise* $\mathrm{depth}_{\Lambda_1}(\mathbf{M}_1)=1$.

(b) *If* $r=2$ *and if* $\mathbf{W}_{K_{\infty,2}}=(e)$ *then* $\mathbf{M}_2$ *is a free* Λ_2*-module, otherwise* $\mathrm{depth}_{\Lambda_2}(\mathbf{M}_2)=2$.

(c) *If* $r\geqslant 3$ *then* $\mathrm{depth}_{\Lambda_r}(\mathbf{M}_r)=3$.

Proof. (a) If $r=1$ then the result follows from the theorem of Iwasawa quoted at the beginning of this paper.

(b) Let $r=2$ and let $H\subset \mathscr{G}_2$ be a subgroup such that $\mathscr{G}_2/H\simeq \mathscr{G}_1$. Then by Lemma 1 we have an exact sequence

$$0\to \mathbf{M}_2/(v-1)\,\mathbf{M}_2\to \mathbf{M}_1\to \mathbf{Z}_p\to 0\cdots(*)$$

of Λ_1-modules, where v is a topological generator of H. Now from (a) we have $\mathrm{depth}_{\Lambda_1}(\mathbf{M}_1)$ is 1 or 2. Also we have $\mathrm{depth}_{\Lambda_1}(\mathbf{Z}_p)=1$. On the other hand if (R,m) is a regular local ring of dimension r and N is a finitely generated R module then

$$\mathrm{depth}_R(N)+\mathrm{hd}_R(N)=\dim(R)=r,$$

where $\mathrm{hd}_R(N)=$ homological dimension of N. (See [S, iv-35]).
Note that $\mathrm{hd}_R(N)=\sup\{i\mid \mathrm{Tor}_i^R(N,R/m)\neq 0\}$. (See [S, iv-34]).

Case (1). Let $\mathbf{W}_{K_{\infty},2}=(e)$.

Then $\mathbf{M}_2$ is a free Λ_2-module (see [W, Theorem 4.1(i)]).

Case (2). Let $\mathbf{W}_{K_{\infty},2}\neq(e)$.

But then $\mathbf{W}_K \neq (e)$(this is because

$$\mathrm{Gal}(K_{\infty,2}/K) \simeq \mathbf{Z}_p^2 \qquad \text{and} \qquad \mathrm{Gal}(\mathbf{Q}_p(\mathbf{W}_n)/\mathbf{Q}_p) \simeq (\mathbf{Z}_p/(p^n))^*$$

for $n \geqslant 1$), hence $\mathbf{W}_{K_{\infty,1}} \neq (e)$. Thus by (a) and the sequence $(*)$ we obtain

$$\mathrm{depth}_{\Lambda_1}(\mathbf{M}_1) = 1 = \mathrm{depth}_{\Lambda_1}(\mathbf{Z}_p) \qquad \text{and} \qquad \mathrm{hd}_{\Lambda_1}(\mathbf{M}_1) = 1 = \mathrm{hd}_{\Lambda_1}(\mathbf{Z}_p)$$

Now tensoring the exact sequence $(*)$ with the residue field k of Λ_1 we obtain an exact sequence,

$$0 \to \mathrm{Tor}_1^{\Lambda_1}(\mathbf{M}_2/(v-1)\,\mathbf{M}_2, k) \to \mathrm{Tor}_1^{\Lambda_1}(\mathbf{M}_1, k) \to \cdots \to k \to 0.$$

This exact sequence together with the fact

$$\mathbf{M}_2/(v-1)\,\mathbf{M}_2 \otimes k \to \mathbf{M}_1 \otimes k$$

is not injective gives $\mathrm{hd}_{\Lambda_1}(\mathbf{M}_2/(v-1)\,\mathbf{M}_2) = 1$ (see [S, iv-28]) hence $\mathrm{depth}_{\Lambda_1}(\mathbf{M}_2/(v-1)\,\mathbf{M}_2) = 1$. Again using the fact that $v-1$ is injective on $\mathbf{M}_2$ we obtain $\mathrm{depth}_{\Lambda_2}(\mathbf{M}_2) = 2$ this proves (b).

(c) Let $r \geqslant 3$. Let H be a subgroup of $\mathscr{G}_r$ such that $\mathscr{G}_r/H \simeq \mathbf{Z}_p^{r-1}$. Again by Lemma 1 we have an exact sequence

$$0 \to \mathbf{M}_r/(v-1)\,\mathbf{M}_r \to \mathbf{M}_{r-1} \to \mathbf{Z}_p \to 0$$

of Λ_{r-1}-modules. This exact sequence gives to a long exact sequence:

$$\mathrm{Ext}^1_{\Lambda_{r-1}}(k, \mathbf{M}_{r-1}) \to \mathrm{Ext}^1_{\Lambda_{r-1}}(k, \mathbf{Z}_p) \to \mathrm{Ext}^2_{\Lambda_{r-1}}(k, \mathbf{M}_r/(v-1)\,\mathbf{M}_r) \to \cdots,$$

where k is the residue field of Λ_{r-1}. Since $r \geqslant 2$ by induction we may assume that $\mathrm{depth}_{\Lambda_{r-1}}(\mathbf{M}_{r-1}) \geqslant 2$. Hence $\mathrm{Ext}^1_{\Lambda_{r-1}}(k, \mathbf{M}_{r-1}) = 0$. But $\mathrm{depth}_{\Lambda_{r-1}}(\mathbf{Z}_p) = 1$ implies $\mathrm{Ext}^1_{\Lambda_{r-1}}(k, \mathbf{Z}_p) \neq 0$ so by the long exact sequence we obtain $\mathrm{Ext}^2_{\Lambda_{r-1}}(k, \mathbf{M}_r/(v-1)\,\mathbf{M}_r) \neq 0$ hence $\mathrm{depth}_{\Lambda_{r-1}}(\mathbf{M}_r/(v-1)\,\mathbf{M}_r) = 2$. Now again by Lemma 1, $v-1$ is injective on $\mathbf{M}_r$, hence we obtain

$$\mathrm{depth}_{\Lambda_r}(\mathbf{M}_r) = 3.$$

This proves (c) and hence the theorem. ▮

COROLLARY. *Notations as above. If* $r \geqslant 3$ *then* $\mathbf{M}_r$ *is not a free* Λ_r*-module.*

Proof. If $r \geqslant 3$ then $\dim \Lambda_r = r+1 \geqslant 4$, on the other hand a free Λ_r-module has depth $r+1 \geqslant 4$. But by Theorem 2 $\mathrm{depth}_{\Lambda_r}(\mathbf{M}_r) = 3$; hence $\mathbf{M}_r$ is not free. ▮

The above corollary has been remarked by both J. P. Wintenberger [W] and Nguyen-Quang-Do [N].

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