Mappings Induced by PGSM-Mappings and Some Recursively Unsolvable Problems of Finite Probabilistic Automata

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A probabilistic event is a fuzzy set of tapes accepted by a finite probabilistic automaton. In this paper, a mapping induced by a PGSM-mapping is introduced and is shown to be one of the operations which preserve a probabilistic event. Using mappings induced by a GSM-mapping, some recursively unsolvable problems with respect to rational finite probabilistic automata are studied.

INTRODUCTION

A probabilistic event is a fuzzy set of tapes accepted by a finite probabilistic automaton. The family of probabilistic events was shown to be closed under some operations, i.e., mean, convex combination and transposition in our previous paper.

In this paper another operation, i.e., a mapping induced by a PGSM-mapping is introduced and is shown to preserve probabilistic events.

Using mappings induced by a GSM-mapping a subfamily of sets of tapes accepted by a finite probabilistic automaton with a cut-point is studied. And by the discussion similar to the one in Bar-Hillel et al., it is shown that some decision problems with respect to rational finite probabilistic automata are recursively unsolvable, that is, it is recursively unsolvable to determine for arbitrary rational finite probabilistic automaton $\mathcal{A}$ and a rational cut-point $\lambda$, (1) whether the set of tapes accepted by $\mathcal{A}$ with a cut-point $\lambda$, $\beta(\mathcal{A}, \lambda)$ is empty, (2) whether $\beta(\mathcal{A}, \lambda)$ is $\Sigma^*$, and (3) whether $\beta(\mathcal{A}, \lambda)$ is regular, and it is recursively unsolvable to determine for arbitrary rational probabilistic events $p$ and $q$ whether $p \lor q$ is a probabilistic event and whether $p \land q$ is a probabilistic event.

1. PRELIMINARIES

Some definitions and results in our previous paper will be described in this section. Let $\Sigma$ be a finite nonempty set and be called an alphabet. The
set of all finite sequences of elements of $\Sigma$ is denoted by $\Sigma^*$. The null sequence $\Lambda$ is also an element of $\Sigma^*$. The elements of $\Sigma^*$ will be called tapes or words. Subsets of $\Sigma^*$ will sometimes be called events or languages.

**Definition 1.** A fuzzy event $f$ is a mapping from $\Sigma^*$ into $[0, 1]$. The family of all fuzzy events is denoted by $\mathcal{F}$.

**Definition 2.** Let $f$ and $g$ be fuzzy events. The union of $f$ and $g$ is defined as the fuzzy event $h$ such that $h(x) = \max(f(x), g(x))$ for all $x \in \Sigma^*$ and is denoted by $f \lor g$. The intersection of $f$ and $g$ is defined as the fuzzy event $i$ such that $i(x) = \min(f(x), g(x))$ for all $x \in \Sigma^*$ and is denoted by $f \land g$.

**Definition 3.** Let $f$ be a fuzzy event. The complement of $f$ is denoted by $\bar{f}$, and is defined by $\bar{f}(x) = 1 - f(x)$ for all $x \in \Sigma^*$.

**Definition 4.** The transpose of a fuzzy event $f$ is denoted by $f^T$ and is defined by $f^T(x) = f(x^T)$ for all $x \in \Sigma^*$, where $x^T$ is the transpose of a tape $x$ and is defined by $A^T = A$ and $(x\cdot\cdot\cdot)^T = x^T$ for $x \in \Sigma^*$ and $\sigma \in \Sigma$.

**Definition 5.** Let $f_1, \ldots, f_k$ be $k$-fuzzy events, and let $\theta_1, \ldots, \theta_k$ be $k$-real numbers such that $\theta_i \geq 0$, $i = 1, \ldots, k$, $\sum_{i=1}^{k} \theta_i = 1$. The mean of $f_1, \ldots, f_k$ by $\theta_1, \ldots, \theta_k$ is denoted by $\sum_{i=1}^{k} \theta_i f_i$ and is defined by $(\sum_{i=1}^{k} \theta_i f_i)(x) = \sum_{i=1}^{k} \theta_i f_i(x)$ for all $x \in \Sigma^*$.

**Definition 6.** Let $f$, $g$ and $h$ be fuzzy events. The convex combination of $f$, $g$ and $h$ is denoted by $(f, g; h)$ and is defined as $(f, g; h)(x) = h(x) \cdot f(x) + h(x) \cdot g(x)$ for all $x \in \Sigma^*$.

**Definition 7.** A mapping $I$ from an enumerable set $B$ into $[0, 1]$ is called a distribution of $B$, if $\sum_{b \in B} I(b) = 1$. Let $\mathcal{S}_B$ be the set of all distributions of $B$. A probabilistic mapping from a set $A$ into $B$ is a mapping from $A$ into $\mathcal{S}_B$.

Let $M$ be a probabilistic mapping from $A$ into $B$, then for $a$ in $A$, $M(a)$ is a distribution of $B$ and the value of $M(a)$ at $b$ in $B$, $M(a)(b)$ is the probability for $a$ to be mapped to $b$. The $M$ is a mapping if it holds for all $a$ in $A$ that $M(a)(b) = 1$ for some $b$ in $B$. In this case we will use the usual notation such that $M(a) = b$.

**Definition 8.** A finite probabilistic automaton, abbreviated by $FPA$, over $\Sigma$ is a system $\mathcal{A} = \langle S, M, I, F \rangle$, where $S$ is a finite nonempty set of internal states, $M$ is a probabilistic mapping from $S \times \Sigma$ into $S$, $I$ is a distribution of $S$ and $F$ is a subset of $S$. When $M$ is a mapping in the usual sense and $I$ concentrates into one state $s_0$ in $S$, $\mathcal{A}$ is a finite automaton and is represented by $\mathcal{A} = \langle S, M, s_0, F \rangle$ as usual.
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It is convenient to represent \( A \) by vectors and matrices for deriving the probability associated with input tapes of \( \Sigma^* \).

**Definition 9.** Let \( S \) have \( n \) states \( s_1, \ldots, s_n \). For \( \sigma \in \Sigma \) and \( x = \sigma_1 \cdots \sigma_m \in \Sigma^* \), \( \sigma_i \in \Sigma \cup \{ A \} \), \( 1 \leq i \leq m \), define \( n \times n \) stochastic matrices \( A(\sigma) \) and \( A(x) \) as

\[
A(\sigma) = [a_{i,j}(\sigma)]_{1 \leq i \leq n, 1 \leq j \leq n}
\]

where

\[
a_{i,j}(\sigma) = M(s_i, \sigma)(s_j),
\]

\[
A(A) = E_n \quad \text{and} \quad A(x) = A(\sigma_1) \cdots A(\sigma_m),
\]

where \( E_n \) is the \( n \times n \) unit matrix.

Define an initial state designator \( \pi \) and a final states designator \( \eta^F \) such that

\[
\pi = (\pi_{s_1}, \ldots, \pi_{s_n}), \quad \pi_{s_i} = I(s_i), \quad 1 \leq i \leq n. \quad \eta^F = (\eta_{s_1}, \ldots, \eta_{s_n}),
\]

\[
\eta_{s_i} = 1 \quad \text{if} \quad s_i \in F \quad \text{and} \quad \eta_{s_i} = 0 \quad \text{if otherwise}, \quad 1 \leq i \leq n.
\]

Since \( A \) can be completely characterized by \( S, \{ A(\sigma) \mid \sigma \in \Sigma \}, \pi, \eta^F \), \( A \) will be represented by \( A = \langle S, \{ A(\sigma) \mid \sigma \in \Sigma \}, \pi, \eta^F \rangle \) in the following discussion.

**Definition 10.** A fuzzy event \( p \) is a probabilistic event if there exists a FPA \( A = \langle S, \{ A(\sigma) \mid \sigma \in \Sigma \}, \pi, \eta^F \rangle \) such that for all \( x \in \Sigma^* \), \( p(x) = \pi A(x)\eta^F \), and \( p \) is said to be realized by \( A \). The family of all probabilistic events is represented by \( \mathcal{P} \).

**Proposition 1.** (Paz) If \( p \in \mathcal{P} \), then \( \tilde{p} \in \mathcal{P} \).

**Proposition 2.** If \( 0 \leq \theta_1, \ldots, \theta_k \), \( \sum_{i=1}^{k} \theta_i = 1 \), and \( p_i \in \mathcal{P}, 1 \leq i \leq k \), then \( \sum_{i=1}^{k} \theta_i p_i \in \mathcal{P} \).

**Theorem 3.** If \( p, q \) and \( r \) are elements of \( \mathcal{P} \), then \( (p, q; r) \in \mathcal{P} \).

**Theorem 4.** If \( p, q \in \mathcal{P} \) and \( \{ x \in \Sigma^* \mid p(x) > q(x) \} \) is a regular set, then \( p \lor q \in \mathcal{P} \) and \( p \land q \in \mathcal{P} \).

**Theorem 5.** If \( p \in \mathcal{P} \), then \( p^T \in \mathcal{P} \).

**Definition 11.** Let \( p \) be realized by a FPA \( A \). For a real number \( 0 \leq \lambda \leq 1 \), a subset of \( \Sigma^* \), \( \{ x \in \Sigma^* \mid p(x) > \lambda \} \) is represented by \( \beta(A, \lambda) \) and is called the set accepted by \( A \) with cutpoint \( \lambda \). This subset is also represented by \( \beta(p, \lambda) \) and is called the set determined by \( p \) and \( \lambda \).

When \( A \) is a finite automaton, \( \beta(A, \lambda) \) is the same for all \( \lambda \) such that \( 0 \leq \lambda < 1 \) and will be denoted by \( \beta(A) \).
2. MAPPINGS INDUCED BY A PGSM-MAPPING

A GSM-mapping is a very important operation which preserves regular sets and context-free languages. In this section we consider the generalized sequential machine which moves in a probabilistic way, say a probabilistic generalized sequential machine, abbreviated by PGSM. The concept of a mapping induced by a PGSM-mapping will be introduced, and it will be shown that a mapping induced by a PGSM-mapping preserves probabilistic events.

**Definition 12.** Let \( \mathcal{F}[\Sigma^*] \) and \( \mathcal{F}[^*] \) be the families of all fuzzy events over alphabets \( \Sigma \) and \( ^* \) respectively. Let \( \Gamma \) be a probabilistic mapping from \( \Sigma^* \) into \( ^* \). For any \( f \in \mathcal{F}[^*] \) define \( f' \in \mathcal{F}[\Sigma^*] \) such that \( f'(x) = \sum_{y \in \Delta^*} \Gamma'(x)(y) \cdot f(y) \) for all \( x \in \Sigma^* \). The mapping from \( \mathcal{F}[\Delta^*] \) into \( \mathcal{F}[\Sigma^*] \) which maps \( f \) to \( f' \) is denoted by \( \hat{\Gamma} \) and called a mapping induced by \( \Gamma \). If \( \Gamma \) is a mapping in the usual sense, \( \hat{\Gamma} \) is determined by the usual notation such that \( \hat{\Gamma}(f)(x) = f(\Gamma(x)), (x \in \Sigma^*) \). When \( \Gamma \) is a deterministic mapping, we will use this notation.

**Definition 13.** A PGSM is a system \( \mathcal{G} = (S, \Sigma, \Delta, \delta, \lambda, J) \), where \( S \) is a finite nonempty set of states, \( \Sigma \) and \( \Delta \) are finite nonempty sets called an input alphabet and an output alphabet respectively, \( \delta \) is a probabilistic mapping from \( S \times \Sigma \) into \( S \), \( \lambda \) is a probabilistic mapping from \( S \times \Sigma \) into \( \Delta^* \), and \( J \) is a distribution of \( S \) called an initial distribution. When \( \delta \) and \( \lambda \) are mappings in the usual sense and \( J \) concentrates into one state, say \( s_0 \), \( \mathcal{G} \) is a GSM and is represented by \( \mathcal{G} = (S, \Sigma, \Delta, \delta, \lambda, s_0) \).

**Definition 14.** For a PGSM \( \mathcal{G} = (S, \Sigma, \Delta, \delta, \lambda, J) \), the probabilistic mapping realized by \( \mathcal{G} \) is defined by

\[
\Gamma(x)(y) = \sum_{s \in S, t \in S} J(s) \cdot f_s(x)(t, y), \quad (x \in \Sigma^*, y \in \Delta^*),
\]

where for \( s \in S \), \( f_s \) is a probabilistic mapping from \( \Sigma^* \) into \( S \times \Delta^* \) defined by the following (i) and (ii) inductively. (i) \( f_s(\Lambda)(s, \Lambda) = 1 \), (ii) \( f_s(x \sigma)(t, y) = \sum_{r \in S, y_1, y_2 = y} f_s(x)(r, y_1) \cdot \lambda(r, \sigma)(y_2) \cdot \delta(r, \sigma)(t), x \in \Sigma^*, \sigma \in \Sigma, y \in \Delta^* \). A probabilistic mapping \( \Gamma \) is called a PGSM-mapping if it is realized by some PGSM and especially \( \Gamma \) is called a GSM-mapping as usual if \( \Gamma \) is realized by a GSM. Let \( \Gamma \) be the PGSM-mapping realized by a PGSM \( \mathcal{G} \), then for \( x \) in \( \Sigma^* \) and \( y \) in \( \Delta^* \), \( \Gamma(x)(y) \) is the probability that the output tape of \( \mathcal{G} \) for an input tape \( x \) is \( y \).

**Theorem 6.** A mapping induced by a PGSM-mapping from \( \Sigma^* \) into \( \Delta^* \) maps a probabilistic event over \( \Delta^* \) to a probabilistic event over \( \Sigma^* \).
The structural image of this theorem is that the cascade connection of a FPA with a PGSM may be considered as a FPA. The proof will be made along this image.

Proof of Theorem 6. For a PGSM-mapping $\Gamma$ from $\Sigma^*$ into $\Delta^*$, $\hat{\Gamma}$ is a mapping from $\mathcal{G}[\Delta^*]$ into $\mathcal{G}[\Sigma^*]$ by the definition. In order to see that $\hat{\Gamma}(p)$ is a probabilistic event for any probabilistic event $p$, we will construct the FPA over $\Sigma$ which realizes $\hat{\Gamma}(p)$.

Let $\mathcal{G} = \langle S = \{s_1, \ldots, s_m\}, \Sigma, \Delta, \delta, \lambda, J \rangle$ be the PGSM which realizes $\Gamma$ and let $\mathcal{F}_\Delta = \langle U = \{u_1, \ldots, u_n\}, \{A(\xi) | \xi \in \Delta\}, \tau, \eta^p \rangle$ be the FPA which realizes $p$. Let $\mathcal{F}_\Sigma$ be the FPA $\langle K, \{D(\sigma) | \sigma \in \Sigma\}, \Pi, \Pi \rangle$, where $K = S \times U$, and $D(\sigma)$'s, $\Pi$ and $\Pi$ are given by the following (i), (ii) and (iii) respectively.

(i) For $(s, \sigma) \in S \times \Sigma$, define the $m \times m$ matrix $B(s, \sigma)$ as
\[
B(s, \sigma) = \sum_{y \in \Delta^*} \lambda(s, \sigma)(y) \cdot A(y).
\]

Let $B(s, \sigma) = (b_{u,v}(s, \sigma))_{u \in U, v \in U}$, then an $mn \times mn$ matrix $D(\sigma)$ is defined by
\[
D(\sigma) = (d_{(s,u),(t,v)}(\sigma))_{(s,u),(t,v) \in K}
\]
\[
d_{(s,u),(t,v)}(\sigma) = \delta(s, \sigma)(t) \cdot b_{u,v}(s, \sigma)
\]

the subscripts $(s, u)$, $(t, v)$ of $D$ are in alphabetical order: $(s, u)$, $(t, v) = (s_1, u_1), (s_1, u_2), \cdots, (s_1, u_n), (s_2, u_1), (s_2, u_2), \cdots, (s_n, u_n)$.

(ii) Let $\pi = (\pi_u)_{u \in U}$, then the $mn$ dimensinal row vector $\Pi$ is defined as
\[
\Pi = (\Pi_{(s,u)})_{(s,u) \in K}, \quad \Pi_{(s,u)} = J(s)\pi_u
\]

(iii) Let $\eta^F = (\eta_u)_{u \in U}$, then $mn$ dimensional column vector $\Pi$ is defined as
\[
\Pi = (H_{(s,u)})_{(s,u) \in K}, H_{(s,u)} = \eta_u
\]

To prove the theorem it suffices to show that for each word $x \in \Sigma^*$
\[
\hat{\Gamma}(p)(x) = \Pi D(x)\Pi.
\]

The $mn \times mn$ matrix $D(x)$ is regarded as an $m \times m$ matrix whose components are $n \times n$ submatrices that is, for $s, t \in S$, $D(x)$ is represented as $D(x) = (C_{s,t}(x))_{s \in S, t \in S}$, where
\[
C_{s,t}(\sigma) = B(s, \sigma) \cdot \delta(s, \sigma)(t).
\]
We will show by induction that
\[ C_{s,t}(x) = \sum_{y \in \Delta^*} f_*(y) \cdot A(y). \]  

(a) Since \( C_{s,t}(A) \) is the \( n \times n \) unit matrix if \( s = t \) and then \( n \times n \) zero matrix if \( s \neq t \), (1) holds when \( x = A \).

(b) Suppose (1) is true for a tape \( x \), then for \( \sigma \in \Sigma \),
\[
\sum_{y \in \Delta^*} f_*(x) \cdot A(y)
\]
\[
= \sum_{y \in \Delta^*} \sum_{r \in S} \sum_{v_1, v_2 = y} f_*(y_1, y) \cdot \lambda(r, \sigma) \cdot \delta(r, \sigma) \cdot A(y)
\]
\[
= \sum_{y \in \Delta^*} \sum_{v_1, v_2 = y} f_*(y_1) \cdot A(y_1) \cdot \lambda(r, \sigma) \cdot \delta(r, \sigma) \cdot A(y_2) \cdot \delta(r, \sigma) \cdot t
\]
\[
= \sum_{r \in S} C_{s,t}(x) \cdot C_{r,t}(\sigma) = C_{s,t}(x). 
\]
Thus (1) holds. It follows from (1) that
\[
\hat{\Gamma}(p)(x) = \sum_{y \in \Delta^*} \Gamma(x) \cdot p(y)
\]
\[
= \sum_{y \in \Delta^*} \sum_{s, t \in S} J(s) \cdot f_*(y) \cdot A(y) \cdot \eta^p
\]
\[
= \pi \sum_{s \in S, t \in S} J(s) \cdot C_{s,t}(x) \cdot \eta^p.
\]

Thus the theorem is proved.

When \( \Gamma \) is a GSM-mapping, the above theorem means the next corollary.

**Corollary 7.** If \( \Gamma \) is a GMS-mapping from \( \Sigma^* \) into \( \Delta^* \), for any \( \mathfrak{M}(\mathfrak{A}_\Delta) \) over \( \Delta \), there exists a FPA \( \mathfrak{M}(\mathfrak{A}_\Sigma) \) over \( \Sigma \) such that for any cut point \( 0 \leq \lambda \leq 1 \)
\[
\Gamma^{-1}(\beta(\mathfrak{A}_\Delta, \lambda)) = \beta(\mathfrak{A}_\Sigma, \lambda).
\]

**Proof.** Let \( p \) be realized by \( \mathfrak{A}_\Delta \). Then \( \hat{\Gamma}(p) \) is a probabilistic event, so that there exists a FPA \( \mathfrak{A}_\Sigma \) which realizes \( \hat{\Gamma}(p) \).
\[
\beta(\mathfrak{A}_\Sigma, \lambda) = \{ x \in \Sigma^* \mid \hat{\Gamma}(p)(x) > \lambda \}
\]
\[
= \{ x \in \Sigma^* \mid p(\Gamma(x)) > \lambda \}
\]
\[
= \Gamma^{-1}(\{ y \in \Delta^* \mid p(y) > \lambda \})
\]
\[
= \Gamma^{-1}(\beta(\mathfrak{A}_\Delta, \lambda)).
\]

Now let us compare this result with the following well-known theorem.

**Theorem** (Ginsburg and Rose). If \( \Gamma \) is a GSM-mapping from \( \Sigma^* \)
into $\Delta^*$, then

(i) For any finite automaton $A_\Delta$ over $\Delta$ there exists a finite automaton $A_\Sigma$ over $\Sigma$ such that

$$\Gamma^{-1}(\beta(A_\Delta)) = \beta(A_\Sigma).$$

(ii) For any finite automaton $A_\Sigma$ over $\Sigma$, there exists a finite automaton $A_\Delta$ over $\Delta$ such that

$$\Gamma(\beta(A_\Sigma)) = \beta(A_\Delta).$$

We see that Corollary 7 is the full extension of (i) of the above theorem. Here we have the new interesting problem whether (ii) of the above theorem is extended to the domain of FPA's in the same way. That is, let us consider the problem (ii').

(ii') For any FPA $A_\Sigma$ over $\Sigma$, is there a FPA $A_\Delta$ over $\Delta$ such that for any cut point $0 \leq \lambda \leq 1$ the following holds?

$$\Gamma(\beta(A_\Sigma, \lambda)) = \beta(A_\Delta, \lambda).$$

To consider this problem in terms of fuzzy events, we introduce some definitions.

**Definition 15.** For each element $a$ in $\Sigma$, let $\tau(a)$ be a word of $\Delta^*$. Let $\tau(\Delta) = \Delta$ and $\tau(x_1 \cdots x_r) = \tau(x_1) \cdots \tau(x_r)$ for each word $x_1, \cdots, x_r$ in $\Sigma^*$. Then $\tau$ is a mapping from $\Sigma^*$ into $\Delta^*$ and is called a homomorphism.

It is easily seen that a homomorphism is a GSM-mapping.

**Definition 16.** Let $\Gamma$ be a mapping from $\Sigma^*$ into $\Delta^*$. For each $f \in [\Sigma^*]$, define $f' \in [\Delta^*]$ by

$$f'(x) = \sup \{f(y) \mid y \in \Sigma^*, \Gamma(y) = x\}, \quad x \in \Delta^*.$$ 

The mapping from $[\Sigma^*]$ into $[\Delta^*]$ which maps $f$ to $f'$ is denoted by $\Gamma_\delta$. If $\Gamma$ is a GSM-mapping (homomorphism) from $\Sigma^*$ into $\Delta^*$, $\Gamma_\delta$ is called a GSM-mapping (homomorphism) from $[\Sigma^*]$ into $[\Delta^*]$.

**Lemma 8.** Let $\Gamma$ be a GSM-mapping. The problem (ii') is equivalent to the following problem (iii).

(iii) Is $\Gamma_\delta(p)$ a probabilistic event over $\Delta^*$ for any probabilistic event $p$ over $\Sigma^*$?

**Proof.** Assume that the answer to (ii') is yes. For any probabilistic event $p$ over $\Sigma^*$, let $p$ be realized by $A_\Sigma$. Then there exists a FPA $A_\Delta$ over
Δ and a probabilistic event q over Δ* which is realized by ΑΔ such that for any cut point 0 ≤ λ ≤ 1,

$$\Gamma(\beta(p, \lambda)) = \Gamma(\beta(\mathcal{A}_\Sigma, \lambda)) = \beta(\mathcal{A}_\Delta, \lambda) = \beta(q, \lambda).$$

For x ∈ Δ*, Γ_θ(p)(x) = Sup {p(y) | y ∈ Σ*, Γ(y) = x} > λ if and only if there exists y ∈ Σ* such that Γ(y) = x and p(y) > λ, that is, if and only if x ∈ Γ(β(p, λ)). This means that for any cut point 0 ≤ λ ≤ 1,

$$\beta(\Gamma_\theta(p), \lambda) = \Gamma(\beta(p, \lambda)).$$

Thus for any cut point 0 ≤ λ ≤ 1, β(Γ_θ(p), λ) = β(q, λ). This implies that Γ_θ(p) = q. Hence Γ_θ(p) is a probabilistic event, and the answer to (iii) is yes.

Assume that the answer to (iii) is yes, then it is clear that the answer to (ii') is yes.

In the followings, we shall see that, in general, the answer to (ii') and (iii) is no.

**Proposition 9.** For a probabilistic event p over Σ* and a homomorphism Γ_θ from $\mathcal{G}[\Sigma^*]$ into $\mathcal{G}[\Delta^*]$, Γ_θ(p) is not generally a probabilistic event.

**Proof.** Let Σ = {a, b, c} and Δ = {0, 1}. Consider the FPA

$$\mathcal{A}_\Sigma = \langle S, \{A(\sigma) | \sigma \in \Sigma\}, \pi, \eta^\pi \rangle$$

where $S = \{s_1, s_2, s_3, s_4\}$,

$$A(a) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A(b) = A(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$$

$$\pi = (\frac{1}{2} \ 0 \ \frac{1}{2} \ 0) \quad \text{and} \quad \eta^\pi = (1 \ 0 \ 0 \ 1)' .$$

For x ∈ Σ*, let $N_a(x)$, $N_b(x)$ and $N_c(x)$ be the numbers of occurrences of a, b and c respectively in x. Let r be the probabilistic event realized by $\mathcal{A}_\Sigma$, then for x in Σ*,

$$r(x) = \frac{1}{2} + \frac{1}{2} ((\frac{1}{2})^{N_a(x)} - (\frac{1}{2})^{N_b(x)+N_c(x)}).$$

Let $p = 1/2 - 2(r - 1/2)^2 = 2r(1-r) = (r, \tilde{r}; \tilde{r})$, then from Theorem 3, p is a probabilistic event, and

$$p(x) = \frac{1}{2} - \frac{1}{2} ((\frac{1}{2})^{N_a(x)} - (\frac{1}{2})^{N_b(x)+N_c(x)})^2.$$
Let $\Gamma$ be the homomorphism defined by $\Gamma(a) = 10$, $\Gamma(b) = 01$ and $\Gamma(c) = 1$.

We will show that $\Gamma_{\delta}(p)$ is not a probabilistic event.

Assume that $\Gamma_{\delta}(p)$ is a probabilistic event over $\Delta^*$, then there exists a FPA $\mathfrak{M}_\Delta$ over $\Delta$ which realizes $\Gamma_{\delta}(p)$. Let $\mathfrak{M}_\Delta = \langle T, \{B(0), B(1)\}, \pi_1, \eta^F \rangle$ and $\mathfrak{M}_\Delta$ have $n$ states. For any $i$ such that $1 \leq i \leq n$,

$$\Gamma(a^{n+i}cb^{n-i}c^{2i-1}) = (10)^{2n+1}2^i,$$

and

$$p(a^{n+i}cb^{n-i}c^{2i-1}) = \frac{1}{2}.$$ 

Thus for any $i$ such that $1 \leq i \leq n$,

$$\Gamma_{\delta}(p)((10)^{2n+1}2^i) = \frac{1}{2}, \quad (2)$$

because $1/2$ is the maximum value of $p$. Since $p(x) < 1/2$ for any $x \in \Sigma^*$ such that $\Gamma(x) = (10)^{2n+1}2^{(n+1)}$,

$$\Gamma_{\delta}(p)((10)^{2n+1}2^{(n+1)}) < \frac{1}{2}. \quad (3)$$

On the other hand, let the characteristic polynomial of $B(1^2)$ be

$$x^n - c_{n-1}x^{n-1} - \cdots - c_1x - c_0.$$

Then

$$c_0 + \cdots + c_{n-1} = 1,$$

and

$$\Gamma_{\delta}(p)((10)^{2n+1}2^{(n+1)}) = c_{n-1}\Gamma_{\delta}(p)((10)^{2n+1}) + \cdots + c_0\Gamma_{\delta}(p)((10)^{2n+1}).$$

Thus by (2)

$$\Gamma_{\delta}(p)((10)^{2n+1}2^{(n+1)}) = \frac{1}{2}.$$ 

This contradicts (3). Hence $\Gamma_{\delta}(p)$ is not a probabilistic event.

**Corollary 10.** For a GSM-mapping $\Gamma_{\delta}$ from $\mathfrak{F}[\Sigma^*]$ into $\mathfrak{F}[\Delta^*]$ and a probabilistic event $p$ over $\Sigma^*$, $\Gamma_{\delta}(p)$ is not generally a probabilistic event.

### 3. SOME RECURSIVELY UNSOLVABLE DECISION PROBLEMS

Consider a FPA $\langle S, \{A(\sigma) | \sigma \in \Sigma\}, \pi, \eta^F \rangle$ where every component of $A(\sigma)(\sigma \in \Sigma)$, $\pi$ and $\eta^F$ are rational numbers. Such a FPA will be called a rational finite probabilistic automaton, abbreviated by RFPA and a probabilistic event realized by a RFPA will be called a rational proba-
bilistic event, abbreviated by RPE. The set of all RPE's will be denoted by $\mathcal{R}$.

It is easily seen that Proposition 1 and Theorems 3, 4 and 5 are also valid if $\mathcal{B}$'s in those theorems are replaced by $\mathcal{B}_R$'s. Proposition 2 is also valid if $\theta_1, \cdots, \theta_k$ in the theorem are assumed to be rational numbers and $\mathcal{B}$'s are replaced by $\mathcal{B}_R$'s. It is clear from the construction of the proof of Theorem 6, that Theorem 6 may be replaced by the following.

If $\Gamma$ is realized by a PGSM $\mathcal{G} = \langle S, \Sigma, \Delta, \delta, \lambda, J \rangle$ such that $\delta(x, \sigma)(t)$, $\lambda(s, \sigma)(u)$, and $J(s)$ are rational numbers for all $s, t, \sigma \in \Sigma$ and $u \in \Delta^*$ and if $p$ is a RPE over $\Delta^*$ then $\Gamma(p)$ is a RPE over $\Sigma^*$.

In the following discussion, we will use those theorems in the above meaning.

**Definition 17.** The set of tapes $\beta$ is called a $P$-set if $\beta = \beta(p, \lambda)$ for some $p$ in $\mathcal{B}_R$ and a rational number $0 \leq \lambda \leq 1$. The $\beta$ is called an $E$-set if $\beta \in \{x \in \Sigma^* \mid p(x) = q(x)\}$ for some $p$ and $q$ in $\mathcal{B}_R$. The $\beta$ is called a $D$-set if $\beta = \{x \in \Sigma^* \mid p(x) \neq q(x)\}$ for some $p$ and $q$ in $\mathcal{B}_R$.

Each of a $P$-set, an $E$-set and a $D$-set is a recursive set, and furthermore recently it was given by Tokura et al. that each family of these is a proper subfamily of context-sensitive languages. Let $\mathcal{F}$, $\mathcal{E}$ and $\mathcal{D}$ be the family of all $P$-sets, $E$-sets and $D$-sets respectively.

**Lemma 11.** $\mathcal{E} \subseteq \mathcal{F}$ and $\mathcal{D} \subseteq \mathcal{F}$

**Proof:** If $p \in \mathcal{B}_R$ and $q \in \mathcal{B}_R$, then from Propositions 1 and 2

$$\frac{1}{2} + \frac{1}{2}(p - q) = \frac{1}{2}p + \frac{1}{2}q \in \mathcal{B}_R.$$

From Theorem 3,

$$2(p - \frac{1}{2})^2 + \frac{1}{2} = (p, \bar{p}; p) \in \mathcal{B}_R.$$

Replacing $p$ with $\frac{1}{2} + \frac{1}{2}(p - q)$, in the above equation we have

$$\frac{1}{2} + \frac{1}{2}(p - q)^2 \in \mathcal{B}_R.$$

Let $r$ be $\frac{1}{2} + \frac{1}{2}(p - q)^2$ and $\mathcal{H}_i = \langle S_i, \{A_i(\sigma) \mid \sigma \in \Sigma\}, \pi_i, \eta_i^R \rangle (i = p$ or $q)$ be a RFPA which realizes $i$ ($i = p$ or $q$). Let $M$ be the least common multiple of the denominators of the components of $A_i(\sigma)$'s and $\pi_i$'s ($i = p, q, \sigma \in \Sigma$). The $s$ in $\mathcal{H}$ defined as below is an element of $\mathcal{B}_R$.

$$s(x) = \frac{1}{2} + \frac{1}{4}(1/M^2)^{1(x)+1} (x \in \Sigma^*),$$
where \( l(x) \) is the length of a tape \( x \). Then we have
\[
s(x) > r(x) \iff \left( M^{l(x)+1} p(x) - M^{l(x)+1} q(x) \right)^2 \iff p(x) = q(x),
\]
because \( \left( M^{l(x)+1} p(x) - M^{l(x)+1} q(x) \right)^2 \) is a non-negative integer for all \( x \in \Sigma^* \). Let \( t \) be \( \frac{1}{2} + \frac{1}{2} (s - r) \), then \( t \) is in \( \mathcal{B}_R \) and
\[
t(x) > \frac{1}{2} \iff p(x) = q(x)
\]
\[
t(x) > \frac{1}{2} \iff p(x) \neq q(x).
\]
Hence each of an \( E \)-set and a \( D \)-set is a \( P \)-set.

Now we shall give a \( P \)-set which is neither an \( E \)-set nor a \( D \)-set. Let \( \Sigma = \{a, b\} \) and let \( p \) in \( \mathcal{B}_R \) be realized by the RFPA \( \mathfrak{A} = \langle S, \{A(a), A(b)\}, \pi, \eta^p \rangle \), where \( S = \{1, 2, 3, 4\} \),
\[
A(a) = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad A(b) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & \frac{1}{2} \\
0 & \frac{1}{2} & 1
\end{bmatrix}
\]
\[\pi = (\frac{1}{2} 0 \frac{1}{2} 0) \text{ and } \eta^p = (1 0 0 1)'.\]

Then for \( x \in \Sigma^* \),
\[
p(x) = \frac{1}{2} + \frac{1}{2} (\frac{1}{2} N_a(x) - \frac{1}{2} N_b(x)),
\]
where \( N_a(x) \) and \( N_b(x) \) are the numbers of occurrences of \( a \) and \( b \) in \( x \) respectively. Let \( \chi_a \) and \( \chi_b \) be the characteristic function of the regular set \( \alpha \) such that \( \alpha = \{a^m b^n \mid m \geq 0, n \geq 0\} \) and the empty set \( \emptyset \) respectively. Let \( q = p \cdot \chi_a = (p, \chi_b ; \chi_a) \), then \( q \in \mathcal{B}_R \). Let \( \beta \) be a \( P \)-set such that \( \beta = \beta(q, \frac{1}{2}) = \{a^m b^n \mid m < n\} \).

We shall show that \( \beta \) is not an \( E \)-set. Assume that \( \beta \) is an \( E \)-set, then there exist \( r \) and \( s \) in \( \mathcal{B}_R \) such that \( \beta = \{x \in \Sigma^* \mid r(x) = s(x)\} \). Let \( \mathfrak{A}_1 = \langle S_1, \{B(a), B(b)\}, \pi_1, \eta^{1} \rangle \) be a RFPA which realizes \( t \) such that \( t = \frac{1}{2} + \frac{1}{2} (r - s) \), and let \( \mathfrak{A}_1 \) have \( n \) states. Then
\[
t(b^{2n-1}) = t(a b^{2n-1}) = \cdots = t(a^{2(n-1)} b^{2n-1}) = \frac{1}{2}, \quad (4)
\]
and
\[
t(a^{2n} b^{2n-1}) \neq \frac{1}{2}. \quad (5)
\]

Let the characteristic polynomial of \( B(a^2) \) be \( U^n - C_{n-1} U^{n-1} - \cdots \)

\footnote{We use \( \iff \) to abbreviate "if and only if".}
- \( C_1 U - C_0 \), then

\[
C_0 + \cdots + C_{n-1} = 1, 
\]

and

\[
t(a^{2n} b^{2n-1}) = C_{n-1} t(a^{2(n-1)} b^{2n-1}) + \cdots + C_0 t(b^{2n-1}).
\]

Thus it follows from (4) that

\[
t(a^{2n} b^{2n-1}) = \frac{1}{2}.
\]

This contradicts (5). Hence \( \beta \) is not an \( E \)-set.

Similarly we can see that \( \bar{\beta} \) is not an \( E \)-set, so that \( \beta \) is not a \( D \)-set.

Matuura et al. and Turakainen studied linear space automata (or generalized automata) and found independently the same fact that the family of stochastic languages is the same as the family of languages accepted by linear space automata (or generalized automata). Matuura et al. gave us a key idea with respect to Lemma 11 in this viewpoint. \( P \)-sets, \( E \)-sets and \( D \)-sets can be studied generally in this viewpoint, and \( E \)-sets and \( D \)-sets can be related to the work of Schüzenberger (1961). The next remark and Lemma 12 follows from Schüzenberger's results but as for Lemma 12 we will give a straightforward proof.

**Remark.**

\[ \emptyset - \mathcal{D} \neq \emptyset \text{ and } \mathcal{D} - \emptyset \neq \emptyset. \]

**Lemma 12.** If \( \alpha, \beta \in \emptyset \), then \( \alpha \cap \beta \in \emptyset \) and \( \bar{\alpha} \in \mathcal{D} \).

**Proof.** It is clear that \( \bar{\alpha} \in \mathcal{D} \). From the assumption there exist \( p_\alpha \), \( q_\alpha \), \( p_\beta \) and \( q_\beta \) in \( \mathcal{P}_\alpha \) such that

\[
\alpha = \{ x \in \Sigma^* | p_\alpha(x) = q_\alpha(x) \} \quad \beta = \{ x \in \Sigma^* | p_\beta(x) = q_\beta(x) \}.
\]

Define \( r \) and \( s \) as

\[
r = \frac{1}{2}[\frac{1}{2} + \frac{1}{2}(p_\alpha - q_\alpha)^2 + \frac{1}{2} + \frac{1}{2}(p_\beta - q_\beta)^2],
\]

\[
s(x) = \frac{1}{2} \quad \text{for all} \quad x \in \Sigma^*,
\]

then \( r \) and \( s \) are in \( \mathcal{P}_\alpha \) and

\[
\alpha \cap \beta = \{ x \in \Sigma^* | r(x) = s(x) \}.
\]

Thus the lemma is proved.
PROPOSITION 13. If S and T are GSM-mappings, then

\[ \{ x \in \Sigma^* \mid S(x) = T(x) \} \in \mathcal{E}, \]
\[ \{ x \in \Sigma^* \mid S(x) \neq T(x) \} \in \mathcal{D}. \]

Proof. Let S and T be GSM-mappings from \( \Sigma^* \) into \( \Delta^* \), where \( \Delta = \{ \delta_1, \ldots, \delta_k \} \). Let \( p \) in \( \mathbb{P}_k \) be realized by the FPA \( \mathcal{A} = \langle \{1, 2\}, \{A(\delta_i) \mid i = 1, \ldots, k\}, \pi, \eta^p \rangle \), where

\[
A(\delta_i) = \begin{pmatrix}
\frac{k + 1 - i}{k + 1} & \frac{i}{k + 1} \\
\frac{k - i}{k + 1} & \frac{i + 1}{k + 1}
\end{pmatrix} \quad i = 1, \ldots, k,
\]

\( \pi = (1, 0) \) and \( \eta^p = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

Then we have

\[ P(\lambda) = 0 \]
\[ P(\delta_{i_1} \cdots \delta_{i_n}) = 0, \quad i_n \cdots i_1, \quad ((k + 1)\text{-adic expansion}). \]

Thus \( p \) is a one to one mapping from \( \Delta^* \) into \([0, 1]\).

From Theorem 6, \( \hat{S}(p) \) and \( \hat{T}(p) \) are RPE's over \( \Sigma^* \). And for \( x \in \Sigma^* \),

\[ \hat{S}(p)(x) = \hat{T}(p)(x) \iff p(S(x)) = p(T(x)) \]
\[ \iff S(x) = T(x) \]

Therefore the proposition holds.

In the following discussion the similar method as the one in the proof of Proposition 13 will be used essentially.

In the following, we will use some notations and results about context free languages according to Ginsburg.

A context free grammar is a 4-tuple \( G = (V, \Sigma, P, \sigma) \) where \( V \) is a finite nonempty set, \( \Sigma \) is a nonempty subset of \( V \) and \( P \) is a finite nonempty set of productions of the form \( \xi \to v \), with \( \xi \in V - \Sigma \) and \( v \in V^* \), and \( \sigma \in V - \Sigma \). For \( w_1 \) and \( w_2 \) in \( V^* \) write \( w_1 \Rightarrow w_2 \) if there exist \( u_1, u_2, \xi, v \) such that \( w_1 = u_1\xi u_2, w_2 = u_1v_2 \) and \( \xi \to v \) is in \( P \). A sequence of words \( w_0, \ldots, w_k \) such that \( w_i \Rightarrow w_{i+1} \) for each \( i \) is called a derivation.

For an arbitrary set \( E \) of symbols, \( E^* \) is the free semigroup with identity generated by \( E \).
of \( w_k \) (from \( w_0 \)) and is denoted \( w_0 \Rightarrow \cdots \Rightarrow w_k \). For \( G = (V, \Sigma, P, \sigma) \), \( L(G) \) is the set of all \( \Sigma \)-words of which a derivation from \( \sigma \) exists. The \( L(G) \) is said to be the language generated by \( G \).

Since \( \beta \) in the proof of Lemma 11 is a linear language acceptable by a deterministic pushdown automaton, a linear language acceptable by a deterministic pushdown automaton is not generally either an \( E \)-set or a \( D \)-set. Here we introduce a subfamily of linear languages acceptable by deterministic pushdown automata and show that it is contained in \( \mathcal{E} \).

**Definition 18.** A grammar \( G = (V, \Sigma, P, \sigma) \) is a deterministic linear grammar (abbreviated \( d.l. \) grammar) if each of its productions is of the following form:

(i) \( v \rightarrow a\xi u \) or \( v \rightarrow b \), where \( v \) and \( \xi \) are in \( V - \Sigma \), and \( b \) in \( \Sigma \) and \( u \) in \( \Sigma^* \), and

(ii) For any two productions \( v_1 \rightarrow a_1x \) and \( v_2 \rightarrow a_2y \), where \( x \) and \( y \) are in \( \{\Lambda\} \cup (V - \Sigma)\Sigma^* \), \( v_1 \) and \( v_2 \) in \( V - \Sigma \) and \( a_1 \) and \( a_2 \) in \( \Sigma \), if \( v_1 = v_2 \) and \( a_1 = a_2 \), then \( x = y \).

\( L \subseteq \Sigma^* \) is called a \( d.l. \) language, if \( L \) is generated by a \( d.l. \) grammar.

**Lemma 14.** Each \( d.l. \) language is an \( E \)-set.

**Proof.** Let a \( d.l. \) language \( L \) be generated by a \( d.l. \) grammar \( G = (V, \Sigma, P, \sigma) \). Let \( \mathcal{G} \) be the GSM \( \langle S, \Sigma, \Delta, \delta, \lambda, \sigma \rangle \), where \( S = (V - \Sigma) \cup \{f, d\} \), \( f \) and \( d \) are not in \( V \), \( \Delta = \Sigma \), and \( \delta \) and \( \lambda \) are defined as follows:

(i) If \( v \rightarrow a\xi u \in P \), where \( v \) and \( \xi \) are in \( V - \Sigma \), \( a \) in \( \Sigma \) and \( u \) in \( \Sigma^* \), let

\[
\delta(v, a) = \xi \quad \text{and} \quad \lambda(v, a) = u^T.
\]

(ii) If \( v \rightarrow a \) where \( v \in V - \Sigma \) and \( a \in \Sigma \), let

\[
\delta(v, a) = f \quad \text{and} \quad \lambda(v, a) = \Lambda.
\]

(iii) For \( v \in V - \Sigma \) and \( a \in \Sigma \) if there is no production of the form \( v \rightarrow ax \), let

\[
\delta(v, a) = d \quad \text{and} \quad \lambda(v, a) = \Lambda.
\]

(iv) Let \( \delta(f, a) = f \) and \( \lambda(f, a) = \Lambda \) for \( a \in \Sigma \).

(v) Let \( \delta(d, a) = d \) and \( \lambda(d, a) = \Lambda \) for \( a \in \Sigma \).

\(^3\) As for a linear language and a deterministic pushdown automaton, see Ginsburg.
Let $G'$ be the GSM $\langle S, \Sigma, \delta, \lambda', \sigma \rangle$. This $G'$ is the same as $G$ but $\lambda'$. The $\lambda'$ is defined as follows:

(i') For $\nu \in V - \Sigma$ and $a \in \Sigma$, let $\lambda'(\nu, a) = \Lambda$

(ii') Let $\lambda'(f, a) = a$ for $a \in \Sigma$.

(iii') Let $\lambda'(d, a) = \Lambda$ for $a \in \Sigma$.

Let $M$ and $N$ be GSM-mappings realized by $G$ and $G'$ respectively. Let $r$ be a one to one mapping from $\Sigma^*$ into $[0, 1]$ and in $\mathcal{P}_\infty$. (It is possible to define such $r$. For example, consider the probabilistic event $p$ in the proof of Proposition 13.) Let $\alpha$ be the set of tapes accepted by the finite automaton $A = \langle S, \Sigma, \delta, \{f\} \rangle$. Let $\chi_\alpha$ be the characteristic function of a set $\alpha$. Let

$$p = \hat{M}(r^x) \vee \chi_\alpha$$

$$q = \hat{N}(r) \wedge \chi_\alpha,$$

then it follows from Theorems 4, 5 and 6 that $p$ and $q$ are in $\mathcal{P}_\infty$.

To prove the theorem it suffices to show that for each $x \in \Sigma^*$, $p(x) = q(x)$ if and only if $x \in L$.

$$p(x) = q(x)$$

$\iff x \in \alpha$ and $\hat{M}(r^x)(x) = \hat{N}(r)(x)$

$\iff x \in \alpha$ and $r^x(M(x)) = r(N(x))$

$\iff x \in \alpha$ and $r(M(x)^x) = r(N(x))$

$\iff \delta(\sigma, x) = f$ and $M(x)^x = N(x)$.

Assume that $x \in L$, then two cases arise.

(1) If $x \in \Sigma$ and $\sigma \rightarrow x \in P$, then $\delta(\sigma, x) = f$, $M(x) = N(x) = \Lambda$

(2) If there exist $a_1, \ldots, a_n$ and $b$ in $\Sigma$, $\xi_1, \ldots, \xi_n$ in $V - \Sigma$ and $u_1, \ldots, u_n$ in $\Sigma^*$ such that

$\sigma \Rightarrow a_1 \xi_1 u_1 \Rightarrow \cdots \Rightarrow a_1 \cdots a_n \xi_n u_n \cdots u_1 \Rightarrow a_1 \cdots a_n b u_n \cdots u_1 = x$

then it follows from the construction of $G$ and $G'$ that

$\delta(\sigma, x) = f$, $M(x) = u_1^x \cdots u_n^x$ and $N(x) = u_n \cdots u_1$

that is,

$$\delta(\sigma, x) = f \quad M(x)^x = N(x).$$
Assume that \( \delta(\sigma, x) = f \) and \( x \) is not in \( L \), then there exist \( a_1, \ldots, a_n \) and \( b \) in \( \Sigma, \xi_1, \ldots, \xi_n \) in \( V - \Sigma \), and \( w, u_1, \ldots, u_n \) in \( \Sigma^* \) such that \( x = a_1 \cdots a_nbw \),

\[
\sigma \Rightarrow a_1\xi_1u_1 \Rightarrow \cdots \Rightarrow a_1 \cdots a_n\xi_nu_n \cdots u_1 \Rightarrow a_1 \cdots a_nbu_n \cdots u_1
\]

and \( w \neq u_n \cdots u_1 \).

Therefore \( M(x)^T \neq N(x) \), since \( M(x)^T = u_n \cdots u_1 \) and \( N(x) = w \). Thus the lemma is proved.

Let \( \Sigma' = \{a, b, c, d, a', b', c'\} \) and let \( h \) be the homomorphism from \( \{a, b, c\}^* \) into \( \{a', b', c'\}^* \) such that

\[
h(a) = a', \quad h(b) = b', \quad h(c) = c'.
\]

**Lemma 15.** \( L_\sigma = \{W_1cW_2 d h(W_2 c W_1)^T \mid W_1 \text{ and } W_2 \text{ in } \{a, b\}^*\} \) is a d.l. language and therefore is in \( \mathcal{E} \).

**Proof.** \( L_\sigma \) is generated by the d.l. grammar \( G = (\{\sigma, \xi, a, b, c, d, a', b', c\}, \{a, b, c, d, a', b', c\}, P, \sigma) \) where \( P = \{\sigma \rightarrow a\xi a', \sigma \rightarrow b\xi b', \sigma \rightarrow c\xi c', \xi \rightarrow a\xi a', \xi \rightarrow b\xi b', \xi \rightarrow d\} \).

Notation: for all \( n \)-tuples \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) of non-\( \Lambda \) words in \( \{a, b\}^* \), denote \( L(x) \) and \( L(x, y) \) the sets defined by

\[
L(x) = \{a^{i_1}b \cdots a^{i_k}bcx_{i_1} \cdots x_{i_k} \mid k \geq 1, 1 \leq i_j \leq n\}
\]

and

\[
L(x, y) = L(x) d h(L(y)^T)
\]

**Lemma 16.** \( L(x) \) is a d.l. language and therefore is in \( \mathcal{E} \).

**Proof.** \( L(x) \) is generated by the d.l. grammar

\[
G = (\{\sigma, \xi_0, \xi_1, \ldots, \xi_n, a, b, c\}, \{a, b, c\}, P, \sigma),
\]

where \( P = \{\sigma \rightarrow a\xi_1, \xi_0 \rightarrow a\xi_1, \xi_1 \rightarrow a\xi_2, \ldots, \xi_{n-1} \rightarrow a\xi_n, \xi_0 \rightarrow c, \xi_1 \rightarrow b\xi_0x_1, \xi_2 \rightarrow b\xi_0x_2, \ldots, \xi_n \rightarrow b\xi_0x_n\} \).

**Remark.** \( L(x) \) is not in \( \mathcal{D} \).

**Lemma 17.** \( L(x, y) \) is in \( \mathcal{E} \).

**Proof.** We will prove this lemma by giving RPE's \( p_0 \) and \( q_0 \) such that

\[
\{w \in \Sigma^* \mid p_0(w) = q_0(w)\} = L(x, y).
\]

At first, we will construct the RFPA' \( \mathcal{A}_p \) and \( \mathcal{A}_q \) which realize \( p \) and \( q \) in \( \mathcal{P}_R \) such that \( L(x) = \{x \in \{a, b, c\}^* \mid p(x) = q(x)\} \). The construction will be made along the
proof of Lemmas 14 and 16. Note that the output words of $\mathcal{G}$ and $\mathcal{G}'$ in Lemma 14 are in $\{a, b\}^*$ in this case. We define the one to one mapping $r$ from $\{a, b\}^*$ into $[0, 1]$ in Lemma 14 as the RPE realized by the RFPA $\mathfrak{A}_r$ such that

$$\mathfrak{A}_r = \langle \{1, 2\}, \{A(a), A(b)\}, \pi, \eta^r \rangle$$

$$A(a) = \begin{pmatrix} 8 & 1 \\ 9 & 9 \end{pmatrix}, \quad A(b) = \begin{pmatrix} 7 & 2 \\ 9 & 9 \end{pmatrix}$$

$$\pi = (1 0) \quad \text{and} \quad \eta^r = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

Then $r(A) = 0,$

$$r(e_1 \cdots e_t) = 0. \theta(e_t) \cdots \theta(e_1) \quad (9\text{-adic expansion})$$

where $e_i$ is $a$ or $b,$ $1 \leq i \leq t,$ and $\theta(a) = 1$ and $\theta(b) = 2.$ Now the construction of RFPA's $\mathfrak{A}_p$ and $\mathfrak{A}_q$ which realize $p$ and $q$ in $\mathfrak{P}_\mathcal{R}$ is straightforward along the proof of Lemma 14. Let us denote

$$\mathfrak{A}_p = \langle S_p, \{B_p(a), B_p(b), B_p(c)\}, \pi_p, \eta^{r_p} \rangle,$$

and

$$\mathfrak{A}_q = \langle S_q, \{B_q(a), B_q(b), B_q(c)\}, \pi_q, \eta^{r_q} \rangle.$$  

It is clear from the form of $p$ and $q$ that

(A) the value of $p(x),$ $x$ in $\{a, b, c\}^*,$ is 1 or of the form

$$0.\theta_1\theta_2 \cdots \theta_s \quad (9\text{-adic expansion})$$

where $\theta_i$'s $(1 \leq i \leq s)$ are 1 or 2, and the value of $q(x),$ $x$ in $\{a, b, c\}^*,$ is 0 or of the form $0.\theta_1\theta_2 \cdots \theta_s \quad (9\text{-adic expansion}),$ where $\theta_i$'s $(1 \leq i \leq s)$ are 1 or 2.

Similarly we will construct the RFPA's $\mathfrak{A}_p'$ and $\mathfrak{A}_q'$ which realize $p'$ and $q'$ in $\mathfrak{P}_\mathcal{R}$ respectively such that

$$h(L(y)) = \{x \in \{a', b', c'\}^* \mid p'(x) = q'(x)\}.$$  

We define the RFPA $\mathfrak{A}_r'$ which realizes $r',$ corresponding to $r$ as follows:
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\[ \mathfrak{A}_r = \langle \{1', 2'\}, \{A'(a'), A'(b')\}, \pi', \eta'' \rangle \]

\[
A'(a') = \begin{pmatrix}
6 & 3 \\
9 & 9
\end{pmatrix}, \quad A'(b') = \begin{pmatrix}
3 & 6 \\
9 & 9
\end{pmatrix}
\]

\[
\eta' = (1, 0) \quad \text{and} \quad \eta'' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Then \( r'(\Lambda) = 0 \)

\[
r'(e_1 \cdots e_t) = 0, \quad \theta(e_i) \cdots \theta(e_1) \quad (9\text{-adic expansion})
\]

where \( e_i \) is \( a' \) or \( b' \), \( 1 \leq i \leq t \), and \( \theta(a') = 3 \) and \( \theta(b') = 6 \). Now we have \( \mathfrak{A}_p' \) and \( \mathfrak{A}_q' \) along the construction of the proof of Lemma 14, and furthermore we have the RFPA’s \( \mathfrak{A}_{p',T} \) and \( \mathfrak{A}_{q',T} \) which realize \( p''_T \) and \( q''_T \) respectively. Let

\[
\mathfrak{A}_{p',T} = \langle T_{p',T}, \{C_{p',T}(a'), C_{p',T}(b'), C_{p',T}(c'), \pi_{p',T}, \eta^{p',T}_T \} \rangle
\]

and

\[
\mathfrak{A}_{q',T} = \langle T_{q',T}, \{C_{q',T}(a'), C_{q',T}(b'), C_{q',T}(c'), \pi_{q',T}, \eta^{q',T}_T \} \rangle.
\]

It is clear in this case that

(B) the value of \( p''_T(y), y \in \{a', b', c'\}^* \), is 1 or of the form

\[
0, \theta_1 \cdots \theta_s \quad (9\text{-adic expansion})
\]

where \( \theta_i \)'s \( (1 \leq i \leq s) \) are 3 or 6, and the value of \( q''_T(y), y \in \{a', b', c'\}^* \), is 0 or of the form \( 0, \theta_1 \cdots \theta_s \quad (9\text{-adic expansion}) \), where \( \theta_i \)'s \( (1 \leq i \leq s) \) are 3 or 6.

Define the RFPA’s \( \mathfrak{A}_{z_1}(z = p \text{ or } q) \) as follows:

\[
\mathfrak{A}_{z_1} = \langle S_z \cup T_{z',T}, \{D_{z_1}(\sigma) \mid \sigma \in \Sigma'\}, \pi_{z_1}, \eta^{z_1}_T \rangle
\]

It may be assumed that \( S_z \cap T_{z',T} = \emptyset \). If \( \sigma \) is in \( \{a, b, c\} \), let

\[
D_{z_1}(\sigma) = \begin{pmatrix}
B_z(\sigma) & 0 \\
0 & E_{z',T}
\end{pmatrix},
\]
where $E_{z,T}$ is the $\#T_z \times \#T_z$ unit matrix. If $\sigma$ is in $\{a', b', c'\}$, let

$$D_{z_1}(\sigma) = \begin{pmatrix} E_z & 0 \\ 0 & C_{z,T}(\sigma) \end{pmatrix},$$

where $E_z$ is the $\#S_z \times \#S_z$ unit matrix. Let $D_{z_1}(d)$ be the $(S_z \cup T_z,T_z)$ $\#(S_z \cup T_z,T_z)$ unit matrix, and let

$$\pi_{z_1} = \frac{1}{2}(\pi_z, \pi_{z,T}),$$

and

$$\eta^{f_{z_1}} = \begin{pmatrix} \eta^{f_z} \\ \eta^{f_{z,T}} \end{pmatrix}.$$

Now let $p_1$ and $q_1$ be the RPE's which are realized by $\mathfrak{A}_p$ and $\mathfrak{A}_q$ respectively. Let

$$\alpha = \{a, b, c\}^* d\{a', b', c'\}^*$$

and let

$$p_0 = p_1 \lor \chi_\alpha \quad \text{and} \quad q_0 = q_1 \land \chi_\alpha,$$

then $p_0$ and $q_0$ are in $\mathfrak{B}_R$.

To prove the lemma it suffices to show that for each word $w$ in $\Sigma^*$

$$p_0(w) = q_0(w) \iff w \in L(x, y).$$

For each word $w$ in $\Sigma^*$, $p_0(w) = q_0(w)$ if and only if $w \in \alpha$ and $p_1(w) = q_1(w)$, that is, there exist $u$ in $\{a, b, c\}^*$ and $v$ in $\{a', b', c'\}^*$ such that $w = uv$ and $p_1(w) = q_1(w)$. From the construction of $\mathfrak{A}_{z_1}$ ($z = p$ or $q$), it follows that

$$p_1(w) = \frac{1}{2}(p(u) + p^{T_T}(v))$$

and

$$q_1(w) = \frac{1}{2}(q(u) + q^{T_T}(v)).$$

Then $p_0(w) = q_0(w)$ if and only if $w = uv$, $u$ in $\{a, b, c\}^*$, $v$ in $\{a', b', c'\}^*$ and

$$p(u) + p^{T_T}(v) = q(u) + q^{T_T}(v). \quad (6)$$

Taking (A) and (B) into consideration and comparing each digit of both sides of the Eq. (6) in the form of 9-adic expansion it can be seen that (6) holds if and only if

$$p(u) = q(u),$$

4 For a set $S$, $\#S$ is the cardinarity of $S$. 
and
\[ p'(v) = q'(v), \quad \text{that is,} \quad p'(v^r) = q'(v^r). \]
Therefore \( p_0(w) = q_0(w) \) if and only if \( w = udv, u \) is in \( L(x) \) and \( v^r \) is in \( h(L(y)) \). Thus \( p_0(w) = q_0(w) \iff w \in L(x, y) \).

Now we can treat some unsolvable problems by the similar way\(^5\) as given by Bar-Hillel et al.

**Lemma 18.** \( L(x, y) \cap L_6 \) is in \( \mathcal{E} \) and \( L(x, y) \cap L_6 \) contains no infinite context-free language.

**Proof.** From Lemmas 12, 15 and 17 \( L(x, y) \cap L_6 \) is in \( \mathcal{E} \). The proof of the second statement of the lemma is essentially the same as Bar-Hillel’s.

**Lemma 19.** Let \( \tau \) be the homomorphism of \( \Sigma'^* = \{a, b, c, d, a', b', c'\}^* \) into \( \{a, b\}^* \) defined by \( \tau(a) = ab, \tau(b) = a^2b, \tau(c) = a^3b, \tau(d) = a^6b, \tau(a') = a^5b, \tau(b') = a^8b \) and \( \tau(c') = a^7b \). Then \( \tau(L(x, y) \cap L_6) \) is in \( \mathcal{E} \) and \( \tau(L(x, y) \cap L_6) \) contains no infinite context-free language.

**Proof.** It follows from Lemma 18 that there exist \( p \) and \( q \) in \( \mathcal{B}_R \) such that
\[ L(x, y) \cap L_6 = \{ x \in \Sigma'^* \mid p(x) = q(x) \}. \]
Let \( \mathcal{G} \) be the GSM defined as
\[ \mathcal{G} = (\{s_0, s_1, \ldots, s_7, t\}, \{a, b\}, \Sigma, \delta, \lambda) \]
where
\[
\begin{align*}
\delta(s_i, a) &= s_{i+1} \text{ for } 0 \leq i \leq 6, \\
\delta(s_i, b) &= s_0 \text{ for } 1 \leq i \leq 7, \\
\lambda(s_1, b) &= a, \lambda(s_2, b) = b, \lambda(s_8, b) = c \\
\lambda(s_4, b) &= d, \lambda(s_5, b) = a', \lambda(s_6, b) = b', \lambda(s_7, b) = c', \\
\delta(s_0, b) &= t, \\
\lambda(s_0, b) &= \Lambda, \delta(s_7, a) = t, \lambda(s_7, a) = \Lambda, \\
\lambda(t, a) &= \lambda(t, b) \end{align*}
\]
Let \( U \) be the GSM-mapping realized by \( \mathcal{G} \) and let \( \alpha = \{ab, a^2b, \ldots, a^7b\}^* \). Let
\[ u = U(p) \land \chi_\alpha \quad \text{and} \quad v = U(q) \land \chi_\alpha. \]
\(^5\) As for the details of the following discussion, refer to Bar-Hillel et al. or Ginsburg.
Then \( u \) and \( v \) are in \( \mathcal{B}_n \) and
\[
\{ x \in \{a, b\}^* \mid u(x) = v(x) \} = \tau[L(x, y) \cap L_s].
\]
Thus the first statement of the lemma is proved.

The proof of the second statement of the lemma, essentially the same as the one in Ginsburg (Lemma 4.2.2), is omitted here.

**Lemma 20.** Each of the following is recursively unsolvable for arbitrary \( L(x, y) \): (a) whether \( L(x, y) \cap L_s \) is empty, (b) whether \( \tau[L(x, y) \cap L_s] \) is empty, \( \tau \) is inLemma 19.

**Proof.** (a) The set \( L(x, y) \cap L_s \) consists of words of the form
\[
a^{i_1} b \cdots a^{i_n} b c x_i \cdots x_{i_k} d h(y_{i_1} \cdots y_{i_k} c b a^{i_1} \cdots b a^{i_k})
\]
where \( k \geq 1, 1 \leq i_k \leq n, \) and \( x_i \cdots x_{i_k} = y_{i_1} \cdots y_{i_k} \). Thus it follows from Post correspondence theorem that \( L(x, y) \cap L_s = \emptyset \) is recursively unsolvable.

(b) \( \tau[L(x, y) \cap L_s] = \emptyset \) if and only if \( L(x, y) \cap L_s = \emptyset \). The result follows from (a).

**Theorem 21.** Let \( \Sigma \) contain at least two elements. It is recursively unsolvable to determine for an arbitrary RFPA \( \mathfrak{F} \) over \( \Sigma \) and arbitrary rational number \( 0 \leq \lambda \leq 1 \) (a) whether \( \beta(\mathfrak{F}, \lambda) \) is empty and (b) whether \( \beta(\mathfrak{F}, \lambda) = \Sigma^* \).

**Proof.** Let \( \Sigma = \{a, b\} \). Since \( \tau[L(x, y) \cap L_s] \in \mathcal{S} \subseteq \mathcal{P} \) and \( \Sigma^* - \tau[L(x, y) \cap L_s] \in \mathcal{D} \subseteq \mathcal{P} \), the result follows from Lemma 20.

**Theorem 22.** Let \( \Sigma \) contain at least two elements. It is recursively unsolvable to determine for an arbitrary RFPA \( \mathfrak{F} \) over \( \Sigma \) and an arbitrary rational number \( 0 \leq \lambda \leq 1 \), (a) whether \( \beta(\mathfrak{F}, \lambda) \) is regular and (b) whether \( \beta(\mathfrak{F}, \lambda) \) is a context-free language.

**Proof.** Let \( \Sigma = \{a, b\} \), \( \tau[L(x, y) \cap L_s] \) is either empty or an infinite set. Therefore from Lemma 19, \( \tau[L(x, y) \cap L_s] \) is regular (a context-free language) if and only if \( \tau[L(x, y) \cap L_s] = \emptyset \). Thus the result follows from Lemma 20.

As we stated in our previous paper, Matuura gave the example which shows that for arbitrary \( p \) and \( q \) in \( \mathcal{B} \), \( p \lor q \) and \( p \land q \) are not always in

\[\text{As for Theorem 21, it reduces to the statement in p. 150 of Schüzenberger (1962) from the viewpoint that we said before Lemma 12 of this paper.}\]
The next lemma states the principle of his example in a slightly generalized form.

**Lemma 23.** For \( p \) and \( q \) in \( \mathcal{B} \), let \( \beta = \{ x \in \Sigma^* \mid p(x) > q(x) \} \). If for any \( n \geq 1 \) there exist \( u, w \) and \( v \) in \( \Sigma^* \) such that \( uw \in \beta \), \( uwv \in \beta \), \( \cdots \), \( uw^{n-1}v \in \beta \) and \( uw^n v \in \beta \) then \( p \lor q \) and \( p \land q \) is not in \( \mathcal{B} \).

**Proof.** Assume \( p \lor q \in \mathcal{B} \), then from Propositions 1 and 2
\[
\frac{1}{2} \lor (\frac{1}{2} + \frac{1}{2}(p - q)) = \frac{1}{2}(q + p \lor q) \in \mathcal{B}.
\]
Let \( r = \frac{1}{2} \lor (\frac{1}{2} + \frac{1}{2}(p - q)) \) and let \( r \) be realized by an \( n \)-state FPA \( \mathcal{A} = (S, \{ A(\sigma) \mid \sigma \in \Sigma \}, \pi, \eta^s) \), then from the assumptions of the lemma there exist \( u, w \) and \( v \) in \( \Sigma^* \) such that \( uw \in \beta \), \( \cdots \), \( uw^{n-1}v \in \beta \) and \( uw^n v \in \beta \).

Thus
\[
r(uw) = \cdots = r(uw^{n-1}v) = \frac{1}{2},
\]
and
\[
r(uw^n v) > \frac{1}{2}.
\]

On the other hand, let the characteristic polynomial of \( A(w) \) be
\[
X^n - C_{n-1}X^{n-1} \cdots - C_1x - C_0,
\]
then \( C_{n-1} + \cdots + C_0 = 1 \) and
\[
r(uw^n v) = C_{n-1}r(uw^{n-1}v) + \cdots + C_0r(uw).\]

Thus from (7)
\[
r(uw^n v) = \frac{1}{2}.
\]

This contradicts (8). Hence \( p \lor q \not\in \mathcal{B} \).

Noting that if \( p \land q \in \mathcal{B} \), \( \frac{1}{2} \land (\frac{1}{2} + \frac{1}{2}(q - p)) = \frac{1}{2}(p \land q) \in \mathcal{B} \),
we can see that \( p \land q \not\in \mathcal{B} \) from the similar discussion as above.

Matsuura's example is considered over a two-symbol alphabet. Over a one-symbol alphabet, for two probabilistic events \( p \) and \( q \), \( p \lor q \) and \( p \land q \) are not always probabilistic events. Paz considered the FPA \( A \) with a single symbol (Paz, A. (1965) p. 31). Let \( t \) be the probabilistic event which is realized by \( A \), then it can be shown from the above lemma that neither \( t \lor 4/11 \) nor \( t \land 4/11 \) is a probabilistic event.

**Theorem 24.** Let \( \Sigma \) contain at least two elements. It is recursively unsolvable to determine for arbitrary RPE's \( p \) and \( q \) over \( \Sigma^* \) (1) whether \( p \lor q \) is in \( \mathcal{B} \), and (2) whether \( p \land q \) is in \( \mathcal{B} \).

**Proof.** Let \( \Sigma = \{a, b\} \). From Lemmas 11 and 19 it follows that there exist \( p \in \mathcal{B}_R \) such that \( \{ x \in \Sigma^* \mid p(x) > \frac{1}{2} \} = \tau[L(x, y) \cap L_a] \).
To prove (1), it suffices to show that \( p \lor 1/2 \in \mathcal{B} \) if and only if \( \tau[L(x, y) \cap L_a] = \emptyset \), because of Lemma 20. If \( \tau[L(x, y) \cap L_a] = \emptyset \), then \( p \lor 1/2 = 1/2 \in \mathcal{B} \). If \( \beta = \tau[L(x, y) \cap L_a] \neq \emptyset \), then any word \( t \) in \( \beta \) is written in such a form as

\[
t = e_1a^2b_2a^4b_3a^7b_4,
\]

and furthermore for any \( n \geq 1 \),

\[
e_1^n a^2b_2^n a^4b_3^n a^7b_4^n \in \beta.
\]

Let \( v_n \) be \( a^2b_2^n a^4b_3^n a^7b_4^n \) for each \( n \geq 1 \), then for any \( n \geq 1 \), \( v_n \in \beta \), \( e_1v_n \in \beta \), \( \cdots \), \( e_1^{-1}v_n \in \beta \) and \( e_1^n v_n \in \beta \). Therefore from Lemma 23, \( p \lor 1/2 \in \mathcal{B} \).

(2) Since \( \bar{p} \lor \bar{q} = p \lor q \), \( \bar{p} \lor \bar{q} \in \mathcal{B} \) if and only if \( p \lor q \in \mathcal{B} \). Therefore the result follows from (1).

In the proof of Theorems 21, 22 and 24, the condition that \( \Sigma \) contains at least two elements is essential. The problem whether these decision problems are solvable or not when \( \Sigma \) consists of one element remains unsolved.

**ACKNOWLEDGMENTS**

The authors are grateful to Dr. Y. Inagaki and Mr. H. Matuuta in Nagoya University for helpful discussions. Especially Dr. Inagaki aroused their interest in the decision problems with respect to finite probabilistic automata.

**RECEIVED:** March 20, 1969

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