Journal of Functional Analysis **185**, 604–680 (2001)
doi:10.1006/jfan.2001.3773, available online at http://www.idealibrary.com on IDEAL®

Magnetic Bottles in Connection with Superconductivity

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Received December 28, 2000; accepted March 5, 2001

Motivated by the theory of superconductivity and more precisely by the problem of the onset of superconductivity in dimension two, a lot of papers devoted to the analysis in a semi-classical regime of the lowest eigenvalue of the Schrödinger operator with magnetic field have appeared recently. Here we mention the works by Bernoff–Sternberg, Lu–Pan and Del Pino–Felmer–Sternberg. This partially recovers questions analyzed in a different context by the authors around the question of the so called magnetic bottles. Our aim is to analyze the former results, to treat them in a more systematic way and to improve them by giving sharper estimates of the remainder. In particular, we improve significantly the lower bounds and as a by-product we solve a conjecture proposed by Bernoff–Sternberg concerning the localization of the ground state inside the boundary in the case with constant magnetic fields.

Contents.

[1] This work has been partially supported by the European Union under the TMR Program EBRFXT 960001.

0022-1236/01 $35.00

1. INTRODUCTION

We would like to discuss the spectrum of various self-adjoint realizations of the Schrödinger operator

$$P_{h,A,\Omega} = (hD_{x_1} - A_1)^2 + (hD_{x_2} - A_2)^2,$$

in an open set $\Omega \subset \mathbb{R}^2$, where $h > 0$ is a small parameter. Note that in the case of the problems considered in superconductivity the parameter h is proportional to the inverse of the intensity of some external field but the question which is posed in this theory is actually a semi-classical question, that is the question of analyzing the behavior of the spectrum as $h \to 0$ (more specifically the asymptotic behavior of the ground state energy and the localization of the ground state).

We have shown in [HeMo2] (note that the former spelling of the second author was Abderemane Mohamed), that, in the case when $\Omega = \mathbb{R}^2$, and in the semiclassical regime, the potential $h\,|B(x)|$ plays the role of an effective electric potential. By this we mean that, very roughly, the analysis of the Schrödinger operator, $-h^2\Delta + h\,|B(x)|$, could give some good insight on the localization of the ground state.

For PDE specialists, this is strongly related to Garding–Melin–Hörmander lower bounds (see [Mel], [HeRo]) for the operator

$$P_{h,AD_t} = (hD_{x_1} - A_1 D_t)^2 + (hD_{x_2} - A_2 D_t)^2,$$

in $\Omega \times \mathbb{R}$.

Although the case with boundary is not explicitly discussed in the previous literature ([HeMo2]), one gets similar results (in the case with boundary), **for the Dirichlet problem**, at least under the additional condition

$$b < b', \tag{1.1}$$

where we have used the notations:

$$\inf_{x \in \bar{\Omega}} |B(x)| = b, \qquad \inf_{x \in \partial\Omega} |B(x)| = b'. \tag{1.2}$$

THEOREM 1.1. *Under the condition* (1.1), *the lowest eigenvalue* $\lambda^{(1)}(h)$ *of the Dirichlet realization* $P^D_{h,A,\Omega}$ *of* $P_{h,A,\Omega}$ *satisfies:*

$$\frac{\lambda^{(1)}(h)}{h} = b + o(1). \tag{1.3}$$

The points where the minima of $|B|$ are obtained are called the magnetic wells. The decay outside the minima of $|B|$ is related to the Agmon distance associated with $|B| - b$.

We recall that the estimate is quite easy to prove, when the dimension is 2 and when B has a constant sign due to the inequality:

$$\pm h \int B(x)\, |u(x)|^2 \leqslant \langle P_{h,A} u \mid u \rangle, \qquad \forall u \in C_0^\infty(\Omega). \tag{1.4}$$

Here $\langle \cdot \mid \cdot \rangle$ denotes the scalar product in $L^2(\Omega)$ and $\| \cdot \|$ will denote the corresponding L^2 norm.

When B does not have a constant sign, a similar estimate is obtained but with a remainder estimate (See [HeMo2, Theorem 3.1]) in $\mathcal{O}(h^{5/4})$. The most difficult case when the dimension is greater than 2 is also treated.

As in the case when $B = 0$, where we discussed various results according to the properties of V near the minimum, one can discuss various results according to the properties of $|B(x)|$ near the minimum (see [HeMo2], [Mon]).

As we shall see later, **this property is no longer true in the case of the Neumann problem**. The infimum of $|B(x)|$ over $\bar{\Omega}$ is no longer the good measure for analyzing the bottom of the spectrum—also when (1.1) is satisfied. When the condition (1.1) is not satisfied, the situation is less clear.

First we observe by comparison between Neumann and Dirichlet that, when (1.1) is satisfied, we have, for the lowest eigenvalue $\mu^{(1)}(h)$ of the Neumann realization $P^N_{h,A,\Omega}$ of $P_{h,A,\Omega}$, the same upper bound as for $\lambda^{(1)}(h)$. But this is also true when the condition is not satisfied (if one comes back for example to the proof by Helffer–Mohamed [HeMo2], which is done, after a suitable gauge transform, by construction of gaussian quasimodes). This was also stated more recently by Lu–Pan [LuPa2] (Lemma 6.1) with a similar proof.

THEOREM 1.2. *The lowest eigenvalue $\mu^{(1)}(h)$ of the Neumann realization $P^N_{h,A,\Omega}$ of $P_{h,A,\Omega}$ satisfies:*

$$\frac{\mu^{(1)}(h)}{h} \leqslant b + o(1). \tag{1.5}$$

The problem is that the corresponding lower bound is no longer always true.

The aim of this paper is to discuss all these results and give some improvements in the case of the Dirichlet realization and of the Neumann realization. We shall in particular show how the technique of the Agmon estimates developed in [HeSj1] and [HeMo2] will give the localization of the groundstate in a rather more accurate form than in the previous works of Lu–Pan [LuPa2] and Del Pino–Felmer–Sternberg [PiFeSt]. Perhaps the more significant result is the proof (cf. Theorems 10.6 and 11.1) of the following conjectured property :

THEOREM 1.3. *Let us assume that the magnetic field is constant and not zero. Then any normalized groundstate of the Neumann realization of $P_{h,A,\Omega}$ is exponentially localized as $h \to 0$ in the neighborhood of the points of the boundary with maximal curvature.*

Contents. The paper is organized as follows.

In Section 2, we establish rough results under very weak assumptions on the potential. This extends a result established for the case with electric field [BeHeVe] and was partially motivated by a question of S. Zelditch. In Sections 3 and 4, we analyze the reference models corresponding to the half plane and to the disk. Section 5 continues the general description of the state of the art and exhibits new questions. In Section 6, we recall the method of the Agmon estimates in the most basic cases and show how it gives at a low price results concerning the concentration of the eigenvalues near the boundary (under specific conditions) or in the interior. We pursue in this section the analysis of the decay inside the boundary when some general condition on the magnetic field is satisfied. Section 7 treats the Dirichlet case or the Neumann case, when one can a priori show that the groundstate is localized inside a compact of the open set Ω. We in particular give a two-term expansion of the groundstate energy, improving a one term expansion given in [HeMo2]. Section 8 discusses the Dirichlet problem when the boundary plays an important role, that is when the minimum of the magnetic field is attained at the boundary. Section 9 comes back to the analysis of the Neumann problem; we recover with a better control error term the upper bounds of [BeSt], [LuPa2], [PiFeSt] (this is not simply an academic problem but could have consequence in the superconductivity problems) and improve the lower bound of [LuPa1] in the generic case. The two last sections 10 and 11 are devoted to the case of the constant magnetic case and to the complete proof of the conjecture of [BeSt].

Finally, the two appendices recall the now well-known results which are frequently used along the paper.

2. VERY ROUGH LOWER BOUNDS

Let us start the analysis of the question with very rough estimates. In the case of Dirichlet and if $B(x) \neq 0$ (say for example $B(x) > 0$), we can use[2] (1.4) which gives a comparison between self-adjoint operators in the form (for any $\rho \in [0, 1]$)

$$P^D_{h,A} \geqslant \rho(P^D_{h,A}) + (1-\rho)\, hB(x). \tag{2.1}$$

In order to find a lower bound for the smallest eigenvalue of the Dirichlet realization, it is enough to apply for a suitable ρ a rough lower bound for the operator:

$$\rho(P^D_{h,A}) + (1-\rho)\, hB(x).$$

Using Kato's inequality (cf for example [CFKS]), we can look for a lower bound for the operator:

$$-\rho h^2 \Delta^D_\Omega + (1-\rho)\, hB(x).$$

We recall that this inequality gives, for any real potential V, the comparison

$$\inf \mathrm{Sp}(P^D_{h,A,\Omega} + V) \geqslant \inf \mathrm{Sp}(-h^2 \Delta^D_\Omega + V), \tag{2.2}$$

and that a similar result is true in the case of Neumann (see [HHOO]):

$$\inf \mathrm{Sp}(P^N_{h,A,\Omega} + V) \geqslant \inf \mathrm{Sp}(-h^2 \Delta^N_\Omega + V). \tag{2.3}$$

We shall show as a quite preliminary result the following proposition.

[2] S. Zelditch transmitted to one of us (July 1999) a question of this type coming from geometry.

PROPOSITION 2.1. *Under the condition that* $x \mapsto B(x)$ *is* $\geqslant 0$, *non constant and analytic, then there exists* $\theta \in]0, \frac{1}{2}]$ *and* $C > 0$ *such that:*

$$\lambda^{(1)}(h) - bh \geqslant \frac{1}{C} h^{\frac{1}{\theta}}. \tag{2.4}$$

Proof. Using well known lower bounds for the Schrödinger operator $-\varepsilon\Delta + V$ (see [BeHeVe]) (related to the Cwickel–Lieb–Rosenblyum estimate) with $\varepsilon = \rho h$ and $V(x) = \frac{1}{2}(B(x) - b)$, we start from the property that there exists $\theta \in]0, \frac{1}{2}]$ and C such that, $\forall \rho \in]0, \frac{1}{2}]$,

$$\lambda^{(1)}(h) - (1-\rho)\, hb \geqslant \frac{1}{C}(\rho h^2)(\rho h)^{-\theta}.$$

This can be rewritten in the form

$$\lambda^{(1)}(h) - hb \geqslant \frac{1}{C} \rho^{1-\theta} h^{2-\theta} - b\rho h$$

or

$$\lambda^{(1)}(h) - hb \geqslant \rho^{1-\theta} h \left(\frac{1}{C} h^{1-\theta} - b\rho^{\theta} \right).$$

If we take $\rho = \gamma h^{(1-\theta)/\theta}$ and γb small enough, we get (2.4) for h small enough.

Remark 2.2. The optimality of this inequality will be discussed later in particular cases. In particular, we will discuss the case when $B(x) = b$ and the case when $B(x) - b$ has a non degenerate minimum.

Remark 2.3. When $b = 0$, we can take $\rho = \frac{1}{2}$, and get, for some $\theta > 0$:

$$\lambda^{(1)}(h) \geqslant \frac{1}{C} h^{2-\theta}.$$

Results in [HeMo2], [Mon], [Ue2] or [LuPa1] show that it is optimal.

Remark 2.4. Note finally that if $B \equiv 0$ in a simply connected open set ω (with $\bar{\omega} \subset \Omega$), then

$$\lambda^{(1)}(h) \leqslant \tilde{\lambda}^{(1)}(\omega)\, h^2,$$

where $\tilde{\lambda}^{(1)}(\omega)$ is the lowest eigenvalue of the Dirichlet Laplacian in ω.

3. THE BASIC MODELS IN THE HALF-PLANE

Let us consider in a regular domain Ω of $\mathbb{R}^2$ the Neumann (or Dirichlet) realization of the operator

$$P_{h,\,bA_0,\,\Omega} := \left(hD_{x_1} - \frac{b}{2}x_2\right)^2 + \left(hD_{x_2} + \frac{b}{2}x_1\right)^2, \tag{3.1}$$

with

$$A_0(x_1, x_2) = (\tfrac{1}{2}x_2, -\tfrac{1}{2}x_1). \tag{3.2}$$

Note that, by Neumann realization, we mean the condition

$$\nu \cdot (h\nabla - ibA_0)\, u = 0 \qquad \text{on} \quad \partial\Omega, \tag{3.3}$$

where ν is the external normal at the boundary.

The parameter h is the semiclassical parameter and b is the constant magnetic field. Of course, as we have seen already, replacing (h, b) by $(1, \frac{b}{h})$ does not really change the problem. We have indeed:

$$P_{h,\,bA_0} = h^2 P_{1,\,bA_0/h}. \tag{3.4}$$

Here b is assumed to be different from 0 and, without loss of generality, we can assume:

$$b > 0. \tag{3.5}$$

So the problem of analyzing (for fixed b) the semiclassical limit $h \to 0$ is the same as the problem of analyzing (for fixed $h > 0$) the large field limit $b \to +\infty$.

Let us denote by $\mu^{(1)}(h, b, \Omega)$ and $\lambda^{(1)}(h, b, \Omega)$ the bottoms of the spectrum of the Neumann or Dirichlet realization of $P_{h,\,bA_0}$ in Ω. Depending on Ω, this may correspond to an eigenvalue (if Ω is bounded) or to a point in the essential spectrum (for example if $\Omega = \mathbb{R}^2$ or $\Omega = \mathbb{R}^2_+$). Let us recall that

$$\mu^{(1)}(h, b, \mathbb{R}^2) = \lambda^{(1)}(h, b, \mathbb{R}^2) = hb, \tag{3.6}$$

where b is an eigenvalue with infinite multiplicity.

We note that in the Dirichlet case, we have by monotonicity the lower bound:

$$\lambda^{(1)}(h, b, \Omega) \geqslant \lambda^{(1)}(h, b, \mathbb{R}^2) = hb. \tag{3.7}$$

Such a lower bound is false in the case of the Neumann problem and this makes the problem more interesting.

The analysis in $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$ is less known and will play an important role in our discussion.

First we observe by construction of quasimodes and (3.7) that:

$$\lambda^{(1)}(h, b, \mathbb{R}^2_+) = hb. \tag{3.8}$$

Let us now analyze the Neumann case.

We take $h = 1$ and first perform a gauge transformation which leads to the analysis of the operator

$$\tilde{S}_b = D_{x_2}^2 + (D_{x_1} - bx_2)^2, \tag{3.9}$$

with Dirichlet or Neumann conditions. Note that we now get the standard Neumann condition on $x_2 = 0$.

In order to make the spectral analysis of the operator, we can perform a partial Fourier transform and we get

$$\hat{S}_b(t, \sigma, D_t; B) = D_t^2 + (\sigma - bt)^2, \tag{3.10}$$

considered as an operator on $\mathbb{R}^2_+ = \{(t, \sigma) \in \mathbb{R}^2 \mid t > 0\}$ with Dirichlet or Neumann conditions on $t = 0$.

A scaling shows that it is enough to consider the case $b = 1$ and then we recover the general case by multiplication by b.

The analysis of the spectrum is then reduced to the spectral analysis of an ξ-family of differential operators

$$H(t, D_t; \xi) := D_t^2 + (t - \xi)^2. \tag{3.11}$$

We shall consider the Dirichlet and Neumann realizations of this operator in $\mathbb{R}^+$ respectively denoted by $H^{D,\xi}$ and $H^{N,\xi}$. Let us observe that the operator is unitary equivalent to the corresponding realization of the harmonic oscillator $D_t^2 + t^2$ in $]-\xi, +\infty[$.

The Dirichlet spectrum of P_{1, A_0} is obtained by

$$\mathrm{Sp}(P^D_{1, A_0}) = \{\lambda \in \mathbb{R}^+ \mid \exists \xi \in \mathbb{R} \text{ s.t. } \lambda \in \mathrm{Sp}(H^{D,\xi})\} \tag{3.12}$$

and similarly for Neumann. The eigenvalues of $H^{D,\xi}$ and $H^{N,\xi}$ are analytic with respect to ξ and not constant. Hence one obtains easily (See Reed–Simon, Vol. IV [ReSi]) that we get a priori a continuous band spectrum in the two cases. The analysis of the bottom of the spectrum is an immediate consequence of the analysis of the lowest eigenvalue of $H^{D,\xi}$ or $H^{N,\xi}$ as a function of ξ.

Here the answer is quite different for Dirichlet and Neumann.

For Dirichlet, the lowest eigenvalue $\hat{\lambda}^{(1)}(\xi)$ decays monotonically from $+\infty$ to 1 as ξ goes from $-\infty$ to $+\infty$.

For Neumann, the lowest eigenvalue $\hat{\mu}^{(1)}(\xi)$ decays monotonically from $+\infty$ to 1 as ξ goes from $-\infty$ to 0. Then one can show [DaHe] that there exists $\xi_0 > 0$ such that $\hat{\mu}^{(1)}(\xi)$ continues to decay monotonically till some value

$$\Theta_0 := \hat{\mu}^{(1)}(\xi_0) < 1 \tag{3.13}$$

and increases again monotonically to 1 as $\xi \to +\infty$ (see Appendix A for more details) and we get

PROPOSITION 3.1. *The spectrum of the Dirichlet realization in $\mathbb{R}^2_+$ is:*

$$\mathrm{Sp}(P^D_{h,\, bA_0,\, \mathbb{R}^2_+}) = [bh, +\infty[. \tag{3.14}$$

The spectrum of the Neumann realization in $\mathbb{R}^2_+$ is

$$\mathrm{Sp}(P^N_{h,\, bA_0,\, \mathbb{R}^2_+}) = [b\Theta_0 h, +\infty[, \tag{3.15}$$

with

$$0 < \Theta_0 < 1. \tag{3.16}$$

The fact that the bottom of the spectrum of the Neumann problem is strictly lower that the bottom of the problem in $\mathbb{R}^2$ will play a quite important role in the discussion.

Remark 3.2. The analysis of the Dirichlet problem in the half-plane appears also in the study of the quantum Hall effect. We refer to [DePu] and to [FrGrWa] for more general domains.

Remark 3.3. The case of a domain with a corner was recently analyzed in [Ja] and [Pan]. It is in particular proved that in domains with right angles, we have

$$\mu^{(1)}(b, \Omega) \leqslant \Theta_1 b + o(b),$$

for some Θ_1 satisfying:

$$0 < \Theta_1 < \Theta_0.$$

This inequality is obtained by the choice of a suitable quasimode. This Θ_1 is the lowest eigenvalue of the Neumann realization in:

$$\mathbb{R}^2_{++} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0, x_2 > 0\}.$$

The localization of the ground state in the corners is also obtained by this author.

4. THE CASE OF THE DISC IN THE CONSTANT MAGNETIC CASE

We first present the results by L. Erdős ([Er1], [Er2]) and give an alternative proof for improving some of the statements. When considering lower bounds, it is useful to mention the following result obtained by L. Erdős :

PROPOSITION 4.1. *For any planar domain Ω and $b>0$, we have*

$$\lambda^{(1)}(b,\Omega)\geqslant\lambda^{(1)}(b,D(0,R)), \tag{4.1}$$

where $D(0,R)$ is the disk with same area as Ω,

$$\pi R^2 = Area(\Omega),$$

and $\lambda^{(1)}(b,\Omega)$ is the lowest eigenvalue of the Dirichlet realization of the Schrödinger operator with constant magnetic field $b>0$.

Moreover the equality in (4.1) occurs if and only if $\Omega = D(0,R)$.

In the case of the disk, L. Erdős observes that $\lambda^{(1)}(b,D(0,R))$ is equal to the lowest eigenvalue of the Dirichlet realization of the two dimensional harmonic oscillator : $-\Delta+r^2/4$ in $D(0,R)$.

This one is equal to $2\hat{\lambda}^{(1)}(]-R,+R[)$ where $\hat{\lambda}^{(1)}(]-R,+R[)$ is the groundstate energy of the one dimensional harmonic oscillator $-(d/(dx^2))+\frac{1}{4}x^2$ in the interval $]-R,+R[$ with Dirichlet condition.

In the one-dimensional case, we can then apply semi-classical techniques developped in this context in [Bo], [BoHe] for getting the following improvement of [Er1] and [Er2]:

PROPOSITION 4.2. *In the case of Dirichlet conditions in $]-R,+R[$, the lowest eigenvalue of the Dirichlet realization of $-d^2/dx^2+x^2/4$ satisfies, in the limit $R\to+\infty$:*

$$\hat{\lambda}^{(1)}(]-R,+R[)-\tfrac{1}{2}\sim 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}R\exp(-\tfrac{1}{2}R^2). \tag{4.2}$$

The proof is quite similar to the proof given in [Bo] (pp. 250–254) for treating the Neuman case [Bo] (Formula (2.23)) and [BoHe]:

PROPOSITION 4.3. *In the case of Neumann conditions in $]-R,+R[$, the lowest eigenvalue of the Neumann realization of $-d^2/dx^2+x^2/4$ satisfies, in the limit $R\to+\infty$:*

$$\hat{\mu}^{(1)}(]-R,+R[)-\tfrac{1}{2}\sim -2^{\frac{1}{2}}\pi^{-\frac{1}{2}}R\exp(-\tfrac{1}{2}R^2). \tag{4.3}$$

As is expected, we observe that the boundary effect is very small as $R \to +\infty$

$$\hat{\lambda}^{(1)}(]-R,+R[)-\hat{\mu}^{(1)}(]-R,+R[) \sim 2^{\frac{3}{2}}\pi^{-\frac{1}{2}}R \,\exp(-\tfrac{1}{2}R^2),$$

as R tends to ∞.

Coming back to our initial problem, this gives using the scaling invariance :

PROPOSITION 4.4. *In the regime $R\sqrt{b}$ large, we have the asymptotic:*

$$\lambda^{(1)}(b, D(0, R))-b \sim 2^{\frac{3}{2}}\pi-\tfrac{1}{2}b^{\frac{3}{2}}R \exp\left(-\frac{bR^2}{2}\right). \tag{4.4}$$

Remark 4.5. It is a natural question to ask for an isoperimetric inequality in the case of Neumann, that is if the inequality $\mu^{(1)}(b,\Omega) \geqslant \mu^{(1)}(b, D(0,R))$ is true. But the recent results by Sternberg and coauthors [BeSt], [PiFeSt] that we shall discuss here, compared with those of Bauman–Phillips–Tang [BaPhTa], contradict this result. Playing with the maximal curvature at the boundary and keeping fixed the area, one can indeed (in the semiclassical regime) get a contradiction with this isoperimetric inequality (see Theorem 5.2 in this paper) in the regime b large.

Remark 4.6. In the paper by Baumann–Phillips–Tang [BaPhTa] (Theorem 6.1, p. 24) (see also [PiFeSt]) the authors propose the following asymptotic as b is large:

$$\mu^{(1)}(b, D(0, R)) = \Theta_0 b - 2M_3\,\frac{1}{R}\,b^{\frac{1}{2}} + \mathcal{O}(1). \tag{4.5}$$

Here we recall that Θ_0 is introduced in (3.13) and $M_3 > 0$ is a universal constant which will be made explicit in Appendix A (cf (A.6) and (A.9)). We shall give an alternative proof in Section 11.

Remark 4.7. Another interesting result is the case of the exterior of the disk. One can first observe that the bottom of the essential spectrum is equal to b. One can show that, when b is large, there is at least one eigenvalue below b. Moreover the groundstate energy will satisfy:

$$\mu^{(1)}(b, D(0, R)^c) = \Theta_0 b + 2M_3\,\frac{1}{R}\,b^{\frac{1}{2}} + \mathcal{O}(1). \tag{4.6}$$

This last result, which seems new, will be proven in Section 11.

Finally, let us observe that, due to the homogeneity, we get for free the following semi-classical corollaries of respectively (4.4) and (4.5).

PROPOSITION 4.8. *The lowest eigenvalue of* $P^D_{h,A_0,D(0,R)}$ *satisfies as* $h \to 0$:

$$\lambda^{(1)}(h, b, D(0, R)) - bh \sim 2^{\frac{3}{2}} \pi^{-\frac{1}{2}} b^{\frac{3}{2}} h^{\frac{1}{2}} R \exp\left(-\frac{bR^2}{2h}\right). \tag{4.7}$$

PROPOSITION 4.9. *The lowest eigenvalue of* $P^N_{h,A_0,D(0,R)}$ *satisfies as* $h \to 0$:

$$\mu^{(1)}(h, b, D(0, R)) = \Theta_0 bh - \frac{2}{R} M_3 b^{\frac{1}{2}} h^{\frac{3}{2}} + \mathcal{O}(h^2). \tag{4.8}$$

5. SEMICLASSICAL QUESTIONS: FORMER RESULTS

We now consider the questions which are considered in superconductivity. In the case of a bounded regular open set, del Pino, Felmer and Sternberg [PiFeSt] obtain the following results.

THEOREM 5.1. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded, open, simply connected domain with a* C^∞ *boundary* $\partial\Omega$. *Let* A *be a potential in* Ω *such that* $B(x) = b > 0$. *If* u_h *is a sequence of groundstates for the Neumann realization of* $P^N_{h,A,\Omega}$ *in* Ω *associated with the lowest eigenvalue* $\mu^{(1)}(h)$. *Then, there exist* $h_0 > 0$ *and constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$|u_h(x)| \leqslant c_1 \, \|u_h\|_{L^\infty(\Omega)} \exp(-c_2 h^{-\frac{1}{2}} \, dist(x, \partial\Omega) \,), \tag{5.1}$$

for all $x \in \Omega$.

Moreover if Ω *is not a disc, then we have:*

$$\lim_{\substack{h \to 0 \\ h > 0}} \left(\min_{x \in \partial\Omega} |u_h(x)|\right) = 0. \tag{5.2}$$

The second theorem proved by Bernoff–Sternberg [BeSt] (see also [LuPa3]) gives an upper bound of $\mu^{(1)}(h)$. The localization (5.2) is a weak version of a much stronger result which will be proved in Section 11.

THEOREM 5.2. *Under the assumptions of Theorem* 5.1, *we have:*

$$\mu^{(1)}(h) \leqslant \Theta_0 h - 2\kappa_{max} M_3 h^{\frac{3}{2}} + o(h^{\frac{3}{2}}). \tag{5.3}$$

Here κ_{max} is the value of the maximal curvature which is assumed to be strictly positive and M_3 is a universal constant (cf (A.9)).

The proof is essentially a consequence of the construction of matched solutions. Let us just mention that in suitable coordinates (s, t) $(t = 0$

defining the boundary) (see Appendix B) in a neighborhood of a point of maximal curvature ($s=0$ defining this point), the solution takes the form:

$$u := \phi(h^{-\frac{1}{2}}t) \exp -\gamma h^{-\frac{1}{4}}s^2 \cdot \exp ih^{-1}[\Phi_0(s, t)+h^{\frac{1}{2}}\Phi_1(s, t)]. \tag{5.4}$$

Note that it is coherent with the result given, in the case of the disc, by Bauman–Phillips–Tang [BaPhTa] which was already discussed in (4.5). We will come back to these two results in Sections 10 and 11.

This result is completed by a theorem by Lu–Pan [LuPa2] which is valid for general magnetic fields:

THEOREM 5.3.

$$\lim_{h\to +0} \frac{\mu^{(1)}(h)}{h} = \min(\Theta_0 b', b). \tag{5.5}$$

Remark 5.4. 1. Although it is not completely proved in these previous contributions, the localization of the Neumann groundstate is far from the boundary when

$$b < \Theta_0 b'.$$

This will be shown in this paper.

2. When $b > \Theta_0 b'$, one will show that the groundstate is localized near the boundary. This is for example the case when the magnetic field is constant (which is analyzed in [PiFeSt]).

3. It is interesting to have a better localization (than for example (5.2) in the constant magnetic field case) inside the boundary. One can wait for example for exponential decay outside the minima of $|B|_{/\partial\Omega}$ assuming they are isolated. Preliminary arguments are given by [PiFeSt] in the case $B = const.$ but we shall come back to the problem in Sections 10 and 11.

4. When $b' > 0$ and $b = 0$, the analysis by [HeMo2] is relevant. The case when $b' = 0$ and $|B(x)| > 0$ in Ω is treated in [LuPa2].

5. Note that Giorgi–Phillips [GiPh] and Ueki [Ue2] give earlier weaker lower bounds which have independent interest.

6. IMS TYPE DECOMPOSITION AND AGMON'S ESTIMATES

6.1. *Preliminaries*

We recall in this section the technique of the Agmon estimates and show how they give rather easily rough results which will be improved later. In the semi-classical context, this technique was mainly developed by Helffer–Sjöstrand [HeSj1], [HeSj4] but the inspiration comes from former results due to many authors: Lithner, Agmon [Ag], and Simon [Si] among others.

In the Dirichlet case, the inequality (1.4) was (at least when the condition $B(x) > 0$ is satisfied) the starting point of the analysis of the decay. This is no longer the case when Neumann boundary conditions are assumed, but we can keep the general strategy as developped in [HeMo2].

We assume that Ω is a bounded, regular open set and that

$$B(x) > 0. \tag{6.1}$$

One could also consider unbounded domains but in this case one has to add some assumption at ∞, permitting the essential spectrum to be localized.

6.2. *Upper Bounds*

Using suitable quasimodes, one can get:

$$\mu^{(1)}(h) \leqslant \min(b, \Theta_0 b')\, h + C h^{\frac{5}{4}}. \tag{6.2}$$

We will show an improved version of this property later but note that Lu–Pan give at least the weaker

$$\mu^{(1)}(h) \leqslant \min(b, \Theta_0 b')\, h + o(h), \tag{6.3}$$

which is enough for our analysis of the decay. Note also that the upper bound involving $\inf B$ can also be obtained by using [HeMo2].

Further improvements of (6.2) will be given in Theorem 9.1.

6.3. *Lower Bounds*

Let $0 \leqslant \rho \leqslant 1$. We first claim that there exists C such that, for any $\varepsilon_0 > 0$, we can, by scaling a standard partition of unity of $\mathbb{R}^2$, and by restricting it to $\bar{\Omega}$, find a partition of unity χ_j^h satisfying in Ω,

$$\sum_j |\chi_j^h|^2 = 1, \tag{6.4}$$

$$\sum_j |\nabla \chi_j^h|^2 \leqslant C \varepsilon_0^{-2} h^{-2\rho}, \tag{6.5}$$

and

$$\operatorname{supp}(\chi_j^h) \subset Q_j = B(z_j, \varepsilon_0 h^\rho), \tag{6.6}$$

where $B(c, r)$ denotes the open disc in $\mathbb{R}^2$ of center c and radius r. Moreover, we can add the property that:

$$\text{either supp } \chi_j \cap \partial\Omega = \emptyset, \qquad \text{either } z_j \in \partial\Omega. \tag{6.7}$$

According to the two alternatives in (6.7), we can decompose the sum in (6.4) in the form

$$\sum = \sum_{int} + \sum_{bnd},$$

where "int" is in reference to the j's such that $z_j \in \Omega$ and "bnd" is in reference to the j's such that $z_j \in \partial\Omega$.

The second point is to implement this partition of unity in the following way:

$$q_h^N(u) = \sum_j q_h(\chi_j^h u) - h^2 \sum_j \| |\nabla \chi_j^h| \, u\|^2, \qquad \forall u \in H^1(\Omega). \tag{6.8}$$

Here q_h^N (or $q_{h,A}^N$, if we want to keep the reference to the magnetic potential) denotes the quadratic form

$$q_{h,A}^N(u) = \int_\Omega |h \nabla u - iAu|^2 \, dx, \tag{6.9}$$

and we recall that $\| \cdot \|$ denotes the L^2-norm in Ω.

This formula is usually called IMS formula (see [CFKS]) but is actually much older (see for example [Mel]).

If $a_{h,A}^N$ is the associated sesquilinear form, (6.8) is the consequence of the identity, for any function $\chi \in C^\infty(\bar{\Omega})$ and any $u \in H^1(\Omega)$:

$$q_{h,A}^N(\chi u) = \operatorname{Re} a_{h,A}^N(u, \chi^2 u) + h^2 \, \| |\nabla \chi| \, u\|_{L^2(\Omega)}^2. \tag{6.10}$$

We will also use later the property that, for any function $\chi \in C^\infty(\bar{\Omega})$ and any u in the domain of $P_{h,A,\Omega}^N$, that is for any u in the space $D(P_{h,A,\Omega}^N) := \{v \in H^2(\Omega) \mid v \cdot (\partial - iA) \, u_{/\partial\Omega} = 0\}$:

$$q_{h,A}^N(\chi u) = \operatorname{Re} \langle P_{h,A,\Omega}^N u \mid \chi^2 u \rangle_{L^2(\Omega)} + h^2 \, \| |\nabla \chi| \, u\|_{L^2(\Omega)}^2. \tag{6.11}$$

We can rewrite the right hand side of (6.8) as the sum of three (types of) terms.

$$q_h(u) = \sum_{int} q_h(\chi_j^h u) + \sum_{bnd} q_h(\chi_j^h u) - h^2 \sum_j \| |\nabla \chi_j^h| \, u\|^2, \qquad \forall u \in H^1(\Omega). \tag{6.12}$$

For the last term in the right hand side of (6.12), we get using (6.5):

$$h^2 \sum_j \| |\nabla \chi_j^h| \, u\|^2 \leqslant C h^{2-2\rho} \varepsilon_0^{-2} \, \|u\|^2. \tag{6.13}$$

This measures the price to pay when using a fine partition of unity: If ρ is large, the error is as big as $h^{2-2\rho}$. We shall see later what could be the best choice of ρ or of ε_0 for our various problems (note that the play with ε_0 large will be only interesting when $\rho = \frac{1}{2}$).

The first term in the right hand side of (6.12) can be estimated from below by using (1.4). The support of $\chi_j^h u$ is indeed contained in Ω. So we have:

$$\sum_{int} q_h(\chi_j^h u) \geqslant h \sum_{int} \int B(x) \, |\chi_j^h u|^2 \, dx. \tag{6.14}$$

The second term in the right hand side of (6.12) is the more delicate and corresponds to the specificity of the Neumann problem. We have to find a lower bound for $q_h(\chi_j^h u)$ for some j such that $z_j \in \partial\Omega$. We emphasize that z_j depends on h, so we have to be careful in the control of the uniformity.

Let z be a point in $\partial\Omega$. The boundary being regular, we can, by a change of coordinates in a small neighborhood of this point, rewrite the form $q_{h,A}$ for u's with support in this neighborhood of z:

$$q_{h,A}(u) = \int_{\tilde{x}_2 > 0} \sum g^{k,\ell}(\tilde{x})(ih\partial_{\tilde{x}_k} \tilde{u} + \tilde{A}_k(\tilde{x}) \, \tilde{u}) \cdot \overline{(ih\partial_{\tilde{x}_\ell} \tilde{u} + \tilde{A}_\ell(\tilde{x}) \, \tilde{u})} \det(g(\tilde{x}))^{\frac{1}{2}} \, d\tilde{x}.$$

Here we can assume that the new cordinates of z are $(0, 0)$ and we can also assume that the matrix g is the identity at z:

$$g^{k,\ell}(0) = \delta_{k,\ell}.$$

Of course g depends on z, but all the estimates we could need on the derivatives of g will be uniform in z.

The game is now to compare for u's with support in a ball of the type $B(z, 2C\varepsilon_0 h^\rho)$ $q_{h,A}$ with the quadratic form:

$$q_{h,\tilde{A}}(\tilde{u}) = \int_{x_2 > 0} |(ih\partial_{x_1} - \tfrac{1}{2} B(z) \, x_2) \, u|^2 + |(ih\partial_{x_2} + \tfrac{1}{2} B(z) \, x_1) \, u|^2 \, dx.$$

We have omitted for simplicity the tildes in the right hand side. The comparison is not direct but as an intermediate step, we have to use a gauge transformation (multiplication by $\exp -i(\phi_j/h)$) associated to a C^∞ function ϕ_j such that

$$\omega_A = \omega_{A_{new,j}} - d\phi_j,$$

with

$$A_{new,j}(z_j) = 0,$$
$$|A_{new,j}(x) - \tfrac{1}{2}(B(z_j)(-x_2, x_1))| \leqslant C\,|x|^2.$$

In this formula, ω_A is the one-form attached to the vector field A: $\omega_A := A_1\,dx_1 + A_2\,dx_2$. Let us emphasize that C is independent of j. Let us also introduce for the next formula: $A_j^{lin} := \frac{1}{2}(B(z_j)(-x_2, x_1))$.

Following line by line the computations of [HeMo2], we get:

$$\begin{aligned} q_{h,A}(\chi_j^h u) &\geqslant (1 - Ch^{2\theta}\varepsilon^2 - C\varepsilon_0 h^\rho)\, q^h[A_j^{lin}]\left(\exp -\frac{i}{h}\phi_j \chi_j^h u\right) \\ &\quad - Ch^{-2\theta}\varepsilon^{-2}\, \||x|^2\, \chi_j^h u\|^2 \\ &\geqslant (1 - Ch^{2\theta}\varepsilon^2 - C\varepsilon_0 h^\rho)\, q^h[A_j^{lin}]\left(\exp -\frac{i}{h}\phi_j \chi_j^h u\right) \\ &\quad - Ch^{4\rho - 2\theta}\varepsilon^{-2}\, \|\chi_j^h u\|^2. \end{aligned}$$

We can now use the result concerning the half -plane in order to get:

$$q_{h,A}(\chi_j^h u) \geqslant (1 - Ch^{2\theta}\varepsilon^2 - C\varepsilon_0 h^\rho)\, h\Theta_0 \int B(z_j)\,|\chi_j^h u|^2\,dx - Ch^{4\rho-2\theta}\varepsilon^{-2}\,\|\chi_j^h u\|^2. \tag{6.15}$$

We now put together all the estimates and obtain:

$$\begin{aligned} q_{h,A}(u) &\geqslant h \sum_{int} \int B(x)\,|\chi_j^h u|^2\,dx \\ &\quad + (1 - Ch^{2\theta}\varepsilon^2 - C\varepsilon_0 h^\rho)\, h\Theta_0 \sum_{bnd} \int B(z_j)\,|\chi_j^h u|^2\,dx \\ &\quad - Ch^{4\rho-2\theta}\varepsilon^{-2} \sum_{bnd} \|\chi_j^h u\|^2 \\ &\quad - C\varepsilon_0^{-2} h^{2-2\rho}\,\|u\|^2. \end{aligned} \tag{6.16}$$

We have now to optimize our choices of ρ, θ and ε, ε_0. If we just want to get a lower bound of the spectrum, we can first write:

$$q_{h,A}(u) \geqslant h \min(b, \Theta_0 b') \|u\|^2 - (Ch^{2\theta+1}\varepsilon^2 + C\varepsilon_0 h^{\rho+1} + Ch^{4\rho-2\theta}\varepsilon^{-2} + C\varepsilon_0^{-2} h^{2-2\rho}) \|u\|^2.$$

Taking $\rho = \frac{3}{8}$, $\theta = \frac{1}{8}$, $\varepsilon = \varepsilon_0 = 1$, we get:

$$q_{h,A}(u) \geqslant (\min(b, \Theta_0 b') h - Ch^{\frac{5}{4}}) \|u\|^2. \tag{6.17}$$

So, taking $u = u_h^1$, where u_h^1 is a groundstate, we obtain from (6.17):

Proposition 6.1. *There exist constants $C > 0$ and $h_0 > 0$ such that, for all $h \in]0, h_0]$:*

$$\mu^{(1)}(h) \geqslant (\min(b, \Theta_0 b')) h - Ch^{\frac{5}{4}}. \tag{6.18}$$

This result is not optimal. We shall show in Theorem 9.1 a stronger estimate of the remainder in $\mathcal{O}(h^{3/2})$. In the case of a constant magnetic field, we shall actually determine the coefficient of $h^{3/2}$ as suggested by the computations of [BeSt] and [PiFeSt]. This will be done in Sections 10 and 11.

But for the control of the decay, we need also to take in (6.16) $\rho = \frac{1}{2}$, $\theta = \frac{1}{8}$, $\varepsilon = 1$ and ε_0 large. This gives an estimate which may look weaker but will be more efficient.

Proposition 6.2. *There exists C and h_0 and, for all $\varepsilon_0 > 0$, there exists $C(\varepsilon_0)$ such that, for $h \in]0, h_0]$, the following inequality*

$$\begin{aligned} q_{h,A}(u) \geqslant\ & h \sum_{int} \int B(x) |\chi_j^h u|^2 \, dx \\ & - C(\varepsilon_0) h \sum_{bnd} \int |\chi_j^h u|^2 \, dx \\ & - \frac{Ch}{\varepsilon_0^2} \sum_{int} \int |\chi_j^h u|^2 \, dx \end{aligned} \tag{6.19}$$

is satisfied, for all $u \in H^1(\Omega)$.

6.4. *Agmon's Estimates*

We continue to follow the proof given in [HeMo2] and will explain the modifications needed in order to treat Neumann problems.

We first observe that if Φ is a real and uniformly Lipschitzian function and if u is in the domain of the Neuman realization of $P_{h,A}$, then we have by a simple integration by part (see (6.10)):

$$\begin{aligned} \operatorname{Re}\left\langle P_{h,A}u, \exp\frac{2\Phi}{h^{\frac{1}{2}}}u\right\rangle &= \operatorname{Re}\left\langle\left(\frac{h}{i}\nabla-A\right)u, \left(\frac{h}{i}\nabla-A\right)\exp\frac{2\Phi}{h^{\frac{1}{2}}}u\right\rangle \\ &= \left\langle\left(\frac{h}{i}\nabla-A\right)\exp\frac{\Phi}{h^{\frac{1}{2}}}u, \left(\frac{h}{i}\nabla-A\right)\exp\frac{\Phi}{h^{\frac{1}{2}}}u\right\rangle - h\left\||\nabla\Phi|\exp\frac{\Phi}{h^{\frac{1}{2}}}u\right\|^2 \\ &= q_h[A]\left(\exp\frac{\Phi}{h^{\frac{1}{2}}}u\right) - h\left\||\nabla\Phi|\exp\frac{\Phi}{h^{\frac{1}{2}}}u\right\|^2. \end{aligned} \tag{6.20}$$

We now take $u = u_h$ an eigenfunction attached to the lowest eigenvalue $\mu^{(1)}(h)$. This gives:

$$\mu^{(1)}(h)\left\|\exp\frac{\Phi}{h^{\frac{1}{2}}}u\right\|^2 = q_{h,A}\left(\exp\frac{\Phi}{h^{\frac{1}{2}}}u\right) - h\left\||\nabla\Phi|\exp\frac{\Phi}{h^{\frac{1}{2}}}u\right\|^2. \tag{6.21}$$

It remains to reimplement the previous inequality in this new one and to use the upper bound (6.2).

Let us take $\Phi(x) = \alpha\max(d(x,\partial\Omega), h^{\frac{1}{2}})$, where $\alpha > 0$ has to be determined. Let us use Proposition 6.2. We first write:

$$\begin{aligned} q_{h,A}\left(\exp\frac{\Phi}{h^{\frac{1}{2}}}u\right) \geqslant{}& h\sum_{int}\int B(x)\left|\exp\frac{\Phi}{h^{\frac{1}{2}}}\chi_j^h u\right|^2 dx \\ &- C(\varepsilon_0)\,h\sum_{bnd}\int\left|\chi_j^h\exp\frac{\Phi}{h^{\frac{1}{2}}}u\right|^2 dx \\ &- \frac{Ch}{\varepsilon_0^2}\sum_{int}\int\left|\exp\frac{\Phi}{h^{\frac{1}{2}}}\chi_j^h u\right|^2 dx. \end{aligned} \tag{6.22}$$

Let us first consider the case $B(x) = b$. The inequality (6.2) becomes:

$$\mu^{(1)}(h) \leqslant \Theta_0 bh + Ch^{\frac{5}{4}}. \tag{6.23}$$

Using (6.20), we finally obtain:

$$\left(b(1-\Theta_0) - Ch^{\frac{1}{4}} - \frac{C}{\varepsilon_0^2} - \alpha^2\right)\sum_{int}\int\left|\exp\frac{\Phi}{h^{\frac{1}{2}}}\chi_j^h u\right|^2 dx \leqslant C(\varepsilon_0)\sum_{bnd}\int|\chi_j^h u|^2\,dx. \tag{6.24}$$

Taking ε_0 large enough and

$$\alpha < \sqrt{b(1-\Theta_0)}$$

we finally obtain the estimate

$$\left\| \exp \alpha \frac{d(x, \partial\Omega)}{h^{\frac{1}{2}}} u_h \right\| \leqslant C \, \|u_h\|, \tag{6.25}$$

for some new constant $C > 0$.

Note that we just need a weak upper bound of $\mu^{(1)}(h)$ in order to make the argument correct. In particular, the remainder $Ch^{\frac{1}{4}}$ in (6.24) can be replaced by $o(1)$ without changing the argument. This is the upper bound obtained in [LuPa2].

Let us now show how to go from an L^2-estimate to an L^∞-estimate. Using (6.21), we first get:

$$q_{h,A}\left(\exp \frac{\Phi}{h^{\frac{1}{2}}} u\right) \leqslant Ch \left\| \exp \frac{\Phi}{h^{\frac{1}{2}}} u \right\|^2. \tag{6.26}$$

This gives, together with (6.25), an estimate in H^1 norm. Coming back to the second order differential equation satisfied by $\exp(\Phi/h^{1/2})$ and using that Φ is constant near the boundary[3], we can use the regularity of the Neumann problem for getting a control in H^2. This gives finally (through the Sobolev injection theorem) the proof in the constant magnetic field case of the following theorem:

THEOREM 6.3. *Let us assume that the condition*

$$\Theta_0 b' < b \tag{6.27}$$

is satisfied.

There exists $C > 0$, $\alpha > 0$ and $\nu \in \mathbb{R}$, such that if u_h is the ground state of $P^N_{A,h,\Omega}$, then:

$$\exp \alpha \frac{d(x, \partial\Omega)}{h^{\frac{1}{2}}} |u_h(x)| \leqslant Ch^{-\nu} \, \|u_h\|_{L^2}, \qquad \forall x \in \Omega. \tag{6.28}$$

The condition (6.27) is always satisfied when B is constant because $b = b'$ and $\Theta_0 < 1$. In this case, this is essentially similar to the result of [PiFeSt],

[3] Actually, we need first to use a regularized Φ in order to make the argument rigorous. Another way, would be to use L^p estimates.

which was recalled in Theorem 5.1 modulo possibly a negative power of h. These authors use indeed a normalization by the L^∞-norm.

The proof when B is not constant is essentially the same. The inequality (6.24) becomes:

$$\left(b-\Theta_0 b'-o(1)-\frac{C}{\varepsilon_0^2}-\alpha^2\right)\sum_{int}\int\left|\exp\frac{\Phi}{h^{\frac{1}{2}}}\chi_j^h u\right|^2 dx\leqslant C(\varepsilon_0)\sum_{bnd}\int|\chi_j^h u|^2\,dx. \tag{6.29}$$

We can then conclude in the same way (modulo the proof of (6.2) which will be given later or using the result of Lu–Pan (6.3)).

Remark 6.4. On the contrary, when $b<\Theta_0 b'$ the ground state decays exponentially outside neighborhoods of points where $B(x)=b$.

Remark 6.5. In the case when the boundary has right corners (i.e. there exists near these points of the boundary, a local diffeomorphism, whose differential is a rotation at the corner), similar results can be proved[4] using for example the distance to the corner points instead of the distance at the boundary. We only need to show that: $\Theta_1<\Theta_0$ in the case of a constant magnetic field in order to have the following condition satisfied

$$\Theta_1 b''<\min(b,\Theta_0 b'), \tag{6.30}$$

where b'' is the infimum of $B(x)$ over the various right corner points. Other results devoted to the case of three dimensional domains with regular boundaries [LuPa5] or with edges and corners [Pan] appear recently. These results were also announced in [LuPa4].

6.5. *Decay Inside the Boundary*

We first treat the "easy" case when (6.27) is satisfied and when $B(x)$ restricted to $\partial\Omega$ has only isolated minima. What we would like is to recover in a more precise way the statements of [LuPa2] (Proposition 7.2 and Theorem 7.3). Let us first remember what we got from (6.16) by taking $\rho=\frac{3}{8}$ and $\theta=\frac{1}{8}$.

$$\begin{aligned}q_{h,A}^N(u)\geqslant h\sum_{int}\int B(x)\,|\chi_j^h u|^2\,dx+(1-Ch^{\frac{1}{4}})\,h\Theta_0\sum_{bnd}\int B(z_j)\,|\chi_j^h u|^2\,dx\\ -Ch^{\frac{5}{4}}\sum_{bnd}\|\chi_j^h u\|^2-Ch^{\frac{5}{4}}\sum_{int}\|\chi_j^h u\|^2.\end{aligned} \tag{6.31}$$

[4] See [Ja]. This is true at least for the H^1 estimates. In the proof we describe above, one has probably to be careful with the regularity theorems in domain with corners.

We now apply this inequality with $u = \exp(\Phi_h/h^{1/2})\, u_h$ where u_h is the ground state.

This gives (cf (6.21)):

$$\mu^{(1)}(h) \left\| \exp \frac{\Phi_h}{h^{\frac{1}{2}}} u_h \right\|^2 \geqslant q_{h,A}\left(\exp \frac{\Phi_h}{h^{\frac{1}{2}}} u_h \right) - \left\| |\nabla \Phi| \exp \frac{\Phi_h}{h^{\frac{1}{2}}} u_h \right\|^2. \tag{6.32}$$

Combining the two estimates and (6.2), we obtain:

$$\begin{aligned} 0 \geqslant h \sum_{int} \int (B(x) - \Theta_0 b' - Ch^{\frac{1}{4}} - |\nabla \Phi|^2) \left| \chi_j^h \exp \frac{\Phi_h}{h^{\frac{1}{2}}} u \right|^2 dx \\ + (1 - Ch^{\frac{1}{4}})\, h\Theta_0 \sum_{bnd} \int (B(x) - b' - Ch^{\frac{1}{4}} - |\nabla \Phi|^2) \left| \chi_j^h \exp \frac{\Phi_h}{h^{\frac{1}{2}}} u \right|^2 dx. \end{aligned} \tag{6.33}$$

We have just to choose a suitable Φ for getting the right estimate. Let us remember that we have already obtained a good control inside Ω under the condition $\Theta_0 b' < b$. So it is enough to give a control in a neighborhood $\mathcal{N}$ of $\partial\Omega$ of size $\sim h^{3/8}$. Outside this neighborhood, we know indeed already that the ground state is exponentially small. In this neighborhood, we can assume that we have local coordinates (t, s) introduced in Appendix B (the neighborhood being described by $\{(t, s) \mid 0 \leqslant t \leqslant Ch^{3/8}\}$). Changing possibly the decomposition $\sum_{int} + \sum_{bnd}$, we can assume that $\sum_{bnd} (\chi_j^h)^2 \geqslant 1$ on this neighborhood and vanishes outside a neighborhood of the same type $\mathcal{N}'$. By taking $\Phi = \alpha_1 \chi(t)\, d(s) = \chi(t)\, \phi(s)$ where $d_{\partial\Omega}$ is the Agmon distance inside $\partial\Omega$ to the set of minima's attached to the degenerate Agmon metric $(B(s, 0) - b')\, ds^2$, α_1 is strictly positive and sufficiently small and $\chi(t)$ localizes near the boundary, we obtain, for some constant $C > 0$, the inequality:

$$0 \geqslant \sum_{bnd} \int (B(s, 0) - b' - Ch^{\frac{1}{4}} - C\, |\phi'(s)|^2) \left| \chi_j^h \exp \frac{\Phi}{h^{\frac{1}{2}}} u_h \right|^2 dx. \tag{6.34}$$

This leads to a decay[5] in the form

$$\sum_{bnd} \int \left| \chi_j^h \exp \frac{\Phi}{h^{\frac{1}{2}}} u_h \right|^2 dx \leqslant C_\varepsilon \exp \frac{\varepsilon}{h^{\frac{1}{2}}} \|u_h\|_{L^2}^2, \tag{6.35}$$

[5] We do not try to be optimal in the measure of the decay.

for any $\varepsilon > 0$, or

$$\int_{\mathcal{N}} \left| \exp \frac{\Phi}{h^{\frac{1}{2}}} u_h \right|^2 dx \leqslant C_\varepsilon \exp \frac{\varepsilon}{h^{\frac{1}{2}}} \|u_h\|_{L^2}^2 . \tag{6.36}$$

Actually, we can extend this inequality to an h-independent tubular neighborhood of $\partial\Omega$. Combining with the decay outside $\partial\Omega$, this leads to the following localization theorem:

THEOREM 6.6. *Let*

$$p(\partial\Omega) = \{\omega \in \partial\Omega \mid B(\omega) = b'\}. \tag{6.37}$$

Let us assume that $\Theta_0 b' < b$. Then, there exists $\alpha_1 > 0$ and, for any $\varepsilon > 0$, constants $C(\varepsilon)$ and $h_0(\varepsilon)$ such that a normalized groundstate u_h satisfies

$$\left\| \exp \frac{\alpha_1 \tilde{d}(z, p(\partial\Omega))}{h^{\frac{1}{2}}} u_h \right\| \leqslant C(\varepsilon) \exp \frac{\varepsilon}{h^{\frac{1}{2}}}, \tag{6.38}$$

for all $h \in]0, h_0(\varepsilon)]$. Here, the "distance" $\tilde{d}$ is defined by:

$$\tilde{d}(z, p(\partial\Omega)) = d(z, \partial\Omega) + \chi(d(z, \partial\Omega))\, d_{\partial\Omega}(s(z)). \tag{6.39}$$

As in [HeSj1], assuming the non degeneracy of the minima of $s \mapsto B(s, 0)$, one can probably improve the estimate and get the existence of $C > 0$ such that:

$$\left\| \exp \frac{\alpha_1 \tilde{d}(z, p(\partial\Omega))}{h^{\frac{1}{2}}} u_h \right\| \leqslant C \exp C h^{-\frac{1}{4}}. \tag{6.40}$$

Remark 6.7. This theorem gives an accurate version of the result given in [LuPa2].

Of course these estimates are not sufficient to treat the problem considered in [PiFeSt], when $B = \text{const.} > 0$, where we would like to prove localization near the points of maximal curvature. When B is constant, the effect of localization is more difficult to show. We shall see that the effective potential creating the localization is $-h^{3/2}\kappa(s)$ where $\kappa(s)$ is the curvature at the point s of the boundary. One hopes to show a decay in $\exp -\psi/h^{1/4}$ for some ψ defined on $\partial\Omega$ and related to the metric $(\kappa_{max} - \kappa(s))\, g(s)\, ds^2$, where g is some strictly positive density. But this supposes finer estimates with remainders in $o(h^{3/2})$, which will be given in Section 10.

7. A SHARP SEMI-CLASSICAL ESTIMATE OF THE GROUNDSTATE ENERGY FOR THE TWO DIMENSIONAL MAGNETIC DIRICHLET PROBLEM

7.1. *Main Theorems*

Let $\Omega \subset \mathbb{R}^2$ be a regular bounded open set and let $A = (A_1, A_2)$ in $C^1(\bar{\Omega}; \mathbb{R}^2)$ such that

$$B(z) := \frac{\partial A_2}{\partial x}(z) - \frac{\partial A_1}{\partial y}(z) \geqslant 0. \tag{7.1}$$

For any $h \in]0, 1]$, $P^D_{h, A, \Omega}$ denotes the Dirichlet self-adjoint operator, defined on $L^2(\Omega)$, associated to $(hD - A)^2$; $(D = D_z = (D_x; D_y))$.

The associated sesquilinear form will be denoted by $q^D_{h, A, \Omega}$:

$$q^D_{h, A, \Omega}(u) = \|hDu - Au\|^2_{L^2(\Omega)}, \qquad \forall u \in H^1_0(\Omega). \tag{7.2}$$

The associated self-adjoint operator on $L^2(\Omega)$ is

$$P_{h, A, \Omega} u = (hD - A)^2 u, \qquad \forall u \in H^2(\Omega) \cap H^1_0(\Omega). \tag{7.3}$$

When $\Omega = \mathbb{R}^2$, this self-adjoint operator on $L^2(\mathbb{R}^2)$ will be denoted by $P_{h, A}$. We recall that $-i[hD_x - A_1, hD_y - A_2] = hB \geqslant 0$ leads to the estimate

$$h \|B^{1/2} u\|^2_{L^2(\Omega)} \leqslant q_{h, A, \Omega}(u), \qquad \forall u \in H^1_0(\Omega). \tag{7.4}$$

THEOREM 7.1. *If $A \in C^1(\bar{\Omega}; \mathbb{R}^2)$ and if $B \geqslant 0$, then for any $\tau \in]0, h^{-1}[$,*

$$h \inf \operatorname{Sp}(-\tau h^2 \Delta^D_\Omega + (1 - \tau h) B) \leqslant \inf \operatorname{Sp}(P^D_{h, A, \Omega}). \tag{7.5}$$

Here $-\Delta^D_\Omega = P^D_{1, 0, \Omega}$ is the Laplace operator on Ω with Dirichlet boundary condition and, for an operator P, $\operatorname{Sp}(P)$ denotes its spectrum. We will omit the superscript D when there is no possible confusion.

Proof. The estimate (7.4) leads to

$$\tau h P_{h, A, \Omega} + (1 - \tau h) hB \leqslant P_{h, A, \Omega}, \qquad \forall \tau \in [0, h^{-1}]. \tag{7.6}$$

Then we use (2.2) and this gives (7.5). ∎

We will be interested in the special case when $B(z) \in C^{3+M}(\bar{\Omega})$ satisfies, for some $z_0 \in \Omega$,

$$B(z) > b := B(z_0) > 0, \qquad \forall z \in \bar{\Omega} \setminus \{z_0\}, \tag{7.7}$$

and we assume that the minimum is non degenerate:

$$\text{Hess } B(z_0) > 0. \tag{7.8}$$

We introduce in this case the notation:

$$a = \text{Tr}(\tfrac{1}{2} \text{ Hess } B(z_0))^{1/2}. \tag{7.9}$$

If we use the estimate (7.5) with $\tau = a^2/(4b^2)$, then the assumptions (7.7) and (7.8) and the well known result giving the semi-classical estimate of the ground state of the Schrödinger operator

$$-\tau h^2 \Delta_\Omega + (1 - \tau h)\, B$$

(see for example [HeSj1], [Si] or [He1]), guarantee that

$$\begin{aligned} \inf \text{Sp}(-\tau h^2 \Delta_\Omega + (1-\tau h)\, B) &= (1-\tau h)\, b + h\tau^{1/2} a + \mathcal{O}(h^2) \\ &= b + \frac{a^2}{4b}\, h + \mathcal{O}(h^2). \end{aligned} \tag{7.10}$$

So from (7.10) and (7.5), we obtain the existence of a constant $C > 0$ such that:

$$hb + \frac{a^2}{4b}\, h^2 - Ch^3 \leqslant \inf \text{Sp}(P_{h, A, \Omega}). \tag{7.11}$$

This estimate is actually not optimal and will be improved by the following theorem.

THEOREM 7.2. *If $A \in C^{4+M}(\bar{\Omega}; \mathbb{R}^2)$, with $M \geqslant 0$, and if the hypotheses* (7.7) *and* (7.8) *are satisfied, then there exists a constant $C > 0$ such that*

$$-Ch^{19/8} \leqslant \inf \text{Sp}(P_{h, A, \Omega}) - \left[b + \frac{a^2}{2b}\, h \right] h \leqslant Ch^{5/2}. \tag{7.12}$$

The upper bound in (7.12) follows from the following theorem.

THEOREM 7.3. *Under the assumptions* (7.7) *and* (7.8), *then the lowest eigenvalue* $\lambda^{(1)}(h)$ *of* $P^D_{h,A,\Omega}$ *satisfies*

$$\lambda^{(1)}(h) \leqslant bh + \frac{a^2}{2b} h^2 + \mathcal{O}(h^{5/2}), \tag{7.13}$$

as $h \to 0^+$.

Here a and b are introduced in (7.7) *and* (7.9).

The left hand side of the estimate (7.12) will follow as in (7.11) from (7.6) but without using (2.2). The magnetic field near z_0 will not be neglected. So in (7.5) we will take $P_{h,A^1,\Omega}$ instead of $-h^2\Delta_\Omega$, where $A^1(z) = (-\frac{1}{2}by, \frac{1}{2}bx)$.

For that, we will use the following theorem.

THEOREM 7.4. *Let* $A \in C^3(\bar{\Omega}; \mathbb{R}^2)$ *and* $V \in C^3(\bar{\Omega}; \mathbb{R})$ *such that, for some* $z_0 \in \Omega$,

$$B(z) = \frac{\partial A_2}{\partial x}(z) - \frac{\partial A_1}{\partial y}(z) \geqslant B(z_0) = b > 0, \qquad \forall z \in \bar{\Omega} \tag{7.14}$$

$$V(z) > V(z_0), \qquad \forall z \in \bar{\Omega} \setminus \{z_0\}, \tag{7.15}$$

$$\text{Hess}\, V(z_0) > 0.$$

Let $\rho \in [0, 1]$ *be given and let us consider the Dirichlet operator*

$$P^D_{h,A,V,\Omega} = P^D_{h,A,\Omega} - hB(z) + h^\rho V(z).$$

Then there exists a constant $C = C_\rho > 0$ *such that*

$$\left| \inf \text{Sp}(P^D_{h,A,V,\Omega}) - h^\rho V(z_0) + h^{1+\rho} \frac{a_V^2}{2b} \right| \leqslant C h^{d(\rho)} \tag{7.16}$$

with

$$d(\rho) = \inf(\tfrac{3}{2} + \tfrac{3}{4}\rho, 2, 1 + 2\rho) \tag{7.17}$$

$$a_V = \text{Tr}(\tfrac{1}{2}\text{Hess}\, V(z_0))^{1/2}, \tag{7.18}$$

As a matter of fact, if we take in (7.6) $\tau = h^{-1/2}$, then we get the following corollary.

COROLLARY 7.5.

$$h^{1/2} \inf \text{Sp}(P^D_{h,A,\Omega} - hB(z) + h^{1/2} V(z) \leqslant \inf \text{Sp}(P^D_{h,A,\Omega})$$

with $V(z) = B(z)$.

So (7.14) and (7.15) are satisfied, when (7.7) and (7.8) are satisfied. Formula (7.16) with $\rho = \frac{1}{2}$ gives the lower bound of (7.12).

Remark 7.6.

1. Theorem 7.4 is valid for a constant magnetic field.
2. Under the assumptions of Theorem 7.2, Theorem 7.4 can also be applied to $P_{h,A,\Omega}$ by taking $\rho = 1$ and $V(z) = B(z)$.

But in this case we get that

$$\inf \mathrm{Sp}(P_{h,A,\Omega}) = hb + \mathcal{O}(h^2).$$

Remark 7.7. Theorem 7.4 is valid if we change the definition of $P_{h,A,V,\Omega}$ into:

$$P_{h,A,V,\Omega} = P_{h,A,\Omega} - bh + h^{\rho} V(z).$$

Remark 7.8. The results of Theorem 7.4, when $\rho \in [0, 1[$, are valid for the modified Neumann operator $P^{N,mod}_{h,A,V,\Omega}$ associated to the quadratic form, defined on $H^1(\Omega)$ by:

$$q^N_{h,A,V,\Omega}(u) = \int_\Omega \{|[hD_x - A_1(z) + i(hD_y - A_2(z))]\, u(z)|^2 + h^{\rho} V(z)\, |(z)|^2\}\, dz.$$

It is also the case for the Neumann operator $P^N_{h,A,\Omega} + h^{\rho} V(z) - hb$ associated to the quadratic form

$$q^N_{h,A,\Omega}(u) = \int_\Omega \{|h\, Du(z) - A(z)\, u(z)|^2 + [h^{\rho} V(z) - hb]\, |u(z)|^2\}\, dz,$$

$$\forall u \in H^1(\Omega).$$

But the reader should not forget that (7.4) is not valid for Neumann operator.

Let us prove this last remark. Let χ be a cut-off function supported in Ω, such that $\chi(z) = 1$ in a neighborhood of z_0.
Let us consider the partition of unity $(\vartheta_j(z))$, $j \in \{0, 1\}$:

$$\vartheta_0(z) = \chi(z)[\chi^2(z) + (1 - \chi(z))^2]^{-1/2},$$
$$\vartheta_1(z) = (1 - \chi(z))[\chi^2(z) + (1 - \chi(z))^2]^{-1/2}.$$

Then, by the so called IMS Formula (see for example [HeMo2] or the discussion in Section 6 (6.8 or (6.10)), there exists a constant $C_1 > 0$, such that

$$|q^N_{h,A,V,\Omega}(u) - [q^N_{h,A,V,\Omega}(\vartheta_0 u) + q^N_{h,A,V,\Omega}(\vartheta_1 u)]| \leqslant C_1 h^2 \, \|u\|^2_{L^2(\Omega)}, \qquad \forall u \in H^1(\Omega),$$

and, thanks to (7.15)), there exists $\delta > 0$ such that:

$$q^N_{h,A,V,\Omega}(\vartheta_1 u) - h^\rho V(z_0) \, \|\vartheta_1 u\|^2_{L^2(\Omega)} \geqslant \delta h^\rho \, \|\vartheta_1 u\|^2_{L^2(\Omega)}, \qquad \forall u \in H^1(\Omega).$$

Combining with the theorem, this gives the lower bound.

But the upper bound is an immediate consequence of the min-max principle. We get indeed easily that

$$\inf \operatorname{Sp}(P^N_{h,A,V,\Omega}) \leqslant \inf \operatorname{Sp}(P_{h,A,V,\mathcal{O}})$$

for any open subset $\mathcal{O}$ of Ω containing z_0.

The proof remains the same for the operator $P^N_{h,A,\Omega} + h^\rho V(z) - hb$.

Later we will assume that $z_0 = 0$ when giving the proof of the above Theorems 7.2 and 7.4.

7.2. *Some Estimates on Eigenfunctions*

We take the hypotheses of Theorem 7.4. The quadratic form associated to $P_{h,A,V,\Omega}$ is defined on $H^1_0(\Omega)$ by:

$$\begin{aligned} q_{h,A,V,\Omega}(u) &= q_{h,A,\Omega} + \int_\Omega [-hB(z) + h^\rho V(z)] \, |u(z)|^2 \, dz \\ &= \int_\Omega \{ |[hD_x - A_1(z) + i(h\,D_y - A_2(z))] \, u(z)|^2 + h^\rho V(z) \, |u(z)|^2 \} \, dz. \end{aligned} \tag{7.19}$$

Let $\lambda(h)$ be an eigenvalue of $P_{h,A,V,\Omega}$ and u_h be an associated eigenfunction. Let $\phi_h(z)$ denotes the Agmon distance of z to the well

$$U_h = \{ w \in \Omega; \, h^\rho V(w) - \lambda(h) \leqslant 0 \},$$

associated to the Agmon metric $[h^\rho V(w) - \lambda(h)]_+ \, dz^2$.

Proposition 7.9. *For any* $\varepsilon \in \,]0, 1]$,

$$\begin{aligned} (1-\varepsilon^2) \int_\Omega [h^\rho V(z) - \lambda(h)]_+ \, e^{2\varepsilon\phi_h(z)/h} \, |u_h(z)|^2 \, dz \\ \leqslant \int_\Omega [h^\rho V(z) - \lambda(h)]_- \, |u_h(z)|^2 \, dz. \end{aligned} \tag{7.20}$$

Proof. As in [HeMo2], we use the formula (already used in (6.21)),

$$q_{h,A,V,\Omega}(e^{\phi}u_h) = \lambda(h)\, \|e^{\phi}u_h\|^2_{L^2(\Omega)} + h^2\, \|e^{\phi}u_h\, \nabla\phi\|^2_{L^2(\Omega)}, \tag{7.21}$$

for any real Lipschitz function ϕ.

Observing that $|\nabla\phi_h(z)|^2 \leqslant [h^{\rho}V(z) - \lambda(h)]_+$ and using (7.4) and (7.21), we get

$$\int_{\Omega} \{h^{\rho}V(z) - \lambda(h) - \varepsilon^2[h^{\rho}V(z) - \lambda(h)]_+\}\, e^{2\varepsilon\phi_h(z)/h}\, |u_h(z)|^2\, dz \leqslant 0,$$

which is exactly (7.20), if we observe that, in the support of $[h^{\rho}V(z) - \lambda(h)]_-$, we have $\phi_h(z) = 0$. ∎

A rough upper bound of the ground state energy. We take a magnetic potential A such that

$$A(z) = \tfrac{1}{2}(-by, bx) + \mathcal{O}(|z|^3). \tag{7.22}$$

This is always possible thanks to (7.14) (see for example [HeMo1]).

Let us consider the test function $u_h^0(z) = \chi(z)e^{-b|z|^2/(2h)}$, where χ a cut-off function with support in Ω and $\chi(z) = 1$ in a neighborhood of $z_0 = 0$. We get easily using (7.15) and (7.22) that

$$q_{h,A,V,\Omega}(u_h^0) - h^{\rho}V(0)\, \|u_h^0\|^2_{L^2(\Omega)} \leqslant Ch^{1+\rho}\, \|u_h^0\|^2_{L^2(\Omega)},$$

so

$$\inf \operatorname{Sp}(P_{h,A,V,\Omega}) - h^{\rho}V(0) \leqslant Ch^{1+\rho}. \tag{7.23}$$

So we have proved that, if $\lambda^{(1)}(h)$ is a groundstate energy, then

$$\lambda^{(1)}(h) = h^{\rho}V(0) + \mathcal{O}(h^{1+\rho}). \tag{7.24}$$

Then, the assumption (7.15) ensures, by comparing the Agmon metric with the one associated to the metric $|z|^2\, dz^2$, that, for C large enough,

$$Ch^{\rho/2}\, |z|^2 \leqslant \phi_h(z), \qquad \text{if} \quad h^{1/2}C \leqslant |z|. \tag{7.25}$$

A direct consequence of (7.20) and (7.25) is the following lemma.

LEMMA 7.10. *For any fixed real* $k \geqslant 0$, $\| |z|^k\, u_h\|_{L^2(\Omega)} \leqslant C(h^{k/2} + h^{(k(2-\rho)+\rho)/4})$ $\|u_h\|_{L^2(\Omega)}$.

Proof of Lemma 7.10. Let $C_1 > 1$ be fixed and large enough, and let us define the following domains

$$\Omega_{0,h} = \{z \in \Omega \mid |z| \leqslant C_1 h^{1/2}\}, \tag{7.26}$$

$$\Omega_{1,h} = \{z \in \Omega \mid C_1 h^{1/2} \leqslant |z| \leqslant 2C_1 h^{(2-\rho)/4}\}, \tag{7.27}$$

and

$$\Omega_{2,h} = \{z \in \Omega \mid 2C_1 h^{(2-\rho)/4} \leqslant |z|\}. \tag{7.28}$$

By (7.24), (7.25), (7.27) and (7.28), if C_1 is large enough, then

$$h^\rho \, |z|^2/C \leqslant h^\rho V(z) - \lambda^{(1)}(h), \qquad \forall z \in \Omega_{1,h} \cup \Omega_{2,h} \tag{7.29}$$

and

$$h^\rho V(z) - \lambda^{(1)}(h) \leqslant C h^\rho \, |z|^2, \qquad \forall z \in \Omega_{1,h}. \tag{7.30}$$

Hence, by immediate comparison, we have for some constant $C > 0$,

$$\phi_h(z) \leqslant C h^{\rho/2} \, |z|^2, \qquad \forall z \in \Omega_{1,h}. \tag{7.31}$$

In particular this implies for some constant $C > 0$,

$$\phi_h(z) \leqslant Ch \qquad \forall \Omega_{1,h}. \tag{7.32}$$

Using (7.29) and (7.20) with $\varepsilon = 1/2$, we first get:

$$\begin{aligned} h^\rho \int_{\Omega_{1,h}} |z|^2 \, |u_h(z)|^2 \, dz &\leqslant C \int_{\Omega_{1,h}} [h^\rho V(z) - \lambda^{(1)}(h)] \, |u_h(z)|^2 \, dz \\ &\leqslant C \int_{\Omega_{1,h}} [h^\rho V(z) - \lambda^{(1)}(h)] \exp \phi_h(z)/h \, |u_h(z)|^2 \, dz \\ &\leqslant C \int_{\Omega} [h^\rho V(z) - \lambda^{(1)}(h)]_+ \exp \phi_h(z)/h \, |u_h(z)|^2 \, dz \\ &\leqslant 2C \int_{\Omega} [h^\rho V(z) - \lambda^{(1)}(h)]_- \, |u_h(z)|^2 \, dz. \end{aligned}$$

Now, we observe that the support of $[h^\rho V(z) - \lambda^{(1)}(h)]_-$ is contained in $\Omega_{0,h}$, and using the behavior of V in $\Omega_{0,h}$ near 0 and (7.24), we get, for $z \in \Omega_{0,h}$:

$$|h^\rho V(z) - \lambda^{(1)}(h)| \leqslant h^\rho \, |V(z) - V(0)| + C h^{1+\rho} \leqslant \tilde{C} h^{1+\rho}.$$

So we have proved, for some suitable constant C

$$h^\rho \int_{\Omega_{1,h}} |z|^2 \, |u_h(z)|^2 \, dz \leqslant C h^{1+\rho} \int_\Omega |u_h(z)|^2 \, dz.$$

Dividing by h^ρ, we obtain:

$$\int_{\Omega_{1,h}} |z|^2 \, |u_h(z)|^2 \, dz \leqslant C h^1 \int_\Omega |u_h(z)|^2 \, dz. \tag{7.33}$$

This estimate and (7.27) leads to

$$\begin{aligned}\int_{\Omega_{1,h}} |z|^{2k} \, |u_h(z)|^2 \, dz &\leqslant C h^{(k-1)(2-\rho)/2} \int_{\Omega_{1,h}} |z|^2 \, |u_h(z)|^2 \, dz \\ &\leqslant \tilde{C} h^{1+(k-1)(2-\rho)/2} \int_\Omega |u_h(z)|^2 \, dz.\end{aligned} \tag{7.34}$$

For the integral on $\Omega_{2,h}$, we use also (7.20) with $\varepsilon = 1/2$. If

$$\tau(h) = \sup\{|z|^{2k} \, [h^\rho V(z) - \lambda^{(1)}(h)]_+^{-1} \, e^{-\phi_h(z)/h}; \, z \in \Omega_{2,h}\},$$

then

$$\begin{aligned}\int_{\Omega_{2,h}} |z|^{2k} \, |u_h(z)|^2 \, dz &\leqslant C\tau(h) \int_{\Omega_{2,h}} [h^\rho V(z) - \lambda h)]_+ \, e^{\phi_h(z)/h} \, |u_h(z)|^2 \, dz \\ &\leqslant \tilde{C}\tau(h) \int_\Omega [h^\rho V(z) - \lambda h)]_- \, |u_h(z)|^2 \, dz \\ &\leqslant \hat{C} h^{1+\rho} \tau(h) \int_\Omega |u_h(z)|^2 \, dz.\end{aligned}$$

For the estimate of $\tau(h)$, we use (7.24), (7.25), (7.28) and (7.29). This leads to:

$$\begin{aligned}\tau(h) &\leqslant C h^{-\rho} \sup\{t^{k-1} e^{-(h^{-1+\rho/2} t)/C}; \, h^{-1+\rho/2} t \geqslant 4C_1^2\} \\ &\leqslant \tilde{C} h^{-\rho+(k-1)(2-\rho)/2}.\end{aligned}$$

So there exists a constant $C > 0$ such that:

$$\int_{\Omega_{2,h}} |z|^{2k} \, |u_h(z)|^2 \, dz \leqslant C h^{1+(k-1)(2-\rho)/2} \int_\Omega |u_h(z)|^2 \, dz. \tag{7.35}$$

For the integral on $\Omega_{0,h}$, we use (7.26) and get:

$$\int_{\Omega_{0,h}} |z|^{2k} |u_h(z)|^2 \, dz \leqslant Ch^k \int_{\Omega_{0,h}} |u_h(z)|^2 \, dz. \tag{7.36}$$

Hence Lemma 7.10 comes from (7.34)–(7.36). ∎

Proof of Theorem 7.4. Let us introduce

$$A^1(z) = \tfrac{1}{2}(-by, bx)$$

and

$$V_2(z) = V(0) + \tfrac{1}{2} z \cdot (\text{Hess } V(0)) \cdot z.$$

We can suppose that $V(0) = 0$, so

$$P_{h, A, V, \Omega} \geqslant 0.$$

Note that we have also

$$V(z) - V_2(z) = \mathcal{O}(|z|^3). \tag{7.37}$$

So using (7.19), (7.22), and (7.37), there exists a constant $C > 0$, such that, for any $h \in]0, 1]$ and any $u \in H_0^1(\Omega)$,

$$\begin{aligned} &|q_{h, A, V, \Omega}(u) - q_{h, A^1, V_2, \Omega}(u)| \\ &\quad \leqslant C[(q_{h, A, V, \Omega}(u))^{1/2} \, \| |z|^3 u\|_{L^2(\Omega)} + \| |z|^3 u\|^2 + h^\rho \, \| |z|^{3/2} u\|^2_{L^2(\Omega)}]. \end{aligned} \tag{7.38}$$

The estimate (7.24) (with $V(0) = 0$) of the ground state energy $\lambda^{(1)}(h)$ of $P_{h, A, V, \Omega}$, Lemma 7.10 applied to the associated ground state u_h and (7.38) show that

$$\begin{aligned} &|q_{h, A^1, V_2, \Omega}(u_h) - \lambda^{(1)}(h) \, \|u_h\|^2_{L^2(\Omega)}| \\ &\quad \leqslant C[(\lambda^{(1)}(h))^{1/2} \, \|u_h\|_{L^2(\Omega)} \times \| |z|^3 u_h\|_{L^2(\Omega)} + \| |z|^3 u_h\|^2 + h^\rho \, \| |z|^{3/2} u_h\|^2_{L^2(\Omega)}] \\ &\quad \leqslant \tilde{C}[h^2 + h^{3(2+\rho)/4}] \, \|u_h\|^2_{L^2(\Omega)}. \end{aligned} \tag{7.39}$$

So

$$\inf \text{Sp}(P_{h, A^1, V_2, \Omega}) \leqslant \inf \text{Sp}(P_{h, A, V, \Omega}) + Ch^{d_0(\rho)},$$

with

$$d_0(\rho) = \inf(\tfrac{3}{2} + \tfrac{3}{4}\rho, 2).$$

As the potentials $A^1(z)$ and $V_2(z)$ satisfy also the assumptions of Theorem 7.4, we have in the same way:

$$\inf \mathrm{Sp}(P_{h, A, V, \Omega}) \leqslant \inf \mathrm{Sp}(P_{h, A^1, V_2, \Omega}) + Ch^{d(\rho)}.$$

Hence:

$$|\inf \mathrm{Sp}(P_{h, A, V, \Omega}) - \inf \mathrm{Sp}(P_{h, A^1, V_2, \Omega})| \leqslant Ch^{d(\rho)}. \tag{7.40}$$

But Proposition 7.9 and Lemma 7.10 are also available for the operator P_{h, A^1, V_2} in $\mathbb{R}^2$. The groundstate of P_{h, A^1, V_2} has an exponential decay.

Lemma 7.10 applied to the groundstates of P_{h, A^1, V_2} and $P_{h, A^1, V_2, \Omega}$ leads also to

$$|\inf \mathrm{Sp}(P_{h, A^1, V_2}) - \inf \mathrm{Sp}(P_{h, A^1, V_2, \Omega})| \leqslant Ch^k, \qquad \forall k > 0. \tag{7.41}$$

But one can explicitly compute the spectrum in the case of Schrödinger operator with constant magnetic field and positive quadratic potential; see for example [Mel] (Theorem 2.4, p. 121) for the lower bound of the spectrum, or [MatUe] (Theorem 2.2, p. 222) and [Par]. In particular we get

$$\inf \mathrm{Sp}(P_{h, A^1, V_2}) = \mu_h(\rho), \tag{7.42}$$

where:

$$\begin{aligned} \mu_h(\rho) = h^\rho V(z_0) - hb &+ \frac{h}{\sqrt{2}} [h^\rho t_V + b^2 - [(h^\rho t_V + b^2)^2 - 4h^{2\rho} d_V]^{1/2}]^{1/2} \\ &+ \frac{h}{\sqrt{2}} [h^\rho t_V + b^2 + [(h^\rho t_V + b^2)^2 - 4h^{2\rho} d_V]^{1/2}]^{1/2}, \end{aligned} \tag{7.43}$$

with

$$t_V = \mathrm{Tr}(\tfrac{1}{2} \operatorname{Hess} V(z_0)), \tag{7.44}$$

$$d_V = \det(\tfrac{1}{2} \operatorname{Hess} V(z_0)). \tag{7.45}$$

A simple computation shows that:

$$\mu_h(\rho) = h^\rho V(z_0) + h^{1+\rho} \frac{a_V^2}{2b} + \mathcal{O}(h^{1+2\rho}). \tag{7.46}$$

So (7.16) comes from (7.40), (7.41) and (7.42). ∎

7.3. *Polynomial Magnetic Approximation*

We begin now the first step of the proof of Theorem 7.3. We recall that, with $\rho = 1$ and $V(z) = B(z)$, we can write

$$P_{h,\,A,\,\Omega} = P_{h,\,A,\,V,\,\Omega}.$$

We assume that the hypotheses of Theorem 7.3 are satisfied. Let $\lambda^{(1)}(h)$ be a lowest eigenvalue of $P_{h,\,A,\,\Omega}$ and u_h an associated eigenfunction. Let us remember that we have proved (7.24) which becomes in our case

$$\lambda^{(1)}(h) = hb + \mathcal{O}(h^2). \tag{7.47}$$

The estimate of Lemma 7.10 becomes

$$\| |z|^k\, u_h\|^2_{L^2(\Omega)} \leqslant C(h^{k/2} + h^{(k+1)/4})\, \|u_h\|^2_{L^2(\Omega)}, \qquad \forall k \geqslant 0. \tag{7.48}$$

We can assume that the magnetic potential satisfies

$$\operatorname{div}(A) = 0 \qquad \text{and} \qquad A(0) = 0. \tag{7.49}$$

Let $m \in \mathbb{Z}$ a fixed integer, $1 < m < 4 + M$, and let us denote by A^m the Taylor expansion (up to order m) of the magnetic potential:

$$A^m(z) = \sum_{|\alpha| \leqslant m} \frac{z^\alpha}{\alpha!} \frac{\partial^\alpha A}{\partial z^\alpha}(0). \tag{7.50}$$

LEMMA 7.11. *Let $\chi \in C_0^\infty(\mathbb{R}^2)$ be a cut-off function such that*

$$\operatorname{supp} \chi \subset \Omega$$

and

$$0 \notin \operatorname{supp}(1 - \chi).$$

Then there exists C and h_0 such that

$$\|(P_{h,\,A^m} - \lambda^{(1)}(h))\, \chi u_h\| \leqslant C h^{(2m+3)/4}\, \|\chi u_h\|, \qquad \forall h \in h_0,$$

where $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^2)}$.

Proof. Let us first prove that, if $k \in \mathbb{Z}^\star$ is fixed, then

$$\| |z|^k\, (hD - A)\, u_h\|_{L^2(\Omega)} \leqslant C h^{(2k+3)/4}\, \|u_h\|_{L^2(\Omega)}. \tag{7.51}$$

Using that

$$(hD-A)[|z|^{2k}(hD-A)u_h]=\lambda^{(1)}(h)|z|^{2k}u_h-2ihk|z|^{2k-2}[z(hD-A)]u_h,$$

and (7.48), we get that

$$\||z|^k(hD-A)u_h\|^2_{L^2(\Omega)}\leqslant C\lambda^{(1)}(h)h^{(2k+1)/2}\|u_h\|^2_{L^2(\Omega)}$$
$$+Ch\||z|^k(hD-A)u_h\|_{L^2(\Omega)}\||z|^{k-1}u_h\|_{L^2(\Omega)}.$$

So we deduce

$$\||z|^k(hD-A)u_h\|^2_{L^2(\Omega)}\leqslant C\lambda^{(1)}(h)h^{(2k+1)/2}\|u_h\|^2_{L^2(\Omega)}+Ch^2\||z|^{k-1}u_h\|^2_{L^2(\Omega)}.$$

Using (7.47) and once more (7.48) we get (7.51).
Let us now write, (with $R^m=A-A^m$). As div $A=0$, then div $R^m=0$ and

$$P_{h,A^m}=P_{h,A}+2R^m(hD-A)+|R^m|^2.$$

As $|z|^{-m-1}R^m(z)$ is bounded in Ω, (7.48) and (7.51) show that

$$\|(P_{h,A^m}-\lambda^{(1)}(h))\chi u_h\|\leqslant Ch^{(2m+3)/4}\|u_h\|_{L^2(\Omega)}.$$

We finally recall that a consequence of (7.48) is that, for any $k\in\mathbb{Z}$,

$$\|\chi u_h\|=(1+\mathcal{O}(h^k))\|u_h\|_{L^2(\Omega)}. \tag{7.52}$$

This ends the proof of Lemma 7.11. ∎

Lemma 7.11 ensures that

$$\text{distance}(\lambda^{(1)}(h),\operatorname{Sp}(P_{h,A^m}))\leqslant Ch^{(2m+3)/4}. \tag{7.53}$$

Let us introduce:

$$B^m(z)=\sum_{|\alpha|\leqslant m-1}\frac{z^\alpha}{\alpha!}\frac{\partial^\alpha B}{\partial z^\alpha}(0)=\frac{\partial A_2^m}{\partial x}(z)-\frac{\partial A_1^m}{\partial y}(z). \tag{7.54}$$

Assumption (7.7) ensures that, if $m>2$, then

$$\sum_{|\alpha|\leqslant m-1}\left|\frac{\partial^\alpha B^m}{\partial z^\alpha}(z)\right|\to+\infty \qquad \text{as} \quad |z|\to\infty. \tag{7.55}$$

So, a general result in the representation theory of nilpotent Lie algebra of [HeNo1] says that P_{h,A^m} has compact resolvent, (see [MoNo] or [HeMo1] for an analytic proof). More precisely, there exists a constant C such that, for all $v \in C_0^\infty(\mathbb{R}^2)$,

$$\sum_{|\alpha| \leqslant m-1} h^{(2|\alpha|+2)/(|\alpha|+2)} \left\| \left| \frac{\partial^\alpha B^m}{\partial z^\alpha}(z) \right|^{1/(|\alpha|+2)} v \right\|^2 \leqslant C \, \|(hD - A^m) \, v\|^2. \tag{7.56}$$

Conversely, if $\lambda_m(h)$ is an eigenvalue of P_{h, A^m} satisfying (7.47), and if u_h^m is an associated eigenfunction satisfying (7.48), then we will get as for (7.53), that

$$\text{distance}(\lambda_m(h), \text{Sp}(P_{h, A, \Omega}) \leqslant C h^{(2m+3)/4}. \tag{7.57}$$

So, if we want to analyze the bottom of the spectrum of $P_{h, A, \Omega}$ modulo an error of order $\mathcal{O}(h^{(2m+3)/4})$, we just have to study the bottom of the spectrum of P_{h, A^m}. In order to determine the coefficient of h^2 in the expansion, we will choose in the next subsections: $m \geqslant 4$.

7.4. *Some Simplifications*

We make some simplifications which do not change the spectrum of $P_{h, A, \Omega}$ or P_{h, A^m}.

We can always take orthonormal coordinates in $\mathbb{R}^2$ such that, in a neighborhood of $z_0 = 0$,

$$B(z) = b + \alpha x^2 + \beta y^2 + \mathcal{O}(|z|^3). \tag{7.58}$$

We can also choose a gauge $A(z)$ such that

$$A_1(z) = 0 \qquad \text{and} \qquad A_2(z) = bx + \frac{\alpha}{3} x^3 + \beta x y^2 + \mathcal{O}(z^4). \tag{7.59}$$

So $A_1^m = 0$ and we can write

$$A_2^m(z) = bx + \sum_{j=3}^{m} S_j(z), \tag{7.60}$$

with

$$S_j(z) = \sum_{|\gamma| = j} S_{j,\gamma} z^\gamma,$$

and in particular:

$$S_3(z) = \frac{\alpha}{3} x^3 + \beta x y^2.$$

By scaling we get that

$$\mathrm{Sp}(P_{h,A^m}) = h\,\mathrm{Sp}(Q_h(z, D_z)), \tag{7.61}$$
$$Q_h(z, D_z) = D_x^2 + \Big[D_y - bx - h \sum_{j=3}^{m} h^{(j-3)/2} S_j(z) \Big]^2.$$

We are now going to apply some metaplectic transformations. We recall that these transformations are unitary and consequently preserve the spectrum of the operators. Note also that they preserve the functions of the form $f(z) = p(z)\,e^{iq_1(z) - q_2(z)}$ with p and the q_j polynomial, the q_j real and homogeneous of degree j, $q_2(z) > 0$ if $z \neq 0$.

Using Fourier-transform and translation and writing

$$Q_h(z, D_z) = Q_h(x, y, D_x, D_y),$$

we get that

$$\mathrm{Sp}(Q_h(z, D_z)) = \mathrm{Sp}(Q_h(x, -D_\xi, D_x, \xi))$$
$$= \mathrm{Sp}\Big(Q_h\Big(x + \frac{\xi}{b}, -D_\xi + \frac{1}{b} D_x, D_x, \xi \Big)\Big). \tag{7.62}$$

So

$$\mathrm{Sp}(P_{h,A^m}) = h \cdot \mathrm{Sp}(H_h(x, \xi, D_x, D_\xi)),$$
$$H_h(x, \xi, D_x, D_\xi) = D_x^2 + \Big[bx + h \sum_{j=3}^{m} h^{(j-3)/2} T_j(x, \xi, D_x, D_\xi) \Big]^2, \tag{7.63}$$

with $T_j(x, \xi, D_x, D_\xi) = S_j(x + \frac{\xi}{b}, -D_\xi + \frac{1}{b} D_x)$. The differential operator T_3 has the following form

$$T_3(x, \xi, D_x, D_\xi) = xL(\xi, D_\xi) + M_0(\xi, D_\xi) + M_1(x, D_x) + M_2(\xi, D_x, D_\xi)$$
$$+ M_3(x, D_x, D_\xi) + M_4(x, \xi, D_x), \tag{7.64}$$

where $L(\xi, D_\xi)$ an operator with compact resolvent,

$$L(\xi, D_\xi) = \frac{\alpha}{b^2}\xi^2 + \beta D_\xi^2,$$

$$M_0(\xi, D_\xi) = \frac{\alpha}{3b^3}\xi^3 + \frac{\beta}{b}\xi D_\xi^2,$$

$$M_1(x, D_x) = \frac{\alpha}{3}x^3 + \frac{\beta}{b^2} x D_x^2,$$

$$M_2(\xi, D_x, D_\xi) = -2\,\frac{\beta}{b^2}\, D_x \xi D_\xi,$$

$$M_3(x, D_x, D_\xi) = -2\,\frac{\beta}{b}\, x D_x D_\xi,$$

$$M_4(x, \xi, D_x) = \frac{\alpha}{b}\, x^2 \xi + \frac{\beta}{b^3}\, \xi D_x^2 .$$

Using the notation $w = (x, \xi)$, we can write

$$H_h(w, D_w) = H_h^0(w, D_w) + W_h(w, D_w), \tag{7.65}$$

with

$$H_h^0(w, D_w) = D_x^2 + x^2[b + hL(\xi, D_\xi)]^2, \tag{7.66}$$

and

$$\begin{aligned} W_h(w, D_w) = {} & +h \sum_{\ell=0}^{4} [x(b + hL(\xi, D_\xi))\, M_\ell(w, D_w) \\ & + M_\ell(w, D_w)\, x(b + hL(\xi, D_\xi))] \\ & + h^{3/2} \sum_{j=4}^{m} h^{(j-4)/2} [x(b + hL(\xi, D_\xi))\, T_j(w, D_w) \\ & + T_j(w, D_w)\, x(b + hL(\xi, D_\xi))] \\ & + h^2 \left[\sum_{\ell=0}^{4} M_\ell(w, D_w) \right]^2 \\ & + h^{\frac{5}{2}} \sum_{\ell, j} h^{(j-4)/2} [M_\ell(w, D_w)\, T_j(w, D_w) + T_j(w, D_w)\, M_\ell(w, D_w)] \\ & + h^3 \left[\sum_{j=4}^{m} h^{(j-4)/2} T_j(w, D_w) \right]^2 . \end{aligned} \tag{7.67}$$

The operator $H_h^0(w, D_w)$ has compact resolvent and

$$\mathrm{Sp}(H_h^0(w, D_w)) = \{\mu_{j,k}(h) = (b + h\mu_j)(2k-1);\ (j, k) \in (\mathbb{N}^\star)^2\}, \tag{7.68}$$

with

$$\left\{ \mu_j = \frac{d}{b}(2j-1);\ j \in \mathbb{N}^\star \right\} = \mathrm{Sp}(L(\xi, D_\xi), \tag{7.69}$$

and

$$d = (\alpha\beta)^{1/2}. \tag{7.70}$$

The eigenfuction of $H_h^0(w, D_w)$ associated to the eigenvalue $\mu_{j,k}(h)$ is

$$\psi_{j,k}^h(w) = (\mu_1)^{1/2}\,(\mu_{j,1}(h))^{1/2}\,\varphi_k((\mu_{j,1}(h))^{1/2}\,x)\,\varphi_j((\mu_1)^{1/2}\,\xi), \qquad (7.71)$$

where the $\varphi_j(x)$ are the normalized eigenfunctions of the harmonic oscillator $D_x^2+x^2$,

$$\varphi_j(x) = p_j(x)\,e^{-x^2/2},$$

and $p_j(x)$ is the Hermite polynomial function of order $j-1$. We recall in particular that:

$$p_1(x) = \pi^{-1/2}.$$

7.5. *End of the Proof of Theorem* 7.3

Then, by (7.65)–(7.71), we have

$$H_h(w, D_w)\,\psi_{1,1}^h(w) = \mu_{1,1}(h)\,\psi_{1,1}^h(w) + W_h(w, D_w)\,\psi_{1,1}^h(w), \qquad (7.72)$$

and we can write, using the expansion of $W_h(w, D_w)$ given above in (7.67):

$$W_h(w, D_w)\,\psi_{1,1}^h(w) = hb[xM(w, D_w) + M(w, D_w)\,x]\,\psi_{1,1}^h(w) + \mathcal{O}(h^{\frac{3}{2}}), \qquad (7.73)$$

in $L^2(\mathbb{R}^2)$, where

$$M(w, D_w) = \sum_{\ell=0}^{4} M_\ell(w, D_w).$$

Because we need only an upper bound, it is enough to consider:

$$\delta\lambda := \langle hb[xM(w, D_w) + M(w, D_w)\,x]\,\psi_{1,1}^h(w) \mid \psi_{1,1}^h(w)\rangle.$$

By (7.71), the function $\psi_{1,1}^h(1, \xi)$ is even in the x variable, so the expansion of the $M_\ell(w, D_w)$ given below (7.64) entails

$$\begin{aligned}
\delta\lambda &= b\langle[xM_1(w, D_w) + M_1(w, D_w)\,x]\,\psi_{1,1}^h \mid \psi_{1,1}^h\rangle \\
&\quad + b\langle[xM_2(w, D_w) + M_2(w, D_w)\,x]\,\psi_{1,1}^h \mid \psi_{1,1}^h\rangle \\
&= b\left\langle\left[\frac{2\alpha}{3}\,x^4 + \frac{\beta}{b^2}\,x^2 D_x^2 + \frac{\beta}{b^2}\,x\,D_x^2 x\right]\psi_{1,1}^h \,\middle|\, \psi_{1,1}^h\right\rangle \\
&\quad - b\left\langle\left[\frac{2\beta}{b^2}\,x\,D_x\xi\,D_\xi + \frac{2\beta}{b^2}\,D_x x\xi\,D_\xi\right]\psi_{1,1}^h \,\middle|\, \psi_{1,1}^h\right\rangle
\end{aligned}$$

$$= 2b \left\langle \left[\frac{\alpha}{3} x^4 + \frac{\beta}{b^2} x^2 D_x^2 - i \frac{\beta}{b^2} x D_x \right] \psi_{1,1}^h \right.$$

$$\left. - \left[\frac{2\beta}{b^2} x\, D_x \xi D_\xi - i \frac{\beta}{b^2} \xi D_\xi \right] \psi_{1,1}^h \,\middle|\, \psi_{1,1}^h \right\rangle. \tag{7.74}$$

As $D_x^2 \psi_{1,1}^h = \mu_{1,1}(h)(1 - \mu_{1,1}(h)\, x^2)\, \psi_{1,1}^h$ and $x\, D_x \psi_{1,1}^h = i\mu_{1,1}(h)\, x^2 \psi_{1,1}^h$, we get from (7.74)

$$\delta\lambda = 2b \left\langle \left[\left(\frac{\alpha}{3} - \frac{\beta}{b^2} \mu_{1,1}^2(h) \right) x^4 + \frac{2\beta}{b^2} \mu_{1,1}(h)\, x^2 \right] \psi_{1,1}^h \,\middle|\, \psi_{1,1}^h \right\rangle$$

$$- i \frac{2\beta}{b} \langle (2\mu_{1,1}(h)\, x^2 - 1)\, \xi\, D_\xi \psi_{1,1}^h \mid \psi_{1,1}^h \rangle. \tag{7.75}$$

Let us recall the formulas

$$2 \int x^2 e^{-x^2}\, dx = \pi \qquad \text{and} \qquad 4 \int x^4 e^{-x^2}\, dx = 3\pi.$$

From the first one and (7.71) with $k = 1$, we get immediately that, for any differential operator $U(\xi, D_\xi)$,

$$\langle (2\mu_{1,1}(h)\, x^2 - 1)\, U(\xi, D_\xi)\, \psi_{1,1}^h \mid \psi_{1,1}^h \rangle = 0.$$

Using again the formulas, we get from (7.75)

$$\begin{aligned} \delta\lambda &= \frac{3b}{2\mu_{1,1}^2(h)} \left(\frac{\alpha}{3} - \frac{\beta}{b^2} \mu_{1,1}^2(h) \right) + 2 \frac{\beta}{b} \\ &= \frac{1}{2b} (\alpha + \beta) + \mathcal{O}(h). \end{aligned} \tag{7.76}$$

Here we have used that $\mu_{1,1}(h) = b + h\mu_1$.

Returning to the original metaplectic coordinates, we have found $u_{1,h}$ in $\mathscr{S}(\mathbb{R}^2)$ and of norm 1 in $L^2(\mathbb{R}^2)$ such that:

$$\left\langle \left[P_{h, A^m} - h\mu_{1,1}(h) - \frac{h^2}{2b} (\alpha + \beta) \right] u_{h,1} \,\middle|\, u_{h,1} \right\rangle \leqslant C h^{5/2}. \tag{7.78}$$

Keeping in mind that (7.68) and (7.69) give that

$$h\mu_{1,1}(h) + \frac{h^2}{2b} (\alpha + \beta) = hb + h^2 \frac{a^2}{2b},$$

with $a=(\alpha)^{1/2}+(\beta)^{1/2}$, (7.77) and (7.57) show that:

$$\inf \mathrm{Sp}(P_{h,A,\Omega}) \leqslant hb + h^2\frac{a^2}{2b} + Ch^{5/2}.$$

The theorem is proved. ∎

8. ON THE DIRICHLET PROBLEM WHEN THE MINIMUM OF THE MAGNETIC FIELD IS ACHIEVED ON THE BOUNDARY

8.1. *Introduction*

In this section, instead of the assumption (7.7), we consider the case when there exists $z_0 \in \partial\Omega$ such that

$$0 < b := B(z_0) \leqslant B(z), \qquad \forall z \in \bar{\Omega}, \tag{8.1}$$

is satisfied.

In particular, we have $b = b'$.

We assume Ω is a bounded open set of class C^3 and that the magnetic potential is of class C^3. Let $\lambda^{(1)}(h)$ be the groundstate energy of the Dirichlet operator and let $u_h(z) \in H^2(\Omega) \cap H^1_0(\Omega)$ be an associated eigenfunction:

$$\begin{aligned} \lambda^{(1)}(h) &= \inf \mathrm{Sp}(P^D_{h,A,\Omega}), \\ P^D_{h,A,\Omega} u_h &= \lambda^{(1)}(h)\, u_h. \end{aligned} \tag{8.2}$$

We know, thanks to (7.4) and (8.1), that

$$hb \leqslant \lambda^{(1)}(h) \tag{8.3}$$

and by (7.20) (with $\rho = 1$ and $V(z) = B(z)$), we know also that

$$\begin{aligned} &(1-\varepsilon^2)\int_\Omega [hB(z)-\lambda^{(1)}(h)]_+\, e^{2\varepsilon\phi_h(z)/h}\, |u_h(z)|^2\, dz \\ &\leqslant \int_\Omega [hB(z)-\lambda^{(1)}(h)]_-\, |u_h(z)|^2\, dz, \end{aligned} \tag{8.4}$$

for any $\varepsilon \in]0,1]$, where $\phi_h(z)$ is the Agmon distance of z to the well

$$U_h = \{\omega \in \bar{\Omega};\, hB(z) - \lambda^{(1)}(h) \leqslant 0\}$$

(associated to the metric $[hB(z)-\lambda^{(1)}(h)]_+\, dz^2$).

The purpose of this subsection is to get bounds of $\lambda^{(1)}(h)$ and then to exploit this upper bound and (8.4) for a more explicit asymptotic localization of the eigenfunction.

8.2. *Rough Lower Bounds*

Let us start with lower bounds. The following estimate give the probably correct behavior as $h \to 0$.

PROPOSITION 8.1. *Let us assume that the inequality*

$$0 < b < B(z), \qquad \forall z \in \Omega, \tag{8.5}$$

is satisfied. Then, if $\nabla B(z_0) \neq 0$, *for any* $z_0 \in \partial\Omega$ *such that* $B(z_0) = b$, *then there exists* $h_0 > 0$ *and* $\eta_0 > 0$ *such that*

$$hb + h^{3/2}\eta_0 \leqslant \inf \operatorname{Sp}(P^D_{h, A, \Omega}), \qquad \forall h \in]0, h_0]. \tag{8.6}$$

As a matter of fact, (7.6) with $\tau = \rho h^{-1/2}$ leads to

$$\rho h^{1/2} P^D_{h, A, \Omega} + (1 - \rho h^{1/2})\, hB \leqslant P^D_{h, A, \Omega}, \qquad \forall \rho \in [0, h^{-1/2}]. \tag{8.7}$$

If we use Kato's inequality (2.2) we get that:

$$h \inf \operatorname{Sp}(-\rho h^{3/2}\, \Delta^D_\Omega + (1 - \rho h^{1/2})\, B) \leqslant \inf \operatorname{Sp}(P_{h, A, \Omega}). \tag{8.8}$$

Here $-\Delta^D_\Omega = P_{1, 0, \Omega}$ is the Laplace operator on Ω with Dirichlet boundary condition.

But the assumptions of Proposition 8.1 lead to existence of $c_0 > 0$ such that

$$c_0 t(z) + b \leqslant B(z) \tag{8.9}$$

for all z in a neighborhood of $\partial\Omega$ in Ω, where $t(z) = \operatorname{dist}(z, \partial\Omega)$.

It is enough to take

$$0 < c_0 < \beta_1 = \min_{x \in m(\partial\Omega)} \partial_\nu B(x),$$

where

$$m(\partial\Omega) := \{x \in \partial\Omega \mid B(x) = b\}. \tag{8.10}$$

So, if $\rho \leqslant h^{-1/2}/2$,

$$h \inf \operatorname{Sp}\left(-\rho h^{3/2}\, \Delta^D_\Omega + \frac{c_0}{2}\, t(z) \right) + h(1 - \rho h^{1/2})\, b \leqslant \inf \operatorname{Sp}(P^D_{h, A, \Omega}). \tag{8.11}$$

But it is easy to see (see for example [Mai]), that, for h small enough,

$$-C_0\rho h^{3/2}+(\rho h^{3/2})^{1/3}\left(\frac{c_0}{2}\right)^{2/3}\omega_0\leqslant \inf \mathrm{Sp}\left(-\rho h^{3/2}\,\Delta_\Omega^D+\frac{c_0}{2}\,t(z)\right), \tag{8.12}$$

where ω_0 is the first eigenvalue of the Dirichlet problem on $L^2(\mathbb{R}_+)$, associated to the Airy differential operator:

$$A(t, D_t) := -\frac{d^2}{dt^2}+t. \tag{8.13}$$

The constant C_0 depends only on c_0 and does not depend on ρ and h.

The estimates (8.11) and (8.12), with for example

$$\rho=\left(\omega_0\left(\frac{c_0}{2}\right)^{2/3}(3b)^{-1}\right)^{3/2},$$

give the proof of (8.6).

8.3. *Rough Upper Bounds*

The purpose of this subsection is to get an upper bound of $\lambda^{(1)}(h)$ and then to exploit this upper bound and (8.4) for a more explicit asymptotic localization of the eigenfunction.

THEOREM 8.2. *If $\Omega\subset\mathbb{R}^2$ is a bounded open connected set of class C^3 and if A belongs to $C^3(\bar{\Omega}, \mathbb{R}^2)$, then, under the assumption* (8.1), *there exists a constant C and h_0 such that*

$$bh\leqslant \inf \mathrm{Sp}(P^D_{h, A, \Omega})\leqslant bh+Ch^{3/2}(\ln(h))^2, \qquad \forall h\in\,]0, h_0[. \tag{8.14}$$

The difficulty to get (8.14) without the logarithmic term comes from the following fact. The model operator at a point of the boundary where B is minimal is $P^+_{h, A^0} := P_{h, A^0, \mathbb{R}_+\times\mathbb{R}}$ with $A^0(z)=\frac{b}{2}(-y, x)$. It is easy to see that

$$\inf \mathrm{Sp}(P^+_{h, A^0})=bh\inf_{\xi\in\mathbb{R}}\inf \mathrm{Sp}(H^{D, \xi})=bh,$$

where $H^{D, \xi}$ is the Dirichlet operator on $\mathbb{R}_+$ associated to

$$-\frac{d^2}{du^2}+(\xi-u)^2.$$

Unlike the Neumann realization $H^{N,\xi}$, the infimum in ξ of the ground state energy of $H^{D,\xi}$, which is equal to 1, is not achieved for finite ξ. So, in order to construct a quasimode, we have to take ξ large and in this case the groundstate eigenfunction of $H^{D,\xi}$ is localized exponentially near $u=\xi$. But, in our model, Ω is bounded. So we can only work in Ω and we will work (due to the scaling) in a domain such that $0 < h^{1/2}u < \varepsilon(\Omega)$.

Here $\varepsilon(\Omega)$ is a geometrical constant such that the distance to the boundary in the domain $\{z \in \bar{\Omega} \mid d(z, \partial\Omega) \leqslant \varepsilon(\Omega)\}$ is regular.

Let us recall from Appendix A that, for the Neumann realization, the infimum was obtained for some ξ_0 in $]0, +\infty[$.

Proof of Theorem 8.2. Let us work in the system of coordinates recalled in Appendix B. The assumption (8.1) and the identities (B.7) and (B.12) show that, in the right gauge,

$$\left|\frac{\partial \tilde{A}_1}{\partial t} + b(1 - t\kappa(0)) + t\frac{\partial \hat{B}}{\partial t}(0, 0)\right| \leqslant C(s^2 + t^2), \tag{8.15}$$

so we can assume that in K,

$$\left|\tilde{A}_1(w) + tb\left(1 - \frac{t}{2}\kappa(0)\right) + \frac{t^2}{2}\frac{\partial \hat{B}}{\partial t}(0, 0)\right| \leqslant Ct(s^2 + t^2). \tag{8.16}$$

Let us take the test function

$$u_h^0(z) = e^{-i\xi_h s/h} a^{-1/2}(w)\, v_h^0(w), \tag{8.17}$$

with $(w = (s, t) = \psi(z))$, $\xi_h \in \mathbb{R}$ and $v_h^0 \in C_0^2(K; \mathbb{R})$ (to be chosen suitably).

So by (B3)–(B11),

$$\begin{aligned} q_{h,A,\Omega}(u_h^0) = \int_K \Bigg\{ & h^2\left[\left(\frac{\partial v_h^0}{\partial t}\right)^2 + (1 - t\kappa(s))^{-2}\left(\frac{\partial v_h^0}{\partial s}\right)^2\right] \\ & + [a^{-2}(\xi_h + \tilde{A}_1)^2 + h^2 W](v_h^0)^2\Bigg\}\, dw. \end{aligned} \tag{8.18}$$

Let $C_0 > 0$ be fixed large enough. Let $\chi \in C_0^\infty(\mathbb{R}; \mathbb{R})$ be a cut-off function such that $\chi(x) = 1$ if $|x| < C_0$ and $\chi(x) = 0$ if $|x| > 2C_0$.

We choose

$$\xi_h = 2C_0^{1/2} h^{1/2} |\ln(h)| \tag{8.19}$$

and, with

$$\begin{aligned}\theta_h(t) &= b^{1/2}h^{-1/2}(t - b^{-1}\xi_h)\\ v_h^0(w) &= b^{1/4}h^{-3/8}\varphi_1(\theta_h(t))\,\chi(|\ln(h)|^{-1}\,\theta_h(t)) \times f(h^{-1/4}s).\end{aligned} \tag{8.20}$$

Here φ_1 is defined by

$$\varphi_1(x) = \pi^{-1/2}e^{-x^2/2}$$

and f is a function in $C_0^\infty(]-\frac{1}{2}, \frac{1}{2}[; \mathbb{R})$ such that $\int f(x)^2\,dx = 1$.

So, if h is small enough, v_h^0 belongs to $C_0^\infty(K; \mathbb{R})$ and it is easy to check from (8.16), (8.18)–(8.20) that there exist $h_0 \in]0, 1/e[$ and $C_1 > 0$ such that

$$|q_{h, A, \Omega}(u_h^0) - hb\,\|u_h^0\|^2_{L^2(\Omega)}| \leqslant C_1 h^{3/2}(\ln(h))^2, \qquad \forall h \in]0, h_0[. \tag{8.21}$$

Moreover,

$$u_h^0 \in H_0^1(\Omega), \qquad \text{and} \qquad |\|u_h^0\|_{L^2(\Omega)} - 1| \leqslant Ch^2. \tag{8.22}$$

The properties (8.22) and (8.21) prove (8.14). ∎

Remark 8.3. If instead of (8.1) we have, for some $z_0 \in \partial\Omega$, assumption (8.5), and if, for any such z_0, we have: $\nabla B(z_0) \neq 0$, then we get easily from (8.4) and (8.14) the following localization of the groundstate eigenfunction. There exist $\delta \in]0, 1[$ $C > 0$ and h_0 such that, for all $h \in]0, h_0]$,

$$\|\exp \delta t(z)^{3/2}/h^{1/2}\,|u_h(z)|\,\| \leqslant Ch^{-C}\,\|u_h\|,$$

where we recall that $t(z) = \operatorname{dist}(z, \partial\Omega)$.

9. ON THE GROUNDSTATE ENERGY OF THE NEUMANN REALIZATION

9.1. *The Upper Bound of the Groundstate Energy*

Let Ω and A be as in Section 8. Let $P^N_{h, A, \Omega}$ be the Neumann operator associated to the quadratic form

$$q^N_{h, A, \Omega}(u) = \int_\Omega |(hD_z - A(z))\,u(z)|^2\,dz \qquad \forall u \in H^1(\Omega). \tag{9.1}$$

Let us recall that we have defined

$$\mu^{(1)}(h) = \inf \mathrm{Sp}(P^N_{h, A, \Omega}) \tag{9.2}$$

and

$$\Theta_0 = \inf_{\xi \in \mathbb{R}} \inf \mathrm{Sp}(H^{N, \xi}). \tag{9.3}$$

The operator $H^{N,\xi}$ is the Neuman operator on $L^2(\mathbb{R}_+)$ associated to the differential operator $D_x^2 + (x-\xi)^2$. We recall from Section 3 and Appendix A that Θ_0 is the lower bound of Neumann operator on $\mathbb{R}_+ \times \mathbb{R}$ associated to $D_x^2 + (D_y \pm x)^2$.

Let us first prove a version of the first part of the theorem by Lu–Pan recalled in Theorem 5.3 with a better control of the remainder.

THEOREM 9.1. *Under the assumptions of Theorem* 8.2, *but with* (8.1) *replaced by*

$$0 < b' = \min_{\omega \in \partial\Omega} B(\omega), \tag{9.4}$$

then there exists $C_0 > 0$ *such that*

$$\mu^{(1)}(h) \leqslant \Theta_0 b' h + C_0 h^{3/2}, \qquad \forall h \in]0, 1]. \tag{9.5}$$

Proof. The proof is similar to the one of Theorem 8.2. The adapted coordinates near the boundary of Appendix B are still valid. We can assume (B.11). Let

$$z_0 \in \partial\Omega \qquad \text{such that} \quad B(z_0) = b'. \tag{9.6}$$

Then (8.15) is valid thanks to (9.4), so we can assume (8.16).

As in the proof of Theorem 8.2, we can take[6] a test function $u_h^0(z)$ of the form (8.17), where

$$v_h^0(w) = (b')^{1/4}\, h^{-3/8} \varphi_0((b')^{1/2}\, h^{-1/2} t)\, \chi(t) \times f(h^{-1/4} s), \tag{9.7}$$

$$\xi_h = (b')^{1/2}\, h^{1/2} \xi_0, \tag{9.8}$$

the function f is as in (8.20) and $\chi(t)$ is a cut-off function such that $\chi(t) = 1$ if $0 \leqslant t \leqslant \varepsilon_0/2$ and $\chi(t) = 0$ if $t \geqslant \varepsilon_0$.

The eigenfunction $\phi_0 = \phi_{\xi_0}$ is introduced in Appendix A. Using formulas (B.3), (B.4) and (8.16), it is then easy to check that (A.1) and (9.7) imply the existence of $C > 0$ such that:

$$\begin{aligned} |q^N_{h, A, \Omega}(u_h^0) - \Theta_0 b' h\, \|u_h^0\|^2_{L^2(\Omega)}| &\leqslant C h^{3/2} \\ |\|u_h^0\|_{L^2(\Omega)} - 1| &\leqslant C h^{3/2}. \end{aligned} \tag{9.9}$$

So (9.9) proves (9.5). ∎

[6] Similar computations can be found in [BeSt].

9.2. *Rough Lower Bound Estimates*

Our aim is to find an equivalent of the estimate (7.4) for the Neumann quadratic form. We will follow the methods of [HeMo2].

Let $(\chi_\gamma(z))_{\gamma\in\Gamma}$ be a partition of unity of $\mathbb{R}^2$. For example we can take

$$\begin{aligned} &\Gamma=\mathbb{Z}^2, \qquad \chi_\gamma\in C^\infty(\mathbb{R}^2;\mathbb{R}) \qquad \text{and} \\ &\operatorname{supp}(\chi_\gamma)\subset\gamma+[-1,1]^2, \qquad \forall\gamma\in\Gamma, \\ &\sum_\gamma \chi_\gamma^2(z)=1 \qquad \text{and} \qquad \sum_\gamma |\nabla\chi_\gamma(z)|^2<\infty. \end{aligned} \tag{9.10}$$

If $\tau(h)$ is a function of h such that $\tau(h)\in\,]0,\varepsilon(\Omega)[$, ($\varepsilon(\Omega)$ is the geometric constant related to $\partial\Omega$, and introduced in Section 8), we will define the functions

$$\chi_{\gamma,\tau(h)}(z)=\chi_\gamma(z/\tau(h)), \qquad \forall\gamma\in\Gamma. \tag{9.11}$$

So we get a new partition of unity such that

$$\sum_\gamma \chi^2_{\gamma,\tau(h)}(z)=1, \qquad \sum_\gamma |\nabla\chi_{\gamma,\tau(h)}(z)|^2\leqslant C\tau(h)^{-2}, \tag{9.12}$$

and $\operatorname{supp}(\chi_{\gamma,\tau(h)})\subset\tau(h)\,\gamma+[-\tau(h),\tau(h)]^2$.

Then, for any $u\in H^1(\Omega)$, (see (6.8)), we have:

$$q^N_{h,A,\Omega}(u)=\sum_\gamma\,[q^N_{h,A,\Omega}(\chi_{\gamma,\tau(h)}u)-h^2\,\||\nabla\chi_{\gamma,\tau(h)}|\,u\|^2_{L^2(\Omega)}]. \tag{9.13}$$

Let us define

$$\begin{aligned} &\Gamma_{\tau(h)}(\Omega)=\{\gamma\in\Gamma;\ \operatorname{supp}(\chi_{\gamma,\tau(h)})\cap\Omega\neq\varnothing\} \\ &\Gamma^0_{\tau(h)}(\Omega)=\{\gamma\in\Gamma_{\tau(h)}(\Omega);\ \operatorname{dist}(\operatorname{supp}(\chi_{\gamma,\tau(h)}),\partial\Omega)>\tau(h)\} \\ &\Gamma^1_{\tau(h)}(\Omega)=\{\gamma\in\Gamma_{\tau(h)}(\Omega);\ \operatorname{dist}(\operatorname{supp}(\chi_{\gamma,\tau(h)}),\partial\Omega)\leqslant\tau(h)\}. \end{aligned} \tag{9.14}$$

We assume that

$$B(z)>0, \qquad \forall z\in\bar{\Omega}. \tag{9.15}$$

Then, for any $u\in H^1(\Omega)$,

$$q^N_{h,A,\Omega}(\chi_{\gamma,\tau(h)}u)\geqslant h\,\|B^{1/2}\chi_{\gamma,\tau(h)}u\|^2_{L^2(\Omega)}, \qquad \forall\gamma\in\Gamma^0_{\tau(h)}(\Omega). \tag{9.16}$$

For any $\gamma \in \Gamma^1_{\tau(h)}(\Omega)$, we use the adapted coordinates and formulas (B.3) and (B.4).

As in [HeMo2], from the identities (B.7) and (B.12), it is easy to find a gauge such that

$$|\tilde{A}(w)+(B(z_{\gamma,\tau(h)})\, t, 0)| \leqslant C\tau(h)^2, \tag{9.17}$$

for all $w = w(z)$ such that $z \in \operatorname{supp}(\chi_{\gamma,\tau(h)})$.

Here $z_{\gamma,\tau(h)}$ is chosen in $\operatorname{supp}(\chi_{\gamma,\tau(h)})$ such that $B(z_{\gamma,\tau(h)})$ is maximum on $\operatorname{supp}(\chi_{\gamma,\tau(h)})$.

Then using that Θ_0 is a lower bound of the Neumann operator associated to $(D_s+t)^2+D_t^2$, so

$$\int_{\mathbb{R}\times\mathbb{R}_+} [|(D_s+t)\, v|^2+|D_t v|^2-\Theta_0\, |v|^2]\, dw \geqslant 0, \qquad \forall v \in C_0^1(\mathbb{R}\times\overline{\mathbb{R}_+}). \tag{9.18}$$

Using also the property, deduced from (B.4), that

$$\left|\int |v|^2\, dz-\int |v|^2\, dw\right| \leqslant C\tau(h) \int |v|^2\, dz, \qquad \text{if} \quad \operatorname{supp}(v) \subset \Omega_{4\tau(h)}, \tag{9.19}$$

we get easily from (9.17) and (9.18) that, $\forall \gamma \in \Gamma^1_{\tau(h)}(\Omega)$, $\forall \varepsilon \in\,]0, 1/2]$, and $\forall u \in H^1(\Omega)$,

$$\begin{aligned}(1+\varepsilon)[1+C\tau(h)]\, &q^N_{h,A,\Omega}\, (\chi_{\gamma,\tau(h)} u)\\ &\geqslant \int [|(hD_s+tB(z_{\gamma,\tau(h)})\, \chi_{\gamma,\tau(h)} u|^2+h^2\, |D_t \chi_{\gamma,\tau(h)} u|^2]\, dw\\ &\quad -C\varepsilon^{-1}(\tau(h))^4\, \|\chi_{\gamma,\tau(h)} u\|^2_{L^2(\Omega)},\end{aligned}$$

for any $z_{\gamma,\tau(h)} \in \operatorname{supp}(\chi_{\gamma,\tau(h)}) \cap \bar{\Omega}$.

So, there exists $C > 0$, such that, for any $\varepsilon > 0$,

$$\begin{aligned}(1+\varepsilon)&[1+C\tau(h)]\, q^N_{h,A,\Omega}(\chi_{\gamma,\tau(h)} u)\\ &\geqslant h\Theta_0[1-C\tau(h)]\, \|B^{1/2}\chi_{\gamma,\tau(h)} u\|^2_{L^2(\Omega)}-C\varepsilon^{-1}(\tau(h))^4\, \|\chi_{\gamma,\tau(h)} u\|^2_{L^2(\Omega)}.\end{aligned} \tag{9.20}$$

We can now get the following proposition.

PROPOSITION 9.2. *Under the assumptions of Theorem* 8.2, *with* (8.1) *replaced by* (9.15), *then*

$$q^N_{h,A,\Omega}(u) \geqslant \int_\Omega W_h(z)\, |u(z)|^2\, dz, \qquad \forall u \in H^1(\Omega), \tag{9.21}$$

for any $h \in]0, 1]$.

Here W_h *is defined, for some constant* $C_0 > 0$, *by*

$$\begin{aligned} W_h(z) &= hB(z) && \text{if} \quad \operatorname{dist}(z; \partial\Omega) > 2h^{3/8} \\ &= h\Theta_0 B(z) - C_0 h^{5/4} && \text{if} \quad \operatorname{dist}(z; \partial\Omega) \leqslant 2h^{3/8}, \end{aligned}$$

and $\Theta_0 \in]0, 1[$ *is defined by* (9.3).

Proof. We take a partition of unity on $\mathbb{R}$ such that

$$\psi^2_{0,\tau(h)}(t) + \psi^2_{1,\tau(h)}(t) = 1, \qquad |\psi'_{j,\tau(h)}(t)| \leqslant C/\tau(h), \quad \text{for } j = 0, 1, \tag{9.22}$$

and

$$\operatorname{supp}(\psi^2_{0,\tau(h)}) \subset [\tau(h)/20, +\infty[, \qquad \operatorname{supp}(\psi^2_{1,\tau(h)}) \subset]-\infty, \tau(h)/10]. \tag{9.23}$$

Then, for $t = t(z) = \operatorname{dist}(z; \partial\Omega)$ in (9.22), we get as (9.13),

$$q^N_{h,A,\Omega}(u) = \sum_{j=0}^{1} [q^N_{h,A,\Omega}((\psi_{j,\tau(h)} \circ t)\, u) - h^2 \, \| |\nabla(\psi_{j,\tau(h)} \circ t)| \, u\|^2_{L^2(\Omega)}], \tag{9.24}$$

for any $u \in H^1(\Omega)$, and as in (9.16) we have also

$$q^N_{h,A,\Omega}((\psi_{0,\tau(h)} \circ t)\, u) \geqslant h \, \|B^{1/2}(\psi_{0,\tau(h)} \circ t)\, u\|^2_{L^2(\Omega)}. \tag{9.25}$$

Moreover by (9.12) and (9.14), we get:

$$\sum_{\gamma \in \Gamma^1_{\tau(h)}(\Omega)} \chi^2_{\gamma,\tau(h)}(z) = 1, \qquad \forall z \in \operatorname{supp}(\psi_{1,\tau(h)} \circ t). \tag{9.26}$$

So, for any $u \in H^1(\Omega)$,

$$\begin{aligned} &q^N_{h,A,\Omega}((\psi_{1,\tau(h)} \circ t)\, u) \\ &\quad = \sum_{\gamma \in \Gamma^1_{\tau(h)}(\Omega)} [q^N_{h,A,\Omega}(\chi_{\gamma,\tau(h)}(\psi_{1,\tau(h)} \circ t)\, u) \\ &\qquad - h^2 \, \|(\psi_{1,\tau(h)} \circ t)\, |\nabla \chi_{\gamma,\tau(h)}| \, u\|^2_{L^2(\Omega)}]. \end{aligned} \tag{9.27}$$

Taking $\tau(h) = h^{3/8}$ and in (9.20) $\varepsilon = h^{1/4}$, we deduce the estimate (9.21) of Proposition 9.2, from (9.12), (9.20) and (9.22)–(9.27). ∎

With Proposition 9.2 and Theorem 9.1, we can recover a more accurate version of the result of [LuPa2]. We recall that:

$$b := \inf_{z \in \Omega} B(z) \qquad \text{and} \qquad b' := \inf_{\omega \in \partial\Omega} B(\omega).$$

COROLLARY 9.3. *Under the assumptions of Proposition* 9.2 *then the first eigenvalue* $\mu^{(1)}(h)$ *of the Neumann operator* $P^N_{h,A,\Omega}$ *satisfies the following estimate.*

Case 1. If

$$b < \Theta_0 b', \tag{9.28}$$

there exists $C_1 > 0$ and $h_0 > 0$ such that

$$-Ch^2 \leqslant \mu^{(1)}(h) - bh \leqslant C_1 h^2, \qquad \forall h \in]0, h_0]. \tag{9.29}$$

Case 2. If

$$b > \Theta_0 b', \tag{9.30}$$

there exists $C_2 > 0$ and h_0 such that

$$-C_1 h^{5/4} \leqslant \mu^{(1)}(h) - \Theta_0 b' h \leqslant C_1 h^{3/2}, \qquad \forall h \in]0, h_0]. \tag{9.31}$$

Case 3. If

$$b = \Theta_0 b', \tag{9.32}$$

there exists $C_2 > 0$ and h_0 such that

$$-C_1 h^{5/4} \leqslant \mu^{(1)}(h) - \Theta_0 b' h \leqslant C_1 h^2, \qquad \forall h \in]0, h_0]. \tag{9.33}$$

Remark 9.4. Using Agmon's estimates (see the next proposition), one can actually improve in Case 1 the left hand side of (9.29) into

$$-C \exp -\frac{\delta}{h} \leqslant \mu^{(1)}(h) - bh, \tag{9.34}$$

for some $\delta > 0$ and for all h in $]0, h_0]$.

In a particular case, this is related to (4.3).

Note also that for the proof of Case 3, we can use the proof of the upper bound used in Case 1. In Case 3, the localization of the groundstate is, as $h \to 0$, in the union of a small neighborhood of the boundary and of the set where $B = b$.

With the estimate (9.21) of Proposition 9.2 we can get also the localization in energy of the groundstate eigenfunction $u_h^1(z)$ of $P^N_{h,A,\Omega}$ as Proposition 7.9.

PROPOSITION 9.5. *Under the assumptions of Proposition 9.2, for any $\varepsilon \in]0, 1]$,*

$$(1-\varepsilon^2)\int_\Omega [W_h(z)-\mu^{(1)}(h)]_+ \, e^{2\varepsilon\phi_h(z)/h}\, |u_h^1(z)|^2\,dz$$

$$\leqslant \int_\Omega [W_h(z)-\mu^{(1)}(h)]_- \, |u_h^1(z)|^2\,dz, \tag{9.35}$$

for any real Lipschitz function ϕ_h on $\bar{\Omega}$ such that

$$|\nabla\phi_h(z)|^2 \leqslant [W_h(z)-\mu^{(1)}(h)]_+ .$$

In Case 1 of Corollary 9.3, the estimate (9.35) and the one (7.20) associated to the ground state of Dirichlet operator show that the first eigenvalue of the Neumann operator $P^N_{h,A,\Omega}$ is exponentially closed to the one of Dirichlet operator $P_{h,A,\Omega}$.

In Case 2 of Corollary 9.3, we can improve the localization of the groundstate eigenfunction (9.35) by the following proposition.

PROPOSITION 9.6. *Under the assumptions of Proposition 9.2 and if*[7]

$$\Theta_0 b' < b, \tag{9.36}$$

then, for any $k \in \mathbb{Z}$, there exists $C_k > 0$ such that, $\forall h \in]0, 1]$,

$$\int_\Omega t(z)^k \, |u_h^1(z)|^2\,dz \leqslant C_k h^{k/2}\, \|u_h^1\|^2_{L^2(\Omega)}, \tag{9.37}$$

and

$$\int_\Omega t(z)^k \, |(hD_z - A)\, u_h^1(z)|^2\,dz \leqslant C_k h^{1+k/2}\, \|u_h^1\|^2_{L^2(\Omega)}, \tag{9.38}$$

where $t(z) = \mathrm{dist}(z; \partial\Omega)$ and u_h^1 denotes an eigenfunction associated to the first eigenvalue of the Neumann operator $P^N_{h,A,\Omega}$.

[7] The condition (9.36) involves only the infimum of B in a compact of Ω. We recall indeed from (3.13) that $\Theta_0 < 1$.

We can always find a function $\tilde{t}(z) \in C^1(\bar{\Omega})$, such that $t(z)/\tilde{t}(z)$ and $\tilde{t}(z)/t(z)$ are bounded on Ω. Actually, we can in addition take $t = \tilde{t}$ in the neighborhood of $\partial\Omega$.

So, forgetting the tilde for simplicity, we can assume that $t(z) \in C^1(\bar{\Omega})$.

We can actually propose two proofs. The first one consists in applying Theorem 6.3 together with (6.26), observing in addition that $u^k \leqslant k!\exp u$, for $u \geqslant 0$. Let us now give an independent and more direct proof. We also observe that the proposition is true for $k = 0$. Let us now consider the case when $k \geqslant 1$.

As $t(\omega) = 0$, $\forall \omega \in \partial\Omega$, we have the formula

$$\begin{aligned} h \int_\Omega t(z)^k B(z) |u_h^1|^2 \, dz \\ = i \int_\Omega t(z)^k [(hD_y - A_2) u_h^1 \overline{(hD_x - A_1) u_h^1} \\ - (hD_x - A_1) u_h^1 \overline{(hD_y - A_2) u_h^1}] \, dz \\ - hk \int_\Omega t(z)^{k-1} \left[\frac{\partial t}{\partial x} (hD_y - A_2) u_h^1 - \frac{\partial t}{\partial y} (hD_x - A_1) u_h^1 \right] \bar{u}_h^1 \, dz. \end{aligned} \tag{9.39}$$

We deduce from (9.39) that

$$\begin{aligned} h \int_\Omega t(z)^k B(z) |u_h^1|^2 \, dz \leqslant \int_\Omega t(z)^k |(hD_z - A) u_h^1|^2 \, dz \\ + Ch \, \|t^{(k-1)/2}(hD_z - A) u_h^1\|_{L^2(\Omega)} \\ \times \|t^{(k-1)/1} u_h^1\|_{L^2(\Omega)}. \end{aligned} \tag{9.40}$$

Writing that

$$\begin{aligned} \int_\Omega t(z)^k |(hD_z - A) u_h^1|^2 \, dz = \int_\Omega (hD_z - A) u_h^1 \cdot \overline{(hD_z - A) t^k u_h^1} \, dz \\ - ihk \int_\Omega t(z)^{k-1} [\nabla t \cdot (hD_z - A) u_h^1] \, \bar{u}_h^1 \, dz, \end{aligned}$$

or

$$\begin{aligned} \int_\Omega t(z)^k |(hD_z - A) u_h^1|^2 \, dz = \int_\Omega P_{h, A, \Omega} u_h^1 \cdot \overline{t^k u_h^1} \, dz \\ - ihk \int_\Omega t(z)^{k-1} [\nabla t \cdot (hD_z - A) u_h^1] \, \bar{u}_h^1 \, dz, \end{aligned}$$

we get

$$\left|\int_\Omega t(z)^k \left[|(hD_z - A)\, u_h^1|^2 - \mu^{(1)}(h)\, |u_h^1|^2\right] dz\right|$$
$$\leqslant Ch\, \|t^{(k-1)/2}(hD_z - A)\, u_h^1\|_{L^2(\Omega)} \times \|t^{(k-1)/2} u_h^1\|_{L^2(\Omega)}. \tag{9.41}$$

So we get from (9.40) and (9.41) that

$$\int_\Omega t(z)^k \left(B(z) - \frac{\mu^{(1)}(h)}{h}\right) |u_h^1|^2\, dz$$
$$\leqslant C\, \|t^{(k-1)/2}(hD_z - A)\, u_h^1\|_{L^2(\Omega)} \times \|t^{(k-1)/2} u_h^1\|_{L^2(\Omega)}. \tag{9.42}$$

The upper bound in (9.31), the assumption (9.36) which gives a lower bound for $B(x) - \mu^{(1)}(h)/h$ and (9.42) give the existence of $h_0 > 0$ such that, for all $h \in\,]0, h_0]$,

$$\|t^{k/2} u_h^1\|^2_{L^2(\Omega)} \leqslant C\, \|t^{(k-1)/2}(hD_z - A)\, u_h^1\|_{L^2(\Omega)} \times \|t^{(k-1)/2} u_h^1\|_{L^2(\Omega)}. \tag{9.43}$$

Now we can prove (9.37) by recursion.

If $k = 1$, we use that

$$\|(hD_z - A)\, u_h^1\|^2_{L^2(\Omega)} = \mu^{(1)}(h)\, \|u_h^1\|^2_{L^2(\Omega)} \leqslant Ch\, \|u_h^1\|^2_{L^2(\Omega)}.$$

We get from (9.43) that

$$\|t^{1/2} u_h^1\|_{L^2(\Omega)} \leqslant Ch^{1/4}\, \|u_h^1\|_{L^2(\Omega)}. \tag{9.44}$$

In the same way, (9.41) and (9.44) give (9.37) for $k = 1$.

Suppose now that $m \in \mathbb{Z}, m > 1$ and that (9.37) is valid for $k = m-1$. Then (9.43) with $k = m$ and (9.37) for $k = m-1$ give

$$\|t^{m/2} u_h^1\|_{L^2(\Omega)} \leqslant Ch^{m/4}\, \|u_h^1\|_{L^2(\Omega)}.$$

This estimate, together with (9.41) and (9.37) for $k = m-1$, prove the estimate (9.37) for $k = m$. ∎

10. THE CASE OF CONSTANT MAGNETIC FIELD IN DOMAINS WITH POSITIVE CURVATURE

The case of constant magnetic field has been intensively studied. In this case, the upper bound in (9.31) was established in [BeSt], the lower bound in [BeSt] is less precise than our one in (9.31). But we can still improve the lower bound in (9.31) when the magnetic field is constant.

10.1. *Bounds modulo $\mathcal{O}(h^{\frac{3}{2}})$*

THEOREM 10.1. *Suppose that Ω is a bounded open and connected set of $\mathbb{R}^2$. If the magnetic field is constant: $B=b$, then there exists a constant $C_0>0$ and $h_0>0$ such that $\mu^{(1)}(h)$, the first eigenvalue of the Neumann operator associated to $(hD_z-A)^2$ satisfies*

$$-C_0 h^{3/2} \leqslant \mu^{(1)}(h) - \Theta_0 bh \leqslant C_0 h^{3/2}, \qquad \forall h \in]0, h_0], \tag{10.1}$$

where $\Theta_0 \in]0, 1[$ is defined by (3.13).

Proof. The upper bound was already obtained in (9.31). For the lower bound, we use the notations of the proof of Proposition 9.2. Let u_h^1 a normalized groundstate. We take

$$\tau(h) = h^{1/4}. \tag{10.2}$$

So (9.22)–(9.23) and (6.10) show that

$$\begin{aligned} &|q^N_{h,A,\Omega}((\psi_{1,\tau(h)} \circ t)\, u_h^1) - \mu^{(1)}(h)\, \|(\psi_{1,\tau(h)} \circ t)\, u_h^1\|^2_{L^2(\Omega)}| \\ &\quad \leqslant h^2\, \||\nabla(\psi_{1,\tau(h)} \circ t)|\, u_h^1\|^2_{L^2(\Omega)}. \end{aligned} \tag{10.3}$$

On the other hand, if

$$\|u_h^1\|_{L^2(\Omega)} = 1,$$

we consequently obtain, thanks to (10.2), the condition on the support (9.22) and (9.37),

$$\|(\psi_{1,\tau(h)} \circ t)\, u_h^1\|_{L^2(\Omega)} = 1 + \mathcal{O}(h^k), \qquad \forall k > 0. \tag{10.4}$$

Similarly, one can show that:

$$\|\nabla\, |(\psi_{1,\tau(h)} \circ t)|\, u_h^1\|^2_{L^2(\Omega)} \leqslant C_k h^k, \qquad \forall k > 0. \tag{10.5}$$

This leads to the existence of $C>0$ such that:

$$|q^N_{h,A,\Omega}((\psi_{1,\tau(h)} \circ t)\, u_h^1) - \mu^{(1)}(h)\, \|(\psi_{1,\tau(h)} \circ t)\, u_h^1\|^2_{L^2(\Omega)}| \leqslant Ch^2. \tag{10.6}$$

The assumption that the magnetic field is constant allows us to choose a gauge (in the adapted coordinates introduced in Appendix B), such that in the formula (B.3) (cf. (B.6)), we have

$$\tilde{A}(w) = -bt\left(1 - \frac{t}{2}\kappa(s), 0\right), \tag{10.7}$$

which implies

$$|\tilde{A}(w) + (bt, 0)| \leqslant Ct^2. \tag{10.8}$$

Using the formula (B.3), (B.4) and (9.37)) for $k=1$ and $k=4$, we get, from (10.2)–(10.7), and the fact that $\mu^{(1)}(h) = \mathcal{O}(h)$, the existence of $C > 0$ and $h_0 > 0$ such that, for any $h \in \,]0, h_0]$,

$$\left| \int_{\mathbb{R}\times\mathbb{R}_+} [|(hD_s + bt)(\psi_{1,\tau(h)} u_h^1)|^2 + h^2 |D_t(\psi_{1,\tau(h)} u_h^1)|^2 - \mu^{(1)}(h) |\psi_{1,\tau(h)} u_h^1|^2] \, dw \right|$$
$$\leqslant Ch^{3/2}. \tag{10.9}$$

As

$$\int_{\mathbb{R}\times\mathbb{R}_+} [|(hD_s + bt)(\psi_{1,\tau(h)}(t)\, u_h^1(s,t))|^2 + h^2 |D_t(\psi_{1,\tau(h)}(t)\, u_h^1(s,t))|^2] \, dw$$
$$\geqslant \Theta_0 bh \int_{\mathbb{R}\times\mathbb{R}_+} |\psi_{1,\tau(h)}(t)\, u_h^1(s,t)|^2 \, dw, \tag{10.10}$$

the estimates (10.4), (10.9) and (10.10) give the existence of $C_0 > 0$ such that

$$\mu^{(1)}(h) \geqslant \Theta_0 bh - C_0 h^{3/2}. \quad \blacksquare$$

10.2. *Refined Lower Bounds*

We are now able to prove the refined lower bound. We will add two elements in comparison with the previous proof.

The first point is that we shall use the lower bound obtained in the case of the disk which is recalled in Proposition 4.9. An alternative will be presented in the next section using the techniques of the previous subsection. The second point will be to improve the remainder estimates.

Let us first come back to the case of the disk in the semi-classical context. Proposition 4.9 (due to [BaPhTa]) and an homogeneity argument give the existence of $C > 0$ such that

$$\mu^{(1)}(h, b, D(0,R)) \geqslant \Theta_0 bh - \frac{2}{R} M_3 b^{\frac{1}{2}} h^{\frac{3}{2}} - Ch^2 R^{-2}, \tag{10.11}$$

if

$$BR^2/h \geqslant C.$$

We observe also, by using the monotonicity with respect to R_0, that the same inequality is true, for any $0 < R_0 < R$, for the problem in $\Omega(R_0, R) = \{z \in D(0, R) \mid |z| > R_0\}$ with the Neumann condition at the external boundary and the Dirichlet condition at the internal boundary. Moreover Agmon's estimates permit to control more precisely the exponentially small variation of the lowest eigenvalue.

This is this second problem which is easier to compare with the general one. So one can reduce rather easily the general problem to the analysis of the problem in a tubular neighborhood of $\partial\Omega$ where the coordinates (t, s) analyzed in Appendix B are defined.

In the case of the disk, we observe the relation

$$r = R - t, \qquad \theta = s/R,$$

which is coherent with

$$r\, dr \wedge d\theta = \left(1 - \frac{1}{R} s\right) ds \wedge dt.$$

In one case we have $\kappa = \text{const.}$ and in general κ is not constant. So we have mainly to compare the two situations. For this we shall introduce near the boundary of Ω a partition of unity and compare in each ball meeting the boundary the general case with the case with constant curvature. So with the notation of Subsection 9.2, we introduce $\tau(h) = h^\rho$, and a corresponding lattice $\Gamma_{\tau(h)}$. For the moment, we just introduce the condition $\rho < \frac{1}{4}$ in order that the error term due to the partition of unity in the IMS formula has the right order.

Let us consider the balls near the boundary. This means that in the sums over γ which will be considered in the next considerations, we shall only sum over the points of the lattice $\Gamma_{\tau(h)}$ such that the corresponding balls are contained in a fixed neighborhood of $\partial\Omega$ where we can use the special coordinates. Let us denote by $\Gamma'_{\tau(h)}$ this sublattice. We would like to use a localized version of Proposition 9.6 in order to control the comparison with the model. This is the aim of the following proposition.

PROPOSITION 10.2. *Under the assumptions of Proposition 9.2, then, for any $k \in \mathbb{N}$, $\forall h \in]0, h_0]$, there exist sequences $(a_k(\gamma, h))_\gamma$ and $(b_k(\gamma, h))_\gamma$ uniformly bounded in ℓ^2 with respect to h, such that, $\forall \gamma \in \Gamma'_{\tau(h)}$,*

$$\int_\Omega t(z)^k \, |\chi_{\gamma, \tau(h)} u_h^1(z)|^2 \, dz \leqslant a_k(\gamma, h)^2 \, h^{k/2} \, \|u_h^1\|^2_{L^2(\Omega)} \tag{10.12}$$

and

$$\int_\Omega t(z)^k \, |(hD_z - A)\, \chi_{\gamma, \tau(h)} \, u_h^1(z)|^2 \, dz \leqslant b_k(\gamma, h)^2 \, h^{1+k/2} \, \|u_h^1\|^2_{L^2(\Omega)}. \tag{10.13}$$

Proof. The assertion (10.12) is a consequence of (9.37) and of the property that there exists C such that:

$$\sum_{\gamma \in \Gamma'_{\tau(h)}} \chi^2_{\gamma, \tau(h)} \leqslant C.$$

For the second one we observe that:

$$\int_\Omega t(z)^k \, |(hD_z - A)\, \chi_{\gamma, \tau(h)}\, u_h^1(z)|^2 \, dz \leqslant 2 \int_\Omega t(z)^k \, |\chi_{\gamma, \tau(h)} (hD_z - A)\, u_h^1(z)|^2 \, dz$$

$$+ 2h \int_\Omega t(z)^k \, |\nabla \chi_{\gamma, \tau(h)} u_h^1(z)|^2 \, dz.$$

The first term of the right hand side is immediate to control using (9.38). The second term is controlled by $h^{2-2\rho+k/2} c(\gamma, h)^2$ with $(c(\gamma, h))$ uniformly in ℓ^2, that is satisfying $\sum_{\gamma \in \Gamma'_{\tau(h)}} |c(\gamma, h)|^2$ is bounded independently of $h \in]0, h_0]$. So this is of the right order when $\rho < \frac{1}{2}$. ∎

Let us now consider the various errors which we have to control when comparing with the case with constant curvature. We work near the boundary in the adapted coordinates introduced in Appendix B. We have to compare more carefully, for each γ, the expressions $\int |(hD_z - A)\, \chi_{\gamma, \tau(h)}\, u_h^1|^2 \, (1 - t\kappa(s))\, ds\, dt$ and $\int |(hD_z - A^\gamma)\, \chi_{\gamma, \tau(h)}\, u_h^1|^2 \, (1 - t\kappa(s_\gamma))\, ds\, dt$, where (using the notations of Appendix B)

$$\tilde{A} = -bt \left(-1 - \frac{t}{2} \kappa(s), 0 \right), \qquad \tilde{A}^\gamma = -bt \left(-1 - \frac{t}{2} \kappa(s_\gamma), 0 \right).$$

These errors are of the following type. The first term to consider is

$$r_1(\gamma, h) = \mathcal{O}(h^\rho) \int t(z) \, |(hD_z - A)\, \chi_{\gamma, \tau(h)}\, u_h^1|^2 \, dz,$$

where we have used that $\kappa(s) - \kappa(s_\gamma) = \mathcal{O}(h^\rho)$.

This should lead (using (10.13) with $k = 1$) to a remainder of order

$$r_1(\gamma, h) = \mathcal{O}(h^{\rho + \frac{3}{2}})\, a_1(\gamma, h)^2,$$

with $\sum_\gamma a_1(\gamma, h)^2 < C$, for some C independent of h.

The next type of term is like

$$r_2(\gamma, h) = \mathcal{O}(h^\rho) \int t^2 \, |(hD_z - A)\, \chi_{\gamma, \tau(h)} u_h^1)| \cdot |\chi_{\gamma, \tau(h)} u_h^1| \, dz.$$

or like

$$r_3(\gamma, h) = \int t^3 \, |(hD_z - A)\, \chi_{\gamma, \tau(h)} u_h^1)| \cdot |\chi_{\gamma, \tau(h)} u_h^1| \, dz.$$

For r_2, we can use Cauchy–Schwarz and (10.13) with $k=4$. This leads to:

$$r_2(\gamma, h) = \mathcal{O}(h^{\rho+\frac{3}{2}})\, a_1(\gamma, h)\, b_4(\gamma, h).$$

For r_3, we can use Cauchy–Schwarz and (10.13) with $k=6$. This leads to:

$$r_3(\gamma, h) = \mathcal{O}(h^2)\, a_1(\gamma, h)\, b_6(\gamma, h).$$

The last type of remainder is

$$r_4(\gamma, h) = \mathcal{O}(h^{2\rho}) \int t^4\, |\chi_{\gamma, \tau(h)} u_h^1|^2\, dz,$$

which can be treated by (10.12) and leads to:

$$r_4(\gamma, h) = \mathcal{O}(h^{2\rho+2})\, a_4(\gamma, h)^2.$$

Summing up over $\gamma \in \Gamma'_{\tau(h)}$, this shows that we will have the general case, if we have the case of the disk with an error given by: $\mathcal{O}(h^{\rho+3/2}) + \mathcal{O}(h^{2-2\rho})$. The optimal ρ seems to be obtained when: $\rho + \frac{3}{2} = 2 - 2\rho$. This leads to the choice $\rho = \frac{1}{6}$ and to an error in $\mathcal{O}(h^{\frac{5}{3}})$.

So we have proved:

Theorem 10.3. *If $B = b$ and if $\kappa(\omega) > 0$ for all $\omega \in \partial\Omega$, then we have:*

$$\mu^{(1)}(h) \geqslant \Theta_0 bh - 2M_3 \kappa_{max} b^{\frac{1}{2}} h^{\frac{3}{2}} + \mathcal{O}(h^{\frac{5}{3}}). \tag{10.14}$$

Remark 10.4. We are obliged for the moment to assume that $\kappa(s) > 0$. This is not natural. We shall eliminate this artificial condition in the next section.

10.3. *Localization of the Groundstate*

Proposition 10.5. *Under the assumptions of Theorem 10.3, we have*

$$q^N_{h, A, \Omega}(u) \geqslant \int_\Omega W^1_h(z)\, |u(z)|^2\, dz, \qquad \forall u \in H^1(\Omega), \tag{10.15}$$

for any $h \in]0, 1]$.

Here W^1_h is defined, for some constant $C_0 > 0$, by:

$$\begin{aligned} W^1_h(z) &= bh && \text{if } \operatorname{dist}(z; \partial\Omega) > 2h^{\frac{1}{6}} \\ &= \Theta_0 bh - 2M_3 b^{\frac{1}{2}} \kappa(s)\, h^{\frac{3}{2}} - C_0 h^{5/3} && \text{if } \operatorname{dist}(z; \partial\Omega) \leqslant 2h^{\frac{1}{6}}. \end{aligned}$$

In particular, we have, using also the upper bound of $\mu^{(1)}(h)$:

$$q^N_{h,A,\Omega}(u_h^1)-\mu^{(1)}(h) \geqslant \int_\Omega (W_h^1(z)-\mu^{(1)}(h))\,|u_h^1(z)|^2\,dz,$$

$$\geqslant 2M_3 b^{\frac{1}{2}} h^{\frac{3}{2}} \int_{\Omega_{\tau(h)}} (\kappa_{max}-\kappa(s)-C_0 h^{\frac{1}{6}})\,|u_h^1|^2\,dz. \tag{10.16}$$

As for the proof of Theorem 6.6 (cf. also (6.36)) we get:

THEOREM 10.6. *Under the assumptions of Theorem 10.3, one has the following localization. There exist $\delta>0$ and for any $\varepsilon>0$, $C_\varepsilon>0$ and $h_\varepsilon>0$ such that, for all $h\in\,]0,h_\varepsilon]$:*

$$\left\| \exp \delta \frac{\hat{d}(x, n(\partial\Omega), h)}{h^{\frac{1}{4}}}\, u_h^1 \right\| \leqslant C_\varepsilon \exp \frac{\varepsilon}{h^{\frac{1}{4}}}. \tag{10.17}$$

Here $n(\partial\Omega)$ is the set

$$n(\partial\Omega)=\{z\in\partial\Omega \mid \kappa(z)=\kappa_{max}\} \tag{10.18}$$

of the points of maximal curvature

$$\hat{d}(x,n(\partial\Omega),h)=\hat{d}_{\partial\Omega}(s(x),n(\partial\Omega))\,\chi(d(x,\partial\Omega))+h^{-\frac{1}{4}}d(x,\partial\Omega), \tag{10.19}$$

and $\hat{d}_{\partial\Omega}(s,n(\partial\Omega))$ is the Agmon distance to $n(\partial\Omega)$ attached to the metric $(\kappa_{max}-\kappa(s))\,ds^2$.

The proof is similar to the proof given in Section 6 (see also the course [He1], Chap. 3).

As an immediate corollary, we have:

COROLLARY 10.7. *Under the assumptions of Theorem 10.3, then, for any neighborhood $\mathscr{V}(\partial\Omega)$ of $n(\partial\Omega)$ in $\bar{\Omega}$, there exists $\eta>0$ and $C>0$ such that, as $h\to 0$,*

$$\int_{\Omega\setminus\mathscr{V}(\partial\Omega)} |u_h^1(x)|^2\,dx \leqslant C\exp -\eta h^{-\frac{1}{4}}.$$

10.4. *Upper Bounds Modulo $\mathcal{O}(h^{\frac{7}{4}})$.*

Let us recall in this section the result obtained in [BeSt] and established more precisely in [PiFeSt].

PROPOSITION 10.8. *When Ω is of class C^4, there exists a constant C_0 such that*

$$\mu^{(1)}(h) \leqslant \Theta_0 bh - 2M_3 \kappa_{max} b^{1/2} h^{3/2} + C_0 h^{7/4}. \tag{10.20}$$

Here κ_{max} is the maximum of the scalar curvature of $\partial\Omega$ and M_3 is defined in (A.6).

Remark 10.9. In [BeSt], the authors propose a formal expansion modulo $o(h^2)$. In [PiFeSt], the authors give a proof of the remainder in $\mathcal{O}(h^2)$ under the stronger assumption that there is one point where the curvature is maximal is non degenerate. In [LuPa1], another estimate of the remainder is obtained in their appendix. Our proof gives a better remainder estimate than in [LuPa1], weaker than in [PiFeSt] but does not use an assumption of non degeneracy.

Proof of Proposition 10.8. To prove (10.20), we take in the proof of Theorem 9.1 a point $z_0 \in \partial\Omega$ such that $\kappa(s(z_0)) = \kappa_{max}$. We take instead of (9.7), with the notations of Appendix A,

$$v_h^0(w) = b^{1/4} h^{-5/16} g^h(b^{1/2} h^{-1/2} t)\, \chi(t) \cdot f(h^{-1/8} s), \tag{10.21}$$

with

$$g^h(x) = \varphi_0(x) - h^{1/2}(b)^{-1/2}\, \kappa_{max} \psi_1(x). \tag{10.22}$$

The functions χ and f are as in (9.7), and

$$\psi_1 = \tilde{R}^{N,\xi_0}[H_1 \varphi_0 - K_3 \varphi_0], \tag{10.23}$$

with

$$H_1 u(x) = (x - \xi_0)^3\, u(x) - \xi_0^2 (x - \xi_0)\, u(x) + u'(x),$$

and

$$K_3 = \int_{\mathbb{R}_+} (H_1 \varphi_0)\, \varphi_0\, dx.$$

Here $\tilde{R}^{N,\xi_0}$ is the regularized resolvent which vanishes on φ_0 and is equal to $[H^{N,\xi_0} - \Theta_0]^{-1}$ on $\{\varphi_0\}^\perp$.

Using the lemma, one gets:

$$K_3 = -2M_3.$$

Then $\psi_1(x)$ is a well defined real function. It is standard to show that $x^k \psi_1(x) \in L^2(\mathbb{R}_+)$, $\forall k \in \mathbb{N}$.

Let us introduce

$$H^h = a_0^{-2}(t)\left[bt = \left(1 - \frac{t}{2}\kappa_{max}\right) - h^{1/2}b^{1/2}\xi_0 \right]^2 + h^2 a_0^{-1}(t)\, D_t[a_0(t)\, D_t u]. \tag{10.24}$$

with $a_0(t) = 1 - \kappa_{max} t$.

We get easily that

$$\left| q^N_{h,A,\Omega}(v_h^0) - h^{-1/2}(b)^{1/2} \int_{\mathbb{R}_+} H^h(U^h(g^h(t)))\, U^h(g^h)(t)\, \chi^2(t)\, dt \right| \leqslant C h^{7/4}, \tag{10.25}$$

and that

$$|\, \|u_h^0\|_{L^2(\Omega)} - 1| \leqslant C h^{1/2}, \tag{10.26}$$

with

$$U^h(g)(t) = (b)^{1/4}\, h^{-1/4} g(b^{1/2} h^{-1/2} t),\ \forall g \in L^2(\mathbb{R}_+).$$

Then, for any $t \in [0, \varepsilon]$ and for any $g \in H^2(\mathbb{R}_+)$,

$$\begin{aligned} |(H^h g - H_0^h g(t) - H_1^h g)(t)| \leqslant C[&t^2(bt - h^{1/2}b^{1/2}\xi_0)^2 + t^3\, |(bt - h^{1/2}b^{1/2}\xi_0)| \\ &+ t^4 + ht\, |(bt - h^{1/2}b^{1/2}\xi_0)| + ht^3]\, |g(t)|, \end{aligned} \tag{10.27}$$

with

$$H_0^h g = (bt - h^{1/2}b^{1/2}\xi_0)^2 + h^2\, D_t^2\, g$$

and

$$H_1^h g = 2t\kappa_{max}(bt - h^{1/2}b^{1/2}\xi_0)^2 - b\kappa_{max} t^2 (bt - h^{1/2}b^{1/2}\xi_0) + ih^2 \kappa_{max}\, D_t g.$$

So

$$(U^h)^\star\, H_0^h U^h = bh[L + \Theta_0],$$

and

$$(U^h)^\star\, H_1^h U^h = b^{1/2} \kappa_{max} h^{3/2} H_1.$$

The estimates (10.25) and (10.27) (applied to $g = g^h$) prove easily the existence of $C > 0$ such that

$$|q^N_{h,A,\Omega}(u_h^0) - [\Theta_0 bh - 2M_3 \kappa_{max} b^{1/2} h^{3/2}]\, \|u_h^0\|^2_{L^2(\Omega)}| \leqslant C h^{7/4}\, \|u_h^0\|^2_{L^2(\Omega)},$$

and (10.20) follows. ∎

11. AN ALTERNATIVE PROOF AND EXTENSION TO NON-CONVEX DOMAINS

The previous proof was based on a result by Baumann–Phillips–Tang [BaPhTa] (see also [PiFeSt] for an alternative proof). We give now a more direct proof which permits to avoid the unnatural technical condition $\kappa(\omega) > 0$. This could in particular be applied to the Neumann problem in the exterior of the disk.

THEOREM 11.1. *When Ω is of class C^4, Theorems 10.3 and 10.6 are true without the assumption that the curvature is always strictly positive.*

We begin as in the proof of the left-hand side of (10.1). Instead of (10.2), we take

$$\tau(h) = h^\delta \quad \text{with} \quad \tfrac{1}{6} \leqslant \delta < \tfrac{1}{2} \tag{11.1}$$

(δ will be fixed later), so (10.6) and (10.4) are still valid. We can no longer use the result of [BaPhTa] and it will not be enough to localize in balls (or squares) and we choose to localize in rectangles of size h^δ in the normal direction and of size $h^{1/6}$ in the tangential direction.

We use the partition of unity defined in (9.10)–(9.12) and we get as in (9.27)

$$\begin{aligned} q^N_{h,A,\Omega}(\psi_{1,\tau(h)} u^1_h) = & \sum_{\gamma \in \Gamma^1_{\tau(h)}(\Omega)} q^N_{h,A,\Omega}(\chi_{\gamma,h^{1/6}} \psi_{1,\tau(h)} u^1_h) \\ & - h^2 \sum_{\gamma \in \Gamma^1_{\tau(h)}(\Omega)} \| |\nabla \chi_{\gamma,h^{1/6}}| \, \psi_{1,\tau(h)} u^1_h \|^2_{L^2(\Omega)} . \end{aligned} \tag{11.2}$$

We deduce from (11.2) that

$$\left| q^N_{h,A,\Omega}(\psi_{1,\tau(h)} u^1_h) - \sum_{\gamma \in \Gamma^1_{\tau(h)}(\Omega)} q^N_{h,A,\Omega}(\chi_{\gamma,h^{1/6}} \psi_{1,\tau(h)} u^1_h) \right| \leqslant C h^{5/3} \, \|\psi_{1,\tau(h)} u^1_h\|^2_{L^2(\Omega)} . \tag{11.3}$$

But it is easy to see from (B.18) that, for any $\gamma \in \Gamma^1_{\tau(h)}(\Omega)$,

$$\begin{aligned} \Big| q^N_{h,A,\Omega}(\chi_{\gamma,h^{1/6}} \psi_{1,\tau(h)} u^1_h) - \int_{K(\gamma,h)} a_\gamma(t) [h^2 \, |D_t(\chi_{\gamma,h^{1/6}} \psi_{1,\tau(h)} u^1_h)|^2 \\ + (1 + 2\kappa_\gamma t) \, |(hD_s - A^\gamma(t))(\chi_{\gamma,h^{1/6}} \psi_{1,\tau(h)} u^1_h)|^2] \, ds \, dt \Big| \\ \leqslant C h^{1/6} \int_\Omega [t \, |(hD - A)(\chi_{\gamma,h^{1/6}} \psi_{1,\tau(h)} u^1_h)|^2 + (t^3 + t^4) \, |\chi_{\gamma,h^{1/6}} \psi_{1,\tau(h)} u^1_h|^2] \, dz. \end{aligned} \tag{11.4}$$

Here $K(\gamma, h) =]-2h^{1/6}+s_\gamma, s_\gamma+2h^{1/6}[\times]0, h^\delta[$, where $s_\gamma = s(z_\gamma)$ for some $z_\gamma \in \partial\Omega$ such that $\forall z \in \operatorname{supp}(\chi_{\gamma, h^{1/6}})$, $|s-s_\gamma| \leqslant \frac{3}{2} h^{1/6}$.

Moreover,

$$\kappa_\gamma = \kappa(s_\gamma),$$

$$a_\gamma(t) = 1-\kappa_\gamma t,$$

$$A^\gamma(t) = -bt\left(1-\frac{\kappa_\gamma}{2}t\right).$$

We proceed like in Subsection 10.2 for the control of this remainder. Summing (11.4) and taking into account (9.26), (11.3), (9.37) (with $k=1, 2$) and (10.4) or alternatively using Proposition 10.2, we get that

$$\Bigg| q^N_{h, A, \Omega}(\psi_{1, \tau(h)} u^1_h) - \sum_{\gamma \in \Gamma^1_{\tau(h)}(\Omega)} \int_{K(\gamma, h)} a_\gamma(t)[h^2 |D_t(\chi_{\gamma, h^{1/6}} \psi_{1, \tau(h)} u^1_h)|^2$$
$$+(1+2\kappa_\gamma t)\,|(hD_s - A^\gamma(t))(\chi_{\gamma, h^{1/6}}\psi_{1, \tau(h)} u^1_h)|^2]\, ds\, dt \Bigg|$$
$$\leqslant Ch^{5/3} \|\psi_{1, \tau(h)} u^1_h\|^2_{L^2(\Omega)}. \tag{11.5}$$

Our result being semi-classical ($h \to 0$), we assume that $2h^{1/6} \leqslant \pi$ for simplicity.

Let us consider, for some $\gamma \in \Gamma^1_{\tau(h)}$, the term appearing in the left hand side of (11.5):

$$\int_{K(\gamma, h)} a_\gamma(t)[h^2 |D_t(\chi_{\gamma, h^{1/6}} \psi_{1, \tau(h)} u^1_h)|^2$$
$$+(1+2\kappa_\gamma t)\,|(hD_s - A^\gamma(t))(\chi_{\gamma, h^{1/6}}\psi_{1, \tau(h)} u^1_h)|^2]\, ds\, dt.$$

We get, by taking the partial Fourier transform $s \mapsto k$, that, for any γ in $\Gamma^1_{\tau(h)}(\Omega)$,

$$\int_{K(\gamma, h)} a_\gamma(t)\, [h^2 |D_t(\chi_{\gamma, h^{1/6}} \psi_{1, \tau(h)} u^1_h)|^2$$
$$+(1+2\kappa_\gamma t)\,|(hD_s - A^\gamma(t))(\chi_{\gamma, h^{1/6}}\psi_{1, \tau(h)} u^1_h)|^2]\, ds\, dt$$
$$\geqslant \mu^\gamma_1(h) \int_{K(\gamma, h)} a_\gamma(t)\, |\chi_{\gamma, h^{1/6}}\psi_{1, \tau(h)} u^1_h|^2\, ds\, dt, \tag{11.6}$$

with

$$\mu^\gamma_1(h) = \inf_{k \in \mathbb{Z}} \inf \operatorname{Sp}(H^{N, D}_{h, \gamma, k}). \tag{11.7}$$

Here $H^{N,D}_{h,\gamma,k}$ is the self-adjoint operator on $L^2(]0,h^\delta[;a_\gamma(t)\,dt)$ associated to the differential operator

$$H_{h,\gamma,k}=h^2a_\gamma^{-1}(t)\,D_t(a_\gamma(t)\,D_t\,.)+(1+2\kappa_\gamma t)(hk-A^\gamma(t))^2, \qquad (11.8)$$

with domain

$$D(H^{N,D}_{h,\gamma,k})=\{v\in H^2(]0,h^\delta[);v'(0)=v(h^\delta)=0\}. \qquad (11.9)$$

As

$$\left|\|\chi_{\gamma,h^{1/6}}\psi_{1,\tau(h)}u_h^1\|^2_{L^2(\Omega)}-\int_{K(\gamma,h)}a_\gamma(t)\,|\chi_{\gamma,h^{1/6}}\,\psi_{1,\tau(h)}u_h^1|^2\,ds\,dt\right|$$
$$\leqslant Ch^{1/6}\,\|t^{1/2}\chi_{\gamma,h^{1/6}}\psi_{1,\tau(h)}u_h^1\|^2_{L^2(\Omega)}, \qquad (11.10)$$

we get from (11.5), (11.6) and (11.10) that

$$q^N_{h,A,\Omega}(\psi_{1,\tau(h)}u_h^1)\geqslant\Big(\inf_{\gamma\in\Gamma^1_{\tau(h)}(\Omega)}\mu_1^\gamma(h)\Big)$$
$$\times\big[\|\psi_{1,\tau(h)}u_h^1\|^2_{L^2(\Omega)}-Ch^{1/6}\,\|t^{1/2}\psi_{1,\tau(h)}u_h^1\|^2_{L^2(\Omega)}\big]$$
$$-Ch^{\frac{5}{3}}\,\|\psi_{1,\tau(h)}u_h^1\|^2. \qquad (11.11)$$

Then (9.37) (with $k=1$), (10.4) and (11.11) show the existence of C and h_0 such that, for all $h\in\,]0,h_0]$,

$$q^N_{h,A,\Omega}(\psi_{1,\tau(h)}u_h^1)\geqslant\Big(\inf_{\gamma\in\Gamma^1_{\tau(h)}(\Omega)}\mu_1^\gamma(h)\Big)[1-Ch^{2/3}]\,\|\psi_{1,\tau(h)}u_h^1\|^2_{L^2(\Omega)}$$
$$-Ch^{\frac{5}{3}}\,\|\psi_{1,\tau(h)}u_h^1\|^2_{L^2(\Omega)}. \qquad (11.12)$$

The estimates (10.6), (10.4) and (11.12) prove that

$$\mu_1(h)\geqslant\inf_{\gamma\in\Gamma^1_{\tau(h)}(\Omega)}\mu_1^\gamma(h)[1-Ch^{2/3}]-Ch^{\frac{5}{3}}. \qquad (11.13)$$

But it is easy to see, noting that γ appears only through $\kappa(s_\gamma)$ which stays in a compact interval, that there exists $C>0$ such that, for all γ,

$$\mu_1^\gamma(h)\leqslant Ch. \qquad (11.14)$$

So (11.13) and (11.14) give us the existence of $C>0$, such that:

$$\mu^{(1)}(h)\geqslant\inf_{\gamma\in\Gamma^1_{\tau(h)}(\Omega)}\mu_1^\gamma(h)-Ch^{5/3}. \qquad (11.15)$$

Therefore it remains to analyze $\mu_1^\gamma(h)$.

By scaling, that is using the change of variables $t = h^{1/2} b^{-1/2}\tilde{t}$, we have just to study, for any $\xi \in \mathbb{R}$ and for any $\alpha \in [-C, C]$, the ground state energy of the self-adjoint operator $H_h^{N,D,\xi}$, which is defined as an unbounded operator on $L^2(]0, b^{1/2}h^{\delta-1/2}[; (1-h^{1/2}\alpha t)\,dt)$, by associating to the differential operator

$$H_h^\xi = D_t^2 + (t-\xi)^2 + ih^{1/2}\alpha(1-h^{1/2}\alpha t)^{-1}\,D_t + 2h^{1/2}\alpha t\left(t-\xi-h^{1/2}\alpha\frac{t^2}{2}\right)^2$$
$$-h^{1/2}\alpha t^2(t-\xi) + h\alpha^2\frac{t^4}{4}, \tag{11.16}$$

the realization whose domain is

$$D(H_h^{N,D,\xi}) = \{v \in H^2(]0, b^{\frac{1}{2}}h^{\delta-1/2}[);\ v'(0) = v(b^{\frac{1}{2}}h^{\delta-1/2}) = 0\}. \tag{11.17}$$

Note that $H_0^{N,D,\xi}$ still depend on h by its domain and that H_h^ξ depend actually on $\alpha h^{1/2}$.

If $\alpha = b^{-1/2}\kappa_\gamma$ and $\xi = -h^{1/2}b^{-1/2}k$, then

$$\mathrm{Sp}(H_{h,\gamma,k}^{N,D}) = bh\,\mathrm{Sp}(H_h^{N,D,\xi}). \tag{11.18}$$

We choose

$$\delta \in]\tfrac{1}{4}, \tfrac{1}{2}[. \tag{11.19}$$

We observe that this is compatible with (11.1) In this case, it is easy to see that

$$|\mu_j(H_h^{N,D,\xi}) - \mu_j(H_0^{N,D,\xi})| \leqslant Ch^{2\delta-\frac{1}{2}}(1+\mu_j(H_0^{N,D,\xi})). \tag{11.20}$$

For a bounded from below self-adjoint operator T with compact resolvent, $(\mu_j(T))$ denotes its increasing sequence of eigenvalues.

But the mini-max principle (by the same argument as for a Dirichlet problem) says that

$$\mu_1(H_0^{N,D,\xi}) \geqslant \mu(\xi) = \mu_1(H^{N,\xi}), \tag{11.21}$$

where $H^{N,\xi}$ is the Neumann operator on $L^2(\mathbb{R}_+)$ defined in (9.3).

The exponential decay at infinity of the eigenfunctions of $H^{N,\xi}$ and the uniform decay of the lowest eigenfunction for $(h, \xi) \in]0, h_0] \times [-J, +J]$

of $H_0^{N,D,\xi}$ show also that, for any fixed $j \in \mathbb{N}$ and $J > 0$, there exists $C_{j,J} > 0$ such that

$$|\mu_j(H_0^{N,D,\xi}) - \mu_j(H^{N,\xi})| \leqslant C_{j,J} h^2, \qquad \forall (h, \xi) \in]0, h_0] \times [-J, J]. \tag{11.22}$$

Now, let us remind (see Appendix A) that the calculus of [DaHe] is valid for $H^{N,\xi}$.

Therefore (11.20), (11.21), (11.22) and (A.4) prove that there exists $\eta > 0$ such that

$$\mu_1(H_h^{N,\xi}) \geqslant \Theta_0 + h^{2\delta - \frac{1}{2}}, \qquad \forall \xi, \quad |\xi - \xi_0| \geqslant \eta h^{\delta - \frac{1}{4}}. \tag{11.23}$$

Now, if $|\xi - \xi_0| \leqslant \eta h^{\delta - 1/4}$, we use the asymptotic expansion of eigenvalues for the perturbation of a self-adjoint operator in Chapter 1 of [MoPa] to determine $\mu(\xi)$ and φ_ξ modulo $|\xi - \xi_0|^3$.

Let us recall how the method works. We expand $H^{N,\xi}$, the eigenvalue $\mu(\xi)$ and the eigenvector φ_ξ around ξ_0 in powers of $(\xi - \xi_0)$. This leads to formal expansions and we keep as approximate eigenvector the expansion of order 2. So we take

$$\begin{aligned} f_\xi(t) = {} & \varphi_{\xi_0}(t) + 2(\xi - \xi_0)\, \tilde{R}^{N,\xi_0}[(t - \xi_0)\, \varphi_{\xi_0}(t)] \\ & + 4(\xi - \xi_0)^2\, \tilde{R}^{N,\xi_0}\{(t - \xi_0)\, \tilde{R}^{N,\xi_0}[(t - \xi_0)\, \varphi_{\xi_0}(t)] - d_2\, \varphi_{\xi_0}(t)\} \end{aligned} \tag{11.24}$$

with

$$d_2 = \int_{\mathbb{R}_+} (t - \xi_0)\, \tilde{R}^{N,\xi_0}[(t - \xi_0)\, \varphi_{\xi_0}(t)] \times \varphi_{\xi_0}(t)\, dt. \tag{11.25}$$

We recall that $\tilde{R}^{N,\xi_0}$ is the regularized inverse introduced during the proof of Proposition 10.8 and we observe for further use that, using the strict positivity of $\tilde{R}^{N,\xi_0}$ on the orthogonal to $\mathbb{R} \cdot \varphi_{\xi_0}$,

$$d_2 > 0. \tag{11.26}$$

We get easily that

$$\|[D_t^2 + (t - \xi)^2]\, f_\xi(t) - [\Theta_0 + (\xi - \xi_0)^2\, (1 - 4d_2)]\, f_\xi(t)\|_{L^2(\mathbb{R}_+)} \leqslant C\, |\xi - \xi_0|^3. \tag{11.27}$$

Let us now consider the construction of an approximate eigenvector for $H_h^{N,\xi}$. We first observe that

$$\begin{aligned} & \int_{\mathbb{R}_+} [iD_t + 2t(t - \xi_0)^2 - t^2(t - \xi_0)]\, \varphi_{\xi_0}(t) \times \varphi_{\xi_0}(t)\, dt \\ & \qquad = -2 \int_{\mathbb{R}_+} (t - \xi_0)^3\, \varphi_{\xi_0}^2(t)\, dt = -2M_3. \end{aligned}$$

We now introduce:

$$f_\xi^h(t) = \chi^h(t)\{f_\xi(t) - h^{1/2}\alpha \tilde{R}^{N,\xi_0}\{[iD_t + 2t(t-\xi_0)^2 - t^2(t-\xi_0) + 2M_3]\,\varphi_{\xi_0}(t)\}\}. \tag{11.28}$$

Here $\chi^h(t) = \chi(h^{-\delta+1/2}t)$ where χ is a smooth cutoff function satisfying $\chi(t) = 1$, if $t < 1/2$, and $\chi(t) = 0$, if $t > 1$.

In the same way as for the proof of (11.27), we get from (11.16) and (11.24):

$$\|[H_h^{N,\xi} - [\Theta_0 + (\xi-\xi_0)^2\,(1-4d_2) - 2\alpha h^{1/2} M_3]\, f_\xi^h(t)\,\|_{L^2(\mathbb{R}_+)} \leqslant C[|\xi-\xi_0|^3 + h^{1/2}\,|\xi-\xi_0|]. \tag{11.29}$$

Note that we have used the exponential decay of φ_{ξ_0} that gives also

$$|\|f_\xi^h\|_{L^2(\mathbb{R}_+)} - 1| \leqslant C[|\xi-\xi_0| + h^{1/2}]. \tag{11.30}$$

The estimates (11.29) and (11.30) show that, if $|\xi-\xi_0| \leqslant \eta h^{\delta-1/4}$, then

$$\operatorname{dist}([\Theta_0 + (\xi-\xi_0)^2\,(1-4d_2) - 2\alpha h^{1/2}M_3];\, \operatorname{Sp}(H_h^{N,\xi})) \leqslant C[|\xi-\xi_0|^3 + h^{1/2}\,|\xi-\xi_0|]. \tag{11.31}$$

The localization of the eigenvalues given by (11.20) and (11.22) shows that the nearest eigenvalue of $H_h^{N,\xi}$ minimizing (11.31) is $\mu_1(H_h^{N,\xi})$, so we have

$$|\mu_1(H_h^{N,\xi}) - [\Theta_0 + (\xi-\xi_0)^2\,(1-4d_2) - 2\alpha h^{1/2}M_3]| \leqslant C[|\xi-\xi_0|^3 + h^{1/2}\,|\xi-\xi_0|]. \tag{11.32}$$

But the asymptotic expansion of $\mu(\xi)$ in powers of $(\xi-\xi_0)$ shows that

$$2(1-4d_2) = \mu''(\xi_0) > 0. \tag{11.33}$$

Note also that, using (11.26), we have:

$$\mu''(\xi_0) < 2. \tag{11.34}$$

Therefore (A.4), (11.32) and (11.33) prove that, if $|\xi-\xi_0| \leqslant \eta h^{\delta-1/4}$, then

$$\mu_1(H_h^{N,\xi}) \geqslant \Theta_0 + (\xi-\xi_0)^2\,\frac{1-4d_2}{2} - 2\alpha h^{1/2}M_3 - Ch^{1/2}\,|\xi-\xi_0|. \tag{11.35}$$

So (A.4), (11.23) and (11.35) show that, for any $\delta \in]\frac{1}{4}, \frac{1}{2}[$,

$$\mu_1(H_h^{N,\xi}) \geqslant \Theta_0 - 2\alpha h^{1/2}M_3 - Ch^{\delta+\frac{1}{4}}. \tag{11.36}$$

We conclude by observing that (11.7), (11.15), (11.18) and (11.36) (with the choice of $\delta = 5/12$) give the lower bound (10.14). ∎

12. CONCLUSION

We have completed the results obtained by Lu–Pan [LuPa1], [LuPa2], Baumann–Phillips–Tang [BaPhTa], Bernoff–Sternberg [BeSt] and Del Pino–Felmer–Sternberg [PiFeSt]. We have in particular achieved the proof that, in the case of a constant magnetic field, the groundstate is localized near the points of maximal curvature. This leaves open the question of localization of the groundstate in the case where there is more than one isolated point of maximal curvature. We have also completed the analysis of the Dirichlet probem initiated in our paper [HeMo2], whose intial motivation was a paper by Montgomery [Mon].

The analysis of the intermediate boundary conditions between Dirichlet and Neumann could be interesting (see [LuPa3], [HoSm1] and [HoSm2]) and is physically relevant.

Let us mention for connected results (relative to excited states) the heuristic results by K. Hornberger and U. Smilansky [HoSm1] and [HoSm2] concerning bulk states and edge states.

As a result of the analysis we have presented, let us also mention that it is interesting to look at the exterior problem. In particular, in the constant magnetic field case, one can show that in an interval of the form to $[\Theta_0 bh + \mathcal{O}(h^{3/2}), hb[$, we have a discrete spectrum corresponding to states localized near the boundary (edge states).

Moreover the case of piecewise C^∞ domains has to be achieved in order to complete the results of Jadallah [Ja] (see [LuPa4] and references therein).

The analysis of the case of dimension 3 has been less analyzed (although see [LuPa4] and [LuPa5]) and it would be interesting to see what replaces the curvature in this case.

As suggested by J. Sjöstrand (and also used in the numerical computations of [HoSm1]), it could be interesting to develop a theory of reduction to the boundary.

Computations performed by K. Hornberger and U. Smilansky[8] for special domains confirm (for $b = 1$ and $h = 0.1$) the semi-classical prediction.

Finally, in the spirit of Lu–Pan, the analysis of the non-linear problem in superconductivity motivating this analysis has to be continued.

[8] Many thanks to U. Smilansky and K. Hornberger for accepting to analyze our problem with their numerical program.

APPENDICES

A. *On an Important Family of Ordinary Differential Equations*

Let us recall for the comfort of the reader the main properties (mainly due to [DaHe] and [BeSt]), concerning the Neumann realization of $H^{N,\xi}$ in $L^2(\mathbb{R}_+)$ associated to $D_x^2+(x-\xi)^2$. We denote by $\hat{\mu}^{(1)}(\xi)$ the lowest eigenvalue of $H^{N,\xi}$ and by φ_ξ the corresponding strictly positive normal eigenvalue. More simply we will write $\mu(\xi)$ instead of $\hat{\mu}^{(1)}(\xi)$ in this appendix. It has been proved that the infimum $\inf_{\xi\in\mathbb{R}} \inf \mathrm{Sp}(H^{N,\xi})$ introduced in (9.3) is actually a minimum [DaHe]. Then one can show [DaHe] that there exists $\xi_0>0$ such that $\mu(\xi)$ continues to decay monotonically till some value $\Theta_0<1$ and is then increasing monotonically and tending to 1 at $+\infty$. So it was obtained that

$$\Theta_0=\inf \mathrm{Sp}(H^{N,\xi_0}), \tag{A.1}$$

and moreover (see [DaHe] or the proof of Lemma A.1 below) that:

$$\Theta_0=\xi_0^2. \tag{A.2}$$

It is indeed proved in [DaHe] that

$$\mu'(\xi)=-[\mu(\xi)-\xi^2]\,\varphi_\xi(0)^2. \tag{A.3}$$

To get (A.3), we observe that, if $\tau>0$, then

$$\begin{aligned} 0&=\int_{\mathbb{R}_+}[D_t^2\varphi_\xi(t)+(t-\xi)^2\,\varphi_\xi(t)]\,\varphi_{\xi+\tau}(t+\tau)\,dt\\ &=-\varphi_\xi(0)\,\varphi'_{\xi+\tau}(\tau)+(\mu(\xi+\tau)-\mu(\xi))\int_{\mathbb{R}_+}\varphi_\xi(t)\,\varphi_{\xi+\tau}(t+\tau)\,dt. \end{aligned}$$

We then take the limit $\tau\to 0$ to get the formula.

From (A.3), it comes that

$$\mu''(\xi_0)=2\xi_0\,\varphi^2_{\xi_0}(0)>0. \tag{A.4}$$

Here we recall that $\mu(\xi_0)=\xi_0^2=\Theta_0$ is the strict minimum of $\xi\mapsto\mu(\xi)$ on $\mathbb{R}$.

Let $\varphi_{\xi_0}(x)$ be the normalized strictly positive eigenfunction of H^{N,ξ_0} associated to the eigenvalue Θ_0.

We use more simply the notation φ_0 instead of φ_{ξ_0}.

It is easy to see that $\varphi_0(x)$ belongs to $\mathscr{S}(\overline{\mathbb{R}_+})$. In particular we have

$$\varphi_0 \in H^2(\mathbb{R}_+), \quad \varphi_0'(0) = 0 \qquad \text{and} \qquad x^k \varphi_0(x) \in L^2(\mathbb{R}_+), \quad \forall k \in \mathbb{N}. \tag{A.5}$$

We now described some formulas appearing in [BeSt]. Let M_k denote the centred moment, of order k of the probability measure $\varphi_0^2(x)\, dx$:

$$M_k = \int_{\mathbb{R}_+} (x - \xi_0)^k \, \varphi_0^2(x)\, dx. \tag{A.6}$$

We begin by recalling how one can calculate the moments M_k, as done in [BeSt].

LEMMA A.1. *The moments can be expressed by the following formulas:*

$$M_0 = 1, \qquad M_1 = 0, \tag{A.7}$$

$$M_2 = \frac{\Theta_0}{2}, \tag{A.8}$$

$$M_3 = \frac{\varphi_0^2(0)}{6} > 0. \tag{A.9}$$

More generally, if $k > 3$, we have

$$4kM_k = (k-1)\{4\xi_0^2 M_{k-2} + (k-2)[(-\xi_0)^{k-3} \varphi_0^2(0) + (k-3)\, M_{k-4}]\}. \tag{A.10}$$

Proof of Lemma A.1. We use the arguments of [BeSt]. Let L be defined by:

$$L = H^{N, \xi_0} - \Theta_0.$$

We first observe the identity

$$L(2p\varphi_0' - p'\varphi_0) = \varphi_0[p^{(3)} - 4((x - \xi_0)^2 - \Theta_0)\, p' - 4(x - \xi_0)\, p], \tag{A.11}$$

for $p \in C^2(\overline{\mathbb{R}_+})$, and

$$\int_0^{+\infty} \varphi_0 \, Lv\, dx = \varphi_0(0)\, v'(0), \qquad \forall v \in H^2(\mathbb{R}_+) \cap L^2(\mathbb{R}_+; x^2\, dx). \tag{A.12}$$

So, for any polynomial function p,

$$\int_0^{+\infty} \varphi_0 L(2p\varphi_0' - p'\varphi_0)\, dx = \varphi_0^2(0)[2p(0)(\xi_0^2 - \Theta_0) - p^{(2)}(0)]. \tag{A.13}$$

When p is the constant polynomial ($=1$), (A.11) and (A.13) show that $M_1 = -\varphi_0^2(0)(\xi_0^2 - \Theta_0)/2$, as we know that $M_1 = 0$ and $\varphi_0(0) \neq 0$, we can recover (A.2): $\xi_0 = \Theta_0^{1/2}$.

For $p(x) = x - \xi_0$ and then for $p(x) = (x-\xi_0)^2$, we get in the same way the value of M_2 and M_3. The general case is obtained by considering $p(x) = (x-\xi_0)^{k-1}$. ∎

B. *Coordinates Near the Boundary*

For the most accurate estimates, we need to introduce rather standard adapted coordinates[9] near the boundary. Let ℓ be the length of the boundary $\partial\Omega$ and $I =]-\frac{\ell}{2}, \frac{\ell}{2}]$. Let $M \in C^3(I; \partial\Omega)$ be a parametrization of $\partial\Omega$ such that $M(0) = z_0$ and s is the distance inside $\partial\Omega$ between $M(s)$ and z_0. We denote by

$$T(s) := M'(s)$$

the unit tangent vector of $\partial\Omega$ at $M(s)$ and the scalar curvature by $\kappa(s)$, which can be defined by

$$T'(s) = \kappa(s)\, N(s),$$

where $N(s)$ is the interior normal unit vector of $\partial\Omega$ at $M(s)$.

Moreover the parametrization is chosen positive:

$$\det(T(s), N(s)) = 1, \qquad \forall s \in I.$$

From $N \cdot T = 0$ and $N \cdot N = 1$, we get first $N' \cdot T + N \cdot T' = 0$ and $N \cdot N' = 0$.

Consequently, we get:

$$N'(s) = -\kappa(s)\, T(s).$$

For any $z \in \bar{\Omega}$, we denote by $t(z)$ the standard distance of z to $\partial\Omega$:

$$t(z) = \inf_{\omega \in \partial\Omega} |z - \omega|.$$

So, there exists $\varepsilon_0 > 0$ and a diffeomorphism of class C^3,

$$\psi\colon \Omega_{\varepsilon_0} \to S^1_{\ell/(2\pi)} \times]0, \varepsilon_0[,$$

such that $\psi(z) = w = (s(z), t(z))$ and $|z - M(s(z))| = t(z)$.

We have denoted, for small enough ε, by Ω_ε the tubular neighborhood of $\partial\Omega$

$$\Omega_\varepsilon := \{z \in \Omega;\ \operatorname{dist}(z, \partial\Omega) < \varepsilon\}$$

and S^1_r is the circle of radius r is identified with $[-\pi r, \pi r[$.

[9] See for example [PiFeSt].

So we have the identity

$$z = M(s(z)) + t(z)\, N(s(z)), \qquad \forall z \in \Omega_{\varepsilon_0}. \tag{B.1}$$

From this equality, it is easy to check that

$$T(s(z)) = [1 - t(z)\, \kappa(s(z))]\, \nabla s(z) \qquad \text{and} \qquad N(s(z)) = \nabla t(z). \tag{B.2}$$

So for all $u \in H^1(\Omega)$ such that $\mathrm{supp}(u) \subset \Omega_{\varepsilon_0}$,

$$\int_\omega |(hD_z - A)\, u|^2\, dz$$
$$= \int_K [|(hD_t - \tilde{A}_2)\, v|^2 + (1 - t\kappa(s))^{-2}\, |(hD_s - \tilde{A}_1)\, v|^2](1 - t\kappa(s))\, dw \tag{B.3}$$

and

$$\int_\omega |u|^2\, dz = \int_K |v|^2\, (1 - t\kappa(s))\, dw \tag{B.4}$$

with $v(w) = u(\psi^{-1}(w))$, $K = I \times]0, \varepsilon_0[$, $w = (s, t)$ and $dw = ds\, dt$.

The magnetic potential $\tilde{A}$ satisfies

$$\tilde{A}_1\, ds + \tilde{A}_2\, dt = A_1\, dx + A_2\, dy.$$

So

$$\left[\frac{\partial \tilde{A}_2}{\partial s}(w) - \frac{\partial \tilde{A}_1}{\partial t}(w)\right] ds \wedge dt = B(z)\, dx \wedge dy = \hat{B}(w)[1 - t\kappa(s)]\, ds \wedge dt, \tag{B.5}$$

with $\psi(z) = w$ and $\hat{B}$ is defined by:

$$\hat{B}(w) = B(z). \tag{B.6}$$

This gives:

$$\frac{\partial \tilde{A}_2}{\partial s}(w) - \frac{\partial \tilde{A}_1}{\partial t}(w) = B(\psi^{-1}(w))[1 - t\kappa(s)] = \hat{B}(t, s)(1 - t\kappa(s)). \tag{B.7}$$

Then we get the identity between differential operators

$$(hD_z - A)^2 = a^{-1}[(hD_s - \tilde{A}_1)\, a^{-1}(hD_s - \tilde{A}_1) + (hD_t - \tilde{A}_2)\, a(hD_t - \tilde{A}_2)], \tag{B.8}$$

where $a(w) = 1 - t\kappa(s)$.

The usual Hilbert space $L^2(\Omega_{\varepsilon_0})$ is transformed to $L^2(K; a\,dw)$. The unitary transform U from $L^2(\Omega_{\varepsilon_0}; dz)$ into $L^2(K; dw)$, defined by:

$$U(u)(w) = a^{1/2}(w)\, u(\psi^{-1}(w)), \tag{B.9}$$

allows us to work in $L^2(K)$ and then we get the new identity between differential operators

$$a^{1/2}(hD_z - A)^2\, a^{-1/2} = (hD_s - \tilde{A}_1)\, a^{-2}(hD_s - \tilde{A}_1) + (hD_t - \tilde{A}_2)^2 + h^2 W, \tag{B.10}$$

where W is a scalar function given by:

$$W(w) = a^{-\frac{1}{2}} \frac{\partial^2}{\partial t^2} a^{1/2} + a^{-5/2} \frac{\partial^2}{\partial s^2} a^{1/2} - 4a^{-3} \left[\frac{\partial}{\partial s} a^{1/2} \right]^2.$$

A small computation gives:

$$W(s, t) = -\frac{1}{4} a^{-2}\kappa^2 - \frac{t}{2} a^{-3}\kappa'' - \frac{5}{4} t^2 a^{-4}(\kappa')^2.$$

In the new coordinates and using a gauge transform, we can always assume that the magnetic potential has no normal component in a neighbourhood of $\partial\Omega$:

$$\tilde{A}_2 = 0. \tag{B.11}$$

In this case, we have

$$\partial_t \tilde{A}_1 = -\hat{B}(t, s)(1 - t\kappa(s)), \tag{B.12}$$

where $\hat{B}$ was introduced in (B.6).

These changes may be useful for analyzing the situation near the boundary.

For example, in the Dirichlet case, with the additional condition that the functions are supported near the boundary, we get the identity

$$q^D_{h, A, \Omega}(u) = \tilde{q}^D_h(v), \tag{B.13}$$

with $v = Uu$, $u \in H_0^1(\Omega)$, supp $u \subset \Omega_{\varepsilon(\Omega)}$ and

$$\tilde{q}_h^D(v) = \int a^{-2} |(hD_s - \tilde{A}_1)\, v|^2 \, dt\, ds + \int |hD_t v|^2 \, dt\, ds + h^2 \int W(t, s)\, |v|^2 \, dt\, ds. \tag{B.14}$$

But one has to be careful in the Neumann case, because U does not respect the Neumann condition. In any case, one can use the identity

$$q_{h, A, \Omega}^N(u) = \tilde{q}_h^D(v), \tag{B.15}$$

with $v = Uu$, $u \in H^1(\Omega)$, supp $u \subset \Omega_{\varepsilon(\Omega)}$ and

$$\tilde{q}_h^N(v) = \int a^{-2} |(hD_s - \tilde{A}_1)\, v|^2 \, dt\, ds + \int |hD_t v|^2 \, dt\, ds + h^2 \int W(t, s) |v|^2 \, dt\, ds - \tfrac{1}{2} \kappa h^2 \int_{t=0} |v(s, 0)|^2 \, ds. \tag{B.16}$$

It could be dangerous to forget the last term (boundary term) in Formula (B.16), when analyzing the asymptotic behavior as $h \to 0$ of the ground state energy. Its contribution is indeed of order $\mathcal{O}(h^{\frac{3}{2}})$. On the contrary, the contribution of the third term appears only in the remainder with the order $\mathcal{O}(h^2)$. We emphasize that the minimizer of the last functional does not satisfy the usual Neumann condition but the distorted Neumann condition:

$$(\partial_t v)(s, 0) = -\tfrac{1}{2} \kappa(s)\, v(s, 0). \tag{B.17}$$

This is actually quite natural if we think of the relation $v = Uu$.

The constant magnetic field case. In the neighborhood of $\partial\Omega$, Ω_{ε_0}, we have (cf (B.8)):

$$P_{h, A, \Omega}^N u = a^{-1} \left\{ \left(hD_s + bt \left(1 - \frac{t}{2} \kappa(s) \right) \right) \times \left[a^{-1} \left(hD_s + bt \left(1 - \frac{t}{2} \kappa(s) \right) \right) u \right] + h^2 D_t (a D_t u) \right\}. \tag{B.18}$$

ACKNOWLEDGMENTS

Preliminary versions of some of these results were presented by the first author at various seminars and meetings: Paris XIII, Leyden, Berlin, Jerusalem, San Feliu de Guixols, Singapore during the academic years 1999-2000 and 2000-2001. We thank all the organizers of these fruitful meetings. Many thanks also to P. Sternberg and X.-B. Pan for communicating to us interesting preprints, to M. Dutour, L. Erdős, X.-B. Pan, and U. Smilansky for useful discussions and to K. Hornberger for the nice pictures he transmitted to us.

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