ACADEMIC PRESS
Available online at www.sciencedirect.com
SCIENCE DIRECT
J. Differential Equations 188 (2003) 221–241
Journal of Differential Equations
http://www.elsevier.com/locate/jde

Remote limit points on surfaces

N.G. Markley[a,*] and M.H. Vanderschoot[b]

[a] Alumni Memorial Building, Lehigh University, 27 Memorial Drive W, Bethlehem,PA, 18015, USA
[b] Department of Mathematics and Computer Science, Concordia College, 901 8th St S, Moorhead, MN 56562, USA

Received November 22, 2001; revised May 7, 2002

Abstract

A flow (continuous real action) on a compact orientable surface M of genus greater than one (a sphere with at least two handles) has sufficient room for orbits to wrap around one of the handles in an exotic fashion. Specifically, an orbit that is wrapping around one handle can, between wraps, spend increasing amounts of time wrapping and unwrapping around a second handle before returning to the first for the next wrap around it. As a result the omega limit set of such an orbit can contain a simple closed curve of fixed points around the second handle in spite of wrapping around the first handle. In an earlier paper (Colloq. Math. 84/85 (2000) 235), the authors constructed such a flow from this perspective and studied its lift to the universal covering space of the surface. In this paper it is shown that many of the properties of the example are consequences of a general theory that extends classical limit cycle theory.

Keywords: Surface flows; Omega limit sets; Sections; Universal coverings; Hausdorff metric

1. Introduction

The analysis of flows on surfaces in this paper is based almost entirely on the universal covering space $\tilde{M}$ of the surface M. For a compact surface of genus at least 2, $\tilde{M}$ is the interior of the unit disk and the covering transformations are hyperbolic linear fractional transformations with 2 fixed points on the unit circle, $\mathbb{K}$. The flow on M lifts to $\tilde{M}$ [9]. The orbits studied are those with lifts to $\tilde{M}$ that limit to a fixed

*Corresponding author.
E-mail addresses: ngm2@lehigh.edu (N.G. Markley), vandersc@cord.edu (M.H. Vanderschoot).

0022-0396/03/$ - see front matter
PII: S0022-0396(02)00065-7

point $a \in \mathbb{K}$ of a covering transformation T. Because of the links between covering transformations and the fundamental group of M, these orbits are wrapping around M following a particular element of the fundamental group.

The circle orthogonal to $\mathbb{K}$ determined by the fixed points a and b of T is called the axis of T. The example in [12] has the property that the hyperbolic distance on $\tilde{M}$ between the lifted orbit and the axis of T is unbounded. This is how wrapping and unwrapping about another handle manifests itself in the covering space. Other examples of flows exhibiting this behavior are known. Anosov discusses such examples in [2,3], and Nikolaev and Zuhzhoma devote a portion of Chapter 10 of [14] to this phenomenon and examples of it. These examples are obtained using existence theorems for flows and generally do not limit to a fixed point of a covering transformation. Whereas the example in [12] that motivated this paper was constructed by modifying the system of differential equations $\dot{x} = 1, \dot{y} = 1 - y^2$ to obtain a flow on a cylinder with 2 holes (a sphere with 4 holes) and attaching 2 handles with simple flows on them.

The points in the omega limit set arising from sequences of points on the positive orbit for which the hyperbolic distance of the lifts to the axis of T goes to infinity will be called remote limit points. Bounded limit points will refer to those with a bounded distance. This does not preclude the possibility that an omega limit point is both bounded and remote. Our key result about these remote limit points is that they must be fixed points.

There are a number of results showing that under certain hypothesis orbits stay a bounded distance from a geodesic. For example, in 1995, Aranson, Grines, and Zhuzhoma showed that if the set of fixed points of the flow on M is finite, then any lifted orbit that limits to a point on $\mathbb{K}$ stays a bounded hyperbolic distance from a geodesic with the same limiting point [7]. Consequently, if the set of fixed points of the flow on M is finite, then the remote limit set is empty.

A second key result links the more usual bounded limit points of an orbit with the Hausdorff limit of a sequence of lifted orbits, when the omega limit set does not consist entirely of fixed points. If T^{-n} is applied to the lifted positive orbit which approaches a as time goes to infinity, we obtain a sequence of lifts of the same positive orbit. (We are assuming a is the attractive fixed point of T.) With the addition of the point a, they become closed and thus compact subsets of the unit disk. The second key result is that this is a convergent sequence of compact sets with respect to the Hausdorff metric on the unit disk. Moreover, the portion of the limit in $\tilde{M}$, the open unit disk, projects onto the bounded limit points of the positive orbit in M. Even when this limit is not the lift of a true limit cycle, the lifted orbit approaches it in the same way that it would approach a lift of a limit cycle.

One application of the key results is the following: If y is in the omega limit set of x and x has a lift that limits to a fixed point of a covering transformation, then either y is periodic or the omega and alpha limit sets of y consist entirely of fixed points. Although this is a well-known result for the sphere, the authors do not know of any reference to it in this context. Moreover, Anosov ([2, Theorem 9]; [4, p. 131]) implies that it was unknown as of 1995.

It is also worth noting the connections with an earlier paper [11] by the first author on toral flows. The existence of non-fixed or moving points in the omega limit set played an important role in that paper and is the critical hypothesis in this one. However, the results for genus 1 and genus at least 2 are strikingly different. On the torus, when there are moving points in the omega limit set, there are no remote limit points ([11, Theorem 5]). Thus what follows is truly a theory for surfaces of genus greater than 1.

2. Main definitions

A *flow* or *continuous real action* on M is a continuous mapping ϕ: $M \times \mathbb{R} \to M$, where $\mathbb{R}$ is the reals, such that $\phi(\phi(x,t),s) = \phi(x, t+s)$ and $\phi(x,0) = x$ for all $x \in M$ and $s, t \in \mathbb{R}$. For convenience we will often follow the convention of writing xt for $\phi(x,t)$. The set of *fixed points* of ϕ is $F = \{x \in M\colon\ xt = x \text{ for all } t \in \mathbb{R}\}$. If $x \notin F$, then we say x is a *moving point*. The *orbit* of x is defined by $\mathcal{O}(x) = \{xt\colon\ t \in \mathbb{R}\}$. The *positive orbit* of x is defined by $\mathcal{O}^+(x) = \{xt\colon\ t \geqslant 0\}$. The ω-*limit set* of x is defined by $\omega(x) = \bigcap_{t\geqslant 0} \overline{\mathcal{O}^+(xt)}$.

In particular, $y \in \omega(x)$ if and only if there exists $t_n \to \infty$ such that $xt_n \to y$. The α-*limit set* is defined similarly.

A *local cross-section* Σ of ϕ at a point $x \in M$ is a closed subset Σ of M containing x such that the map $(x,t) \to xt$ is a homeomorphism of $\Sigma \times [-\varepsilon, \varepsilon]$ onto the closure of an open neighborhood V of x for some $\varepsilon > 0$. If x is a moving point then there exists a local cross section at x [13]. When M is a compact connected surface, Σ is a closed arc [17].

Throughout this paper M will be a compact orientable surface of genus $g > 1$. Thus the universal cover $\tilde{M}$ of M is the Poincaré disk: the open unit disk with the hyperbolic metric d_h derived from the differential

$$ds = \frac{2\sqrt{dx^2 + dy^2}}{1 - x^2 - y^2}.$$

The flow on M lifts to a unique flow $\tilde{\phi}$ on $\tilde{M}$ such that the covering projection π: $\tilde{M} \to M$ is a homomorphism of flows, i.e., $\pi(\tilde{\phi}(\tilde{x}, t)) = \phi(\pi(\tilde{x}), t)$, and every covering transformation T of $\tilde{M}$ is an automorphism of the flow $\tilde{\phi}$. Moreover, $\pi(\tilde{x}) \in F$ if and only if $\tilde{x} \in \tilde{F}$, where $\tilde{F}$ denotes the fixed points of $\tilde{\phi}$. These results are a consequence of the homotopy lifting theorem and can be found in [9].

Furthermore, the group of covering transformations is a discrete group of hyperbolic linear fractional transformations Γ, and M is homeomorphic to the quotient space $\tilde{M}/\Gamma$. Each $T \in \Gamma$ has exactly two fixed points, one is attracting and the other is repelling. The fixed points of Γ lie on the unit circle, denoted by $\mathbb{K}$, and are called the set of *rational points*. The segment of the Euclidean circle that passes through the fixed points of $T \in \Gamma$ and is orthogonal to $\mathbb{K}$ is called the *axis* of T. (All diameters of $\mathbb{K}$ and segments of Euclidean circles that are orthogonal to $\mathbb{K}$ are geodesics for the hyperbolic metric.) Each $T \in \Gamma$ is an orientation preserving isometry. A transformation $T \in \Gamma$ is called *primitive* if $T = S^j$, $S \in \Gamma$, implies that $|j| = 1$.

Let $\tilde{x} \in \tilde{M}$ and let $\mathscr{U}$ be the closed unit disk with Euclidean metric d, so $\mathscr{U} = \tilde{M} \cup \mathbb{K}$. The following definitions can be found in [1,6]. The lifted positive orbit $\mathcal{O}^+(\tilde{x})$ is *unbounded* if $\overline{\lim}_{t\to\infty} d_h(\tilde{x}, \tilde{x}t) = \infty$. Note that if $\lim_{t\to\infty} d_h(\tilde{x}, \tilde{x}t) = \infty$, then its limit set does not belong to $\tilde{M}$. In order to study the asymptotic behavior of $\mathcal{O}^+(\tilde{x})$, we can extend the lifted flow to $\mathscr{U}$ by taking $\mathbb{K}$ to be fixed points of $\tilde{\phi}$.

Suppose $\mathcal{O}^+(\tilde{x})$ limits to a rational point, i.e., $d(\tilde{x}t, a) \to 0$ as $t \to \infty$, where $a \in \mathbb{K}$ is a fixed point of some $T \in \Gamma$. Let $x = \pi(\tilde{x})$, and let A be the axis of T. We define the following two subsets of $\omega(x)$: $\omega_{\mathrm{R}}(x)$, the *remote limit set* of x, and $\omega_{\mathrm{B}}(x)$, the *bounded limit set* of x.

$$\omega_{\mathrm{R}}(x) = \{y \in \omega(x) | \text{there exists } \{t_n\}, t_n \to \infty \text{ as } n \to \infty, \\ \text{such that } xt_n \to y \text{ and } d_h(\tilde{x}t_n, A) \to \infty\},$$

$$\omega_{\mathrm{B}}(x) = \{y \in \omega(x) | \text{there exists } \{t_n\}, t_n \to \infty \text{ as } n \to \infty, \\ \text{such that } xt_n \to y \text{ and } d_h(\tilde{x}t_n, A) \text{ is bounded}\}.$$

Note that $\omega_{\mathrm{R}}(x) \cup \omega_{\mathrm{B}}(x) = \omega(x)$, $\omega_{\mathrm{B}}(x) \cap \omega_{\mathrm{R}}(x) \neq \emptyset$ is not precluded, and $\omega_{\mathrm{R}}(x) = \emptyset$ if and only if $\mathcal{O}^+(\tilde{x})$ stays a bounded hyperbolic distance from A.

We will use the following notation found in [11] for segments of curves and orbits. If C is a simple curve, hence homeomorphic to an interval, and a and b lie on C, then $(a, b)_C$ will denote the open segment of C between a and b. If $s, \tau \in \mathbb{R}$, then $[xs, x\tau]_\phi$ will denote $\{xt\colon s \leqslant t \leqslant \tau\}$ or $\{xt\colon \tau \leqslant t \leqslant s\}$, according as $s < \tau$ or $\tau < s$. Then $[a, b]_C$ and $(xs, x\tau)_\phi$ have the obvious meanings.

If $T \in \Gamma$ is primitive and $Ta = a$, then $Sa = a$ for $S \in \Gamma$ implies $S = T^n$ [8]. We will often make use of the following consequence to determine the behavior of lifted orbits.

Rational Boundary Point Principle: Let R be a positively invariant region in $\mathscr{U}$ such that $\bar{R} \cap \mathbb{K} = \{a\}$ and $Ta = a$ for some primitive $T \in \Gamma$. Suppose $\tilde{x} \in R$ and $\tilde{x}t \to a$ as $t \to \infty$. If $S \in \Gamma$ and $S\tilde{x} \in \bar{R}$, then $(S\tilde{x})t \to a$ as $t \to \infty$, $Sa = a$, and $S = T^n$ for some n. Thus any time we know that $S \neq T^n$ for all n we have a contradiction.

3. Main results and examples

The two main results in this paper establish the fundamental properties of the remote and bounded limit points. The first main theorem appears in Section 7.

Theorem 10. *Let ϕ be a continuous flow on M. If there exists $x \in M$ with lift $\tilde{x}$ such that $\tilde{x}t \to a \in \mathbb{K}$, where a is rational, then $\omega_{\mathrm{R}}(x) \subset F$.*

In Section 8 we show that the set of bounded limit points can be described using positive orbits in the covering space $\tilde{M}$ and the Hausdorff metric on $\mathscr{U}$. The key idea is a geometric limit set which is the Hausdorff limit of a sequence of copies of $\overline{\mathcal{O}^+(\tilde{x})}$ by distinct elements of Γ. The second main theorem is in Section 8.

Theorem 18. *Suppose $\tilde{x}t \to a \in \mathbb{K}$ as $t \to \infty$ and a is the attractive fixed point of $T \in \Gamma$. If $\omega(x) \not\subset F$, then the sequence $\{T^{-n}\mathcal{O}^+(\tilde{x})\}_{n=0}^{\infty}$ converges on $\mathscr{U}$ to a geometric limit set L with the following properties*:

(a) $TL = L$,
(b) *L is an invariant set of the flow $\tilde{\phi}$*,
(c) $\pi(L \cap \tilde{M}) = \omega_{\mathrm{B}}(x)$,
(d) *If K is a geometric limit set of $\mathcal{O}^+(\tilde{x})$ and $(K \cap \tilde{M}) \not\subset \tilde{F}$, then $K = SL$ for some $S \in \Gamma$.*

The simplest example to which these theorems apply is a limit cycle $\mathcal{O}(y)$ which is not null homotopic. If $\omega(x) = \mathcal{O}(y)$, then $\omega_{\mathrm{R}}(x) = \emptyset$, $\omega_{\mathrm{B}}(x) = \mathcal{O}(y)$, and L is just a universal covering of the periodic orbit $\mathcal{O}(y)$.

A slightly more interesting case is $\omega(x) = \{y_0\} \cup \mathcal{O}(y_1) \cup \mathcal{O}(y_2)$ such that $y_0 \in F$ and $\alpha(y_i) = \{y_0\} = \omega(y_i)$ for $i = 1$ and 2. Assume that the simple closed curve $\overline{\mathcal{O}(y_1)}$ is not null homotopic. Again, $\omega_{\mathrm{R}}(x) = \emptyset$. If $\overline{\mathcal{O}(y_2)}$ is a null homotopic loop, then L will consist of a universal cover of $\overline{\mathcal{O}(y_1)}$, i.e., a component of $\pi^{-1}(\overline{\mathcal{O}(y_1)})$ with a copy of $\mathcal{O}(\tilde{y}_2)$ attached at each element of $\pi^{-1}(y_0)$ of it. (L looks like an infinite string of Christmas lights.) In particular, L is still a component of $\pi^{-1}(\omega(x))$ but is not simply connected.

If we assume both $\overline{\mathcal{O}(y_1)}$ and $\overline{\mathcal{O}(y_2)}$ are not null homotopic and not homotopic, the picture gets more complex. (Picture a figure eight on top of a two holed doughnut with y_0 lying between the holes, with $\overline{\mathcal{O}(y_1)}$ going around the hole on the left, and with $\overline{\mathcal{O}(y_2)}$ going around the hole on the right.) Given $\tilde{y}_0 \in \pi^{-1}(y_0)$, there exists $\tilde{y}_1$ such that $\alpha(\tilde{y}_1) = \tilde{y}_0$ and $\omega(\tilde{y}_1) = S_1(\tilde{y}_0)$. Similarly, there exists $\tilde{y}_2$ such that $\alpha(\tilde{y}_2) = S_1(\tilde{y}_0)$ and $\omega(\tilde{y}_2) = S_2S_1(\tilde{y}_0)$. Set $L_0 = \overline{\mathcal{O}(\tilde{y}_1)} \cup \overline{\mathcal{O}(\tilde{y}_2)}$. Then $L = \bigcup_{n=-\infty}^{\infty}(S_2S_1)^n(L_0)$. In this case L is not a component of $\pi^{-1}(\omega(x))$. In fact the component of $\pi^{-1}(\omega(x))$ containing $\tilde{y}_0$ is a universal cover of $\omega(x)$ and is invariant under the subgroup Γ_0 of Γ generated by S_1 and S_2. Thus Γ_0 is a free group on two generators. It follows that L alternatively follows lifts of the orbits of y_1 and y_2 with points from $\pi^{-1}(y_0)$ in between. Moreover, L is a simple curve in $\mathscr{U}$ joining the two fixed points of S_2S_1 on $\mathbb{K}$, and $\tilde{x}t \to a$ where a is the fixed point of S_2S_1. Finally, the component of $\pi^{-1}(\omega(x))$ containing $\tilde{y}_0$ contains infinitely many geometric limit sets of the form SL, $S \in \Gamma_0$, and two of the geometric limit sets can intersect at a point in $\pi^{-1}(y_0)$.

The methodology in [12] can be used to construct a wide variety of examples with $\omega_{\mathrm{R}}(x) \neq \emptyset$. In the specific example in [12], $\omega_{\mathrm{R}}(x)$ is a simple closed curve and $\omega(x)$ is not locally connected at every point of $\omega_{\mathrm{R}}(x)$. In this case L is of the form

$\bigcup_{n=-\infty}^{\infty} T^n(\mathcal{O}(\tilde{y}))$, where $\tilde{y}t$ converges to rational points as t goes to both ∞ and $-\infty$. It follows that $\omega_B(x)$ is a single orbit. We believe it is possible to construct a similar example with $\omega_B(x) = \mathcal{O}(y)$ and $\tilde{y}t$ converging to irrational points of $\mathbb{K}$ as t goes to both ∞ and $-\infty$, but the details have not been completely worked out. In this example $\omega_R(x)$ would be a set of fixed points homeomorphic to the space of a Denjoy minimal set.

4. Control curves

In this section, we will examine control curves, the primary tools we use to study $\omega_R(x)$ and $\omega_B(x)$ when x is not periodic. Let ϕ be a continuous flow on M and let Σ be a section at a moving point $\tilde{x} \in \tilde{M}$. If there exist $\tau > 0$ and a covering transformation $T \in \Gamma \backslash I$ such that $\tilde{x}\tau \in T\Sigma$ and $(\tilde{x}, \tilde{x}\tau)_{\tilde{\phi}} \cap T^n\Sigma = \emptyset$ for all $n \in \mathbb{Z}$, then we can construct a simple curve J in $\tilde{M}$ called a *control curve* defined by

$$J = \bigcup_{n \in \mathbb{Z}} T^n([\tilde{x}, \tilde{x}\tau]_{\tilde{\phi}} \cup (\tilde{x}, T^{-1}\tilde{x}\tau)_{\Sigma}).$$

The curve J divides $\tilde{M}$ into two regions; one is positively invariant and the other is negatively invariant. We denote these regions J^+ and J^-, respectively. If we extend J to $\mathcal{U}$, then the limiting points of J are the fixed points of T.

Suppose an orbit consecutively crosses two equivalent section Σ and $H\Sigma$, where $H = K^j$ for some $j > 0$ and primitive transformation K. In the main result of this section, Theorem 6, we will show that between crossing Σ and $H\Sigma$ the orbit must successively cross the sections $K\Sigma, K^2\Sigma, \dots, K^{j-1}\Sigma$. Moreover, the pull-back of these crossings to Σ is strictly monotone on Σ. This is an analog for surfaces of the planar result which states that if a non-periodic orbit crosses one section more than once, then the crossings are strictly monotone. The proof will use several results about control curves; the underlying issue in these results will only be the simplicity of J, and the simplicity of $\pi(J)$ will not be assumed. The results in this section play an absolutely essential role in the main theorems.

Throughout this section we will assume that Σ is a local section at some moving point of $\tilde{\phi}$.

Lemma 1. *Suppose there exist $\kappa \in \mathbb{Z}$, $\kappa > 1$, and times $\sigma, \tau \in \mathbb{R}$, $\sigma < \tau$, such that $\tilde{x}\sigma \in \Sigma$, $\tilde{x}\tau \in T^\kappa\Sigma$, and $(\tilde{x}\sigma, \tilde{x}\tau)_{\tilde{\phi}} \cap T^{n\kappa}\Sigma = \emptyset$ for all $n \in \mathbb{Z}$. Let J be the control curve defined by $J = \bigcup_{n \in \mathbb{Z}} T^{n\kappa}([\tilde{x}\sigma, \tilde{x}\tau]_{\tilde{\phi}} \cup (\tilde{x}\sigma, T^{-\kappa}\tilde{x}\tau)_{\Sigma})$. Then $T^\lambda J \cap J \neq \emptyset$ for all λ* where $1 \leqslant \lambda \leqslant \kappa - 1$.

Proof. Since $(\tilde{x}\sigma, \tilde{x}\tau)_{\tilde{\phi}} \cap T^{n\kappa}\Sigma = \emptyset$ for all $n \in \mathbb{Z}$, J is a simple curve. Suppose there exists λ, $1 \leqslant \lambda \leqslant \kappa - 1$, such that $T^\lambda J \cap J = \emptyset$. Since J is connected, either $T^\lambda J \subset J^+$ or

$T^{\lambda}J \subset J^-$. Since M is orientable, and the extensions of J and $T^{\lambda}J$ to $\mathcal{U}$ have the same endpoints on $\mathbb{K}$, $T^{\lambda}J \subset J^+$ implies that $T^{\lambda}J^+ \subsetneq J^+$. Thus $T^{\lambda\kappa}J^+ \subsetneq J^+$. But $T^{\lambda\kappa}J = J$ and hence $T^{\lambda\kappa}J^+ = J^+$, which is a contradiction. A similar analysis applies when $T^{\lambda}J \subset J^-$. Thus $T^{\lambda}J \cap J \neq \emptyset$ for $1 \leqslant \lambda \leqslant \kappa - 1$. $\square$

An immediate consequence of Lemma 1 is the following corollary, whose proof we leave for the reader.

Corollary 2. *Suppose there exist $\kappa \in \mathbb{Z}$, $\kappa > 1$, and $\sigma, \tau \in \mathbb{R}$, $\sigma < \tau$, such that $\tilde{x}\sigma \in \Sigma$, $\tilde{x}\tau \in T^{\kappa}\Sigma$, and $(\tilde{x}\sigma, \tilde{x}\tau)_{\tilde{\phi}} \cap T^{n\kappa}\Sigma = \emptyset$ for all $n \in \mathbb{Z}$. For each λ, with $1 \leqslant \lambda \leqslant \kappa - 1$, there exists s, $\sigma < s < \tau$, such that $\tilde{x}s \in T^{p}\Sigma$, where $p \equiv^{+}_{-} \lambda \pmod{\kappa}$.*

Lemma 3. *Suppose $\tilde{x}t \to a \in \mathbb{K}$ as $t \to \infty$. If $\mathcal{O}^+(\tilde{x}) \cap \Sigma = \{\tilde{x}\}$, then $\mathcal{O}^+(\tilde{x})$ crosses $H\Sigma$ at most once for any $H \in \Gamma$.*

Proof. Suppose not. Then there exists $H \in \Gamma$ and $s_1, s_2 \in \mathbb{R}$, $0 < s_1 < s_2$, such that $\tilde{x}s_1 \in H\Sigma$ and $\tilde{x}s_2 \in H\Sigma$. Without loss of generality, we may assume that $(\tilde{x}s_1, \tilde{x}s_2)_{\tilde{\phi}} \cap H\Sigma = \emptyset$. Let G be the Jordan curve defined by $G = [\tilde{x}s_1, \tilde{x}s_2]_{\tilde{\phi}} \cup (\tilde{x}s_1, \tilde{x}s_2)_{H\Sigma}$. Since $\mathcal{O}^+(\tilde{x})$ is unbounded, $Int(G)$ is negatively invariant and $Ext(G)$ is positively invariant. Hence $\tilde{x} \in Int(G)$. Moreover, since $\mathcal{O}^+(\tilde{x}) \cap \Sigma = \{\tilde{x}\}$, $\Sigma \subset Int(G)$ and thus $H^{-1}\tilde{x}s_1 \in Int(G)$ and $H^{-1}\tilde{x}s_2 \in Int(G)$.

Next we check that $H^{-1}G \subset Int(G)$. If not, there exists s', $s_1 < s' < s_2$, such that $H^{-1}\tilde{x}s' \in (\tilde{x}s_1, \tilde{x}s_2)_{H\Sigma}$ and thus $\mathcal{O}^+(H^{-1}\tilde{x}s') \subset Ext(G)$. But this contradicts $H^{-1}\tilde{x}s_2 \in Int(G)$.

Since $\tilde{x}t \to a$ as $t \to \infty$, $H^{-1}(\tilde{x}s_2)t \to H^{-1}a$ as $t \to \infty$, and so $Int(H^{-1}G)$ is negatively invariant. Since H^{-1} is an isomorphism of flows, it must map negatively invariant regions to negatively invariant regions. Thus $H^{-1}(Int(G)) = Int(H^{-1}G) \subset Int(G)$. By Brouwer Fixed Point Theorem, H^{-1} has a fixed point in $Int(G)$. This is impossible: the fixed points of a transformation of Γ lie on $\mathbb{K}$. Thus $\mathcal{O}^+(\tilde{x})$ crosses $H\Sigma$ at most once. $\square$

Proposition 4. *Suppose there exist $\kappa \in \mathbb{Z}$, $\kappa > 0$, and $\sigma, \tau \in \mathbb{R}$, $\sigma < \tau$, such that $\tilde{x}\sigma \in \Sigma$ and $\tilde{x}\tau \in T^{\kappa}\Sigma$.*

(1) *If $(\tilde{x}\sigma, \tilde{x}\tau)_{\tilde{\phi}} \cap T^{n}\Sigma = \emptyset$ for all $n \in \mathbb{Z}$, then $\kappa = 1$.*
(2) *If $\mathcal{O}^+(\tilde{x}\sigma) \cap \Sigma = \{\tilde{x}\sigma\}$, then $(\tilde{x}\sigma, \tilde{x}\tau)_{\tilde{\phi}} \cap T^{m}\Sigma = \emptyset$ for all $m < 0$.*
(3) *If $\kappa = 1$ and $\mathcal{O}^+(\tilde{x}\sigma) \cap \Sigma = \{\tilde{x}\sigma\}$, then $(\tilde{x}\sigma, \tilde{x}\tau)_{\tilde{\phi}} \cap T^{n}\Sigma = \emptyset$ for all $n \in \mathbb{Z}$.*

Proof. (1) Suppose $\kappa > 1$. By Corollary 2, there exists a time between σ and τ where $\mathcal{O}^+(\tilde{x}\sigma)$ crosses a section which is a copy of Σ by a power of T, contradicting $(\tilde{x}\sigma, \tilde{x}\tau)_{\tilde{\phi}} \cap T^{n}\Sigma = \emptyset$. Thus $\kappa = 1$.

(2) Suppose not. Then there exist s, $\sigma < s < \tau$, and $M < 0$ such that $\tilde{x}s \in T^{M}\Sigma$. Without loss of generality, s is the first time less than τ that $\mathcal{O}^+(\tilde{x})$ crosses a copy of Σ by a negative power of T, i.e., $(\tilde{x}s, \tilde{x}\tau)_{\tilde{\phi}} \cap T^{j}\Sigma = \emptyset$ for all $j < 0$.

Let τ' be the first time greater than s that $\mathcal{O}^+(\tilde{x})$ crosses a copy of Σ by a power of T, say $\tilde{x}\tau' \in T^{M'}\Sigma$. Note that $s < \tau' \leqslant \tau$. It follows that $M' > 0$. Thus, by Part 1, $M' - M = 1$. But $M' > 0$ and $M = M' - 1$ imply that $M \geqslant 0$, which is a contradiction. Thus $(\tilde{x}\sigma, \tilde{x}\tau)_{\tilde{\phi}} \cap T^m\Sigma = \emptyset$ for all $m < 0$.

(3) Suppose not. Then there exist $j \in \mathbb{Z}$ and $s_1 \in \mathbb{R}$, such that $\sigma < s_1 < \tau$, and $\tilde{x}s_1 \in T^j\Sigma$. Choose s_1 to be the first time greater than σ that this occurs, i.e., $(\tilde{x}\sigma, \tilde{x}s_1)_{\tilde{\phi}} \cap T^n\Sigma = \emptyset$ for all $n \in \mathbb{Z}$. By Lemma 3, $\mathcal{O}^+(\tilde{x}\sigma)$ crosses any translate of Σ at most once. Thus either $j > 1$ or $j < 0$: but $j > 1$ is impossible by Part 1 and $j < 0$ by Part 2. □

Observation 5. *If J is a simple control curve defined by*

$$J = \bigcup_{n \in \mathbb{Z}} T^n([\tilde{x}t_0, \tilde{x}t_1]_{\tilde{\phi}} \cup (\tilde{x}t_0, T^{-1}\tilde{x}t_1)_{\Sigma}).$$

Then $\tilde{x}t_0$ and $T^{-1}\tilde{x}t_1$ divide Σ into three disjoint pieces:

(1) $[\tilde{x}t_0, T^{-1}\tilde{x}t_1]_{\Sigma}$, *which is in J;*
(2) *the half open interval from $T^{-1}\tilde{x}t_1$ to the endpoint of Σ that does not contain $\tilde{x}t_0$, which is in J^+ since $t_1 > t_0$; and*
(3) *the half open interval from $\tilde{x}t_0$ to the endpoint of Σ that does not contain $T^{-1}\tilde{x}t_1$, which in J^- since $t_0 < t_1$.*

Theorem 6. *Suppose there exist $\kappa \in \mathbb{Z}$, $\kappa > 1$, and $t_0, t_\kappa \in \mathbb{R}$, $t_0 < t_\kappa$, such that $\tilde{x}t_0 \in \Sigma$, $\tilde{x}t_\kappa \in T^\kappa\Sigma$, and $\mathcal{O}^+(\tilde{x}t_0) \cap \Sigma = \{\tilde{x}t_0\}$. Then there exists $\{t_j\}_{j=1}^{\kappa-1}$, $t_0 < t_1 < \cdots < t_\kappa$, such that $\tilde{x}t_j \in T^j\Sigma$, $\{T^{-j}\tilde{x}t_j\}$ is strictly monotone on Σ, and $(\tilde{x}t_j, \tilde{x}t_{j+1})_{\tilde{\phi}} \cap T^n\Sigma = \emptyset$ for all $n \in \mathbb{Z}$.*

Proof. We will argue by induction on κ. First consider the case where $\kappa = 2$. Then there exist times $t_0 < t_2$ such that $\tilde{x}t_0 \in \Sigma$, $\tilde{x}t_2 \in T^2\Sigma$, and $\mathcal{O}^+(\tilde{x}t_0) \cap \Sigma = \{\tilde{x}t_0\}$. Let s_1 be the first time greater than t_0 that $\mathcal{O}^+(\tilde{x})$ crosses a translate of Σ by a positive power of T, say $\tilde{x}s_1 \in T^{N_1}\Sigma$, where $N_1 \geqslant 1$. Note that N_1 and s_1 exist since $\tilde{x}t_2 \in T^2\Sigma$. There exist a time s_0, $t_0 \leqslant s_0 < s_1$, and $N_0 \in \mathbb{Z}$ such that $\tilde{x}s_0 \in T^{N_0}\Sigma$ and $(\tilde{x}s_0, \tilde{x}s_1)_{\tilde{\phi}} \cap T^n\Sigma = \emptyset$ for all $n \in \mathbb{Z}$. Note that $N_0 \leqslant 0$. By Proposition 4, Part 1, $N_1 - N_0 = 1$. Thus $N_1 = 1$ and $N_0 = 0$. Since $\mathcal{O}^+(\tilde{x}t_0) \cap \Sigma = \{\tilde{x}t_0\}$, it follows that $s_0 = t_0$. We have found a time s_1, $t_0 < s_1 < t_2$, such that $\tilde{x}s_1 \in T\Sigma$. By Proposition 4, Part 3, $(\tilde{x}t_0, \tilde{x}s_1)_{\tilde{\phi}} \cap T^n\Sigma = \emptyset$ and $(\tilde{x}s_1, \tilde{x}t_2)_{\tilde{\phi}} \cap T^n\Sigma = \emptyset$ for all $n \in \mathbb{Z}$.

It remains to be shown that $T^{-1}\tilde{x}s_1 \in (\tilde{x}t_0, T^{-2}\tilde{x}t_2)_{\Sigma}$. Let p denote the endpoint of Σ such that $T^{-2}\tilde{x}t_2 \in (\tilde{x}t_0, p)_{\Sigma}$. Let $J_1 = \bigcup_{n \in \mathbb{Z}} T^n([\tilde{x}t_0, \tilde{x}s_1]_{\tilde{\phi}} \cup (\tilde{x}t_0, T^{-1}\tilde{x}s_1)_{\Sigma})$. Note that J_1 is a simple control curve and $TJ_1^+ = J_1^+$. Since $t_2 > s_1$, it follows that $\tilde{x}t_2 \in J_1^+$ and so $T^{-2}\tilde{x}t_2 \in J_1^+$. By Observation 5, $T^{-2}\tilde{x}t_2$ lies on the side of $T^{-1}\tilde{x}s_1$ on Σ which does not contain $\tilde{x}t_0$. Since we also know that $T^{-2}\tilde{x}t_2 \in (\tilde{x}t_0, p)_\Sigma$, it follows that $T^{-1}\tilde{x}s_1 \in (\tilde{x}t_0, T^{-2}\tilde{x}t_2)_\Sigma$. Thus the result holds for $k=2$.

Assume that the result holds for $\kappa = m > 2$, i.e., if there exist $t_0, t_m \in \mathbb{R}$, $t_0 < t_m$, such that $\tilde{x}t_0 \in \Sigma$, $\tilde{x}t_m \in T^m\Sigma$, and $\mathcal{O}^+(\tilde{x}t_0) \cap \Sigma = \{\tilde{x}t_0\}$, then there exists $\{t_j\}_{j=1}^{m-1}$, $t_0 < t_1 < \cdots < t_m$, such that $\tilde{x}t_j \in T^j\Sigma$, $\{T^{-j}\tilde{x}t_j\}$ is strictly monotone on Σ, and $(\tilde{x}t_j, \tilde{x}t_{j+1})_{\tilde{\phi}} \cap T^n\Sigma = \emptyset$ for all $n \in \mathbb{Z}$.

Now suppose that there exist times t_0, t_{m+1}, $t_0 < t_{m+1}$, such that $\tilde{x}t_0 \in \Sigma$, $\tilde{x}t_{m+1} \in T^{m+1}\Sigma$, and $\mathcal{O}^+(\tilde{x}t_0) \cap \Sigma = \{\tilde{x}t_0\}$. Let p denote the endpoint of Σ such that $T^{-(m+1)}\tilde{x}t_{m+1} \in (\tilde{x}t_0, p)_\Sigma$. Let t_m be the first time less than t_{m+1} such that $\mathcal{O}^+(\tilde{x})$ crosses any translate of Σ by a nonnegative power of T less than $m+1$, say $\tilde{x}t_m \in T^N\Sigma$, where $0 \leqslant N \leqslant m$. Note that N and t_m exist since $\tilde{x}t_0 \in \Sigma$. Let τ be the first time greater than t_m that $\mathcal{O}^+(\tilde{x})$ crosses a T-translate of Σ, say $\tilde{x}\tau \in T^M\Sigma$, i.e., $(\tilde{x}t_m, \tilde{x}\tau)_{\tilde{\phi}} \cap T^n\Sigma = \emptyset$ for all $n \in \mathbb{Z}$. Either $M \geqslant m+1$ or $M < 0$. By Proposition 4, Part 2, M cannot be negative. Hence $M \geqslant m+1$. By Proposition 4, Part 1, $M - N = 1$. Since $M \geqslant m+1$ and $0 \leqslant N \leqslant m$, it follows that $N = m$, $M = m+1$, and hence $\tau = t_{m+1}$. Thus we have found a time t_m, $t_0 < t_m < t_{m+1}$, such that $\tilde{x}t_m \in T^m\Sigma$. Note $(\tilde{x}t_m, \tilde{x}t_{m+1})_{\tilde{\phi}} \cap T^n\Sigma = \emptyset$ for all $n \in \mathbb{Z}$ by Proposition 4, Part 3.

We will now show that $T^{-m}\tilde{x}t_m \in (\tilde{x}t_0, T^{-(m+1)}\tilde{x}t_{m+1})_\Sigma$. Let

$$\hat{J} = \bigcup_{n \in \mathbb{Z}} T^n([T^{-m}\tilde{x}t_m, T^{-m}\tilde{x}t_{m+1}]_{\tilde{\phi}} \cup (T^{-m}\tilde{x}t_m, T^{-(m+1)}\tilde{x}t_{m+1})_\Sigma).$$

The control curve $\hat{J}$ is simple. Since $t_0 < t_m$, $\tilde{x}t_0 \in \hat{J}^-$. Thus $\tilde{x}t_0$ lies on the side of $T^{-m}\tilde{x}t_m$ which does not contain $T^{-(m+1)}\tilde{x}t_{m+1}$. We also have $T^{-(m+1)}\tilde{x}t_{m+1} \in (\tilde{x}t_0, p)_\Sigma$. It follows that $T^{-m}\tilde{x}t_m \in (\tilde{x}t_0, T^{-(m+1)}\tilde{x}t_{m+1})_\Sigma$.

We may now apply the induction hypothesis to find a sequence of times $\{t_j\}_{j=1}^{m-1}$, $t_0 < t_1 < \cdots < t_m$, such that $\tilde{x}t_j \in T^j\Sigma$, $\{T^{-j}\tilde{x}t_j\}$ is strictly monotone on Σ, and $(\tilde{x}t_j, \tilde{x}t_{j+1})_{\tilde{\phi}} \cap T^n\Sigma = \emptyset$ for all $n \in \mathbb{Z}$. $\square$

5. Recurrence and limit points

A point x in M is *positively recurrent* if $x \in \omega(x)$. On a surface of genus 0 we know from classical Poincaré–Bendixson theory that positively recurrent points must be periodic. The same is true for the Klein bottle, but not the torus and all surfaces of genus greater than 1. It follows from the next theorem that when $\tilde{x}t \to a$ as $t \to \infty$ and a is rational, then positively recurrent implies periodic.

The first author showed in [10] that when x was positively recurrent and not periodic, then $\lim_{t\to\infty} \tilde{x}t = a \in \mathbb{K}$. This result may also have been known to D. Anosov about the same time. Aranson and Grines were the first to show that a was irrational [6]. However, their paper assumes the flow is differentiable and the proof uses this assumption when the author refers to a lemma [5, Lemma 3] in an earlier paper on structural instability. We include a new proof using the Rational Boundary Point Principle that does not require differentiability. It then plays a critical role in the main theorem of the next section and helps bring the paper to closure in the last section.

Theorem 7. *Suppose $\tilde{x}t \to a \in \mathbb{K}$ as $t \to \infty$. If x is positively recurrent and not periodic, then a is irrational.*

Proof. Let Σ be a local section x and let $\tilde{\Sigma}$ be the lift of Σ containing $\tilde{x}$. Because $\tilde{x}t \to a \in \mathbb{K}$ as $t \to \infty$, we can assume that $\mathcal{O}^+(\tilde{x}) \cap \tilde{\Sigma} = \{\tilde{x}\}$.

Because x is positively recurrent and not periodic, there exists $S \in \Gamma$ such that $\tilde{x}\tau \in S\tilde{\Sigma}$ for some $\tau > 0$ and $\tilde{x}\tau \neq S\tilde{x}$. By Theorem 6, we can assume S is primitive and $J = \bigcup_{n\in\mathbb{Z}} S^n([\tilde{x}, \tilde{x}\tau]_{\tilde{\phi}} \cup (\tilde{x}, S^{-1}\tilde{x}\tau)_{\tilde{\Sigma}})$ is a control curve. In particular, $(\tilde{x}, \tilde{x}\tau)_{\tilde{\phi}} \cap S^n\tilde{\Sigma} = \emptyset$ for all n.

Now suppose a is rational and the attracting fixed point of the primitive transformation $T \in \Gamma$. Then there are two cases: either $S = T^{\pm 1}$ or $S \neq T^{\pm 1}$.

First suppose $S = T^{\pm 1}$ and form the region R bounded by $\mathcal{O}^+(S^{-2}\tilde{x}\tau)$, $\mathcal{O}^+(S\tilde{x}\tau)$, and the piece of J joining $S^{-2}\tilde{x}\tau$ and $S\tilde{x}\tau$. It is easy to check that R is positively invariant and a is the only boundary point of R on $\mathbb{K}$. Because x is positively recurrent, $\mathcal{O}^+(H\tilde{x})$ crosses $(\tilde{x}, S^{-1}\tilde{x}\tau)_{\tilde{\Sigma}}$ and enters R for some $H \in \Gamma$. Clearly, $H \neq T^n$ for all n. But this contradicts the Rational Boundary Point Principle. Thus $S \neq T^{\pm 1}$.

Suppose $x\tau$ is the first crossing of Σ so $\pi(J)$ is a simple closed curve and $HJ \cap J = \emptyset$ for all $H \neq S^n$ for some n. From the above we know that $S \neq T^{\pm 1}$. Because x is positively recurrent, there exists a sequence of distinct $H_n \in \Gamma$ such that $\mathcal{O}^+(\tilde{x})$ crosses H_nJ as $t \to \infty$. Clearly $H_nJ \to a$ in the Hausdorff metric. Hence for large n, the fixed points a and b of T lie on opposite sides of H_nJ.

Replacing $\tilde{x}$ with $\tilde{x}t \in H_nJ$ and relabeling, we can assume that the fixed points of T lie on opposite sides of J, i.e., the axes of T and S intersect. It follows that $T^kJ \subset J^+$, $a \in T^kJ^+$, and $\mathcal{O}^+(\tilde{x})$ intersects T^kJ for all $k > 0$. In particular, $\mathcal{O}^+(\tilde{x})$ intersects $T^kS^{m_k}(\tilde{x}, S^{-1}\tilde{x}\tau)_{\tilde{\Sigma}}$ for some $m_k \in \mathbb{Z}$.

Now if $m_k = m_{k'}$ for some $k < k'$, we can use Theorem 6 to construct a new J' and $S' = T^{\pm 1}$, which is impossible. So there exists $|m_k| \geqslant 3$. For convenience suppose $m_k \geqslant 3$. Consider the region R bounded by $\mathcal{O}^+(\tilde{x}\tau), \mathcal{O}^+(T^{-k}\tilde{x}\sigma)$, where $\tilde{x}\sigma \in T^kS^{m_k}(\tilde{x}, S^{-1}\tilde{x}\tau)_{\tilde{\Sigma}}$, and the piece of J joining $\tilde{x}\tau$ and $T^{-k}\tilde{x}\sigma$. As before R is positively invariant and a is the only boundary point on $\mathbb{K}$. Finally, $S\tilde{x}t \in R$ for $t > \tau$ and hence, by the Rational Boundary Point Principle, $S = T^n$ for some n, a contradiction. □

6. The presence of moving points in $\omega_{\mathbf{B}}(x)$

Let x be a point on the surface M and let $\tilde{x}$ be a lift of x in the universal covering space $\tilde{M}$ of M. Throughout this section we will assume that $\tilde{x}t \to a \in \mathbb{K}$ as $t \to \infty$, and that a is rational, that is, a is the attracting fixed point of some primitive $T \in \Gamma$, the group of covering transformations for $\tilde{M}$. Clearly if this hypothesis holds for one lift of x it holds for all of them.

The presence of non-fixed or moving points in $\omega(x)$ will be a crucial hypothesis in the main theorems of this paper because it guarantees a local section at a point in $\omega(x)$ and control curves constructed using lifts of the section and the positive orbit of x. The key theorem in this section ensures that some of the moving points are in $\omega_B(x)$ and consequently that there are T-invariant control curves.

Theorem 8. *Suppose $\tilde{x}t \to a \in \mathbb{K}$ as $t \to \infty$ and a is rational. If $\omega(x)$ contains a moving point, i.e., $\omega(x) \not\subset F$, then $\omega_B(x)$ contains a moving point and is not empty.*

Proof. The proof proceeds by assuming $\omega_B(x) \subset F$, dividing it into two cases, and deriving a contradiction in each of them.

Let y be a moving point in $\omega(x)$ and let Σ be a local section at y. Since we are assuming $\omega_B(x) \subset F$, it follows that $y \in \omega_R(x)$ and $\Sigma \cap \omega(x) = \Sigma \cap \omega_R(x)$. If $\Sigma \cap \omega(x)$ has interior in Σ, then x is positively recurrent and not periodic. By Theorem 7 this is impossible because a is rational. Either $\Sigma \cap \omega(x)$ is perfect nowhere dense or contains an isolated point; these are the two cases.

Let $\tilde{\Sigma}_\lambda$, $\lambda \in \Lambda$, be the set of lifts of Σ to $\tilde{M}$. Let $T \in \Gamma$ be the primitive transformation that has a as its attracting fixed point. Then $\mathcal{O}^+(\tilde{x})$ can intersect only a finite number of the section $\tilde{\Sigma}_\lambda$ which are a bounded distance β from A the axis of T because $\omega_B(x) \subset F$. In particular, $\mathcal{O}^+(\tilde{x})$ can intersect only a finite number of $T^n\tilde{\Sigma}_\lambda$ because $T^n\tilde{\Sigma}_\lambda$ are equidistant from A. This observation will play a crucial role in both cases.

Case 1: If $\Sigma \cap \omega(x)$ contains an isolated point, then, without loss of generality, $\Sigma \cap \omega_R(x) = \{y\}$. Then there exists a consecutive sequence of crossings $x_k = x\tau_k \in \mathcal{O}^+(x)$ such that $\tau_k \nearrow \infty$ and $x_k \in \mathcal{O}^+(x) \cap \Sigma_0$, where Σ_0 is one of the two pieces of Σ when y is removed. Clearly x_k converges to y.

For each k form the curve $J_k = [x_k, x_{k+1}]_\phi \cup (x_{k+1}, x_k)_\Sigma$ on M. Each J_k is a simple closed curve and $J_k \cap J_{k'} = \emptyset$ if $k - k'$ is large. Let $\tilde{\Sigma}_k$ be the lift of Σ containing $\tilde{x}_k = \tilde{x}\tau_k$ and let $\sigma_k = d_h(\tilde{\Sigma}_k, A)$. Since $\tilde{x}t \to a \in \mathbb{K}$ as $t \to \infty$ we may assume, without loss of generality, that $\mathcal{O}^+(\tilde{x}_0) \cap \tilde{\Sigma}_0 = \{\tilde{x}_0\}$. By our initial observation, the sequence σ_k goes to infinity as k goes to infinity. Furthermore, we can find a subsequence such that $\sigma_{k_i} < \sigma_{k_{i+1}}$ for all i. By the above we can assume $J_{k_i} \cap J_{k_j} = \emptyset$ for all $i \neq j$.

Since there are at most a finite number of non-homotopic disjoint simple closed curves on M, we can assume the J_{k_i} are all homotopic. Finally, the J_{k_i} are not null homotopic because $\mathcal{O}^+(\tilde{x}_0)$ crosses any lift $\tilde{\Sigma}_k$ at most once by Lemma 3.

Next let J'_k be the loop following the orbit of x to x_{k+1}, going along Σ to x_k and back to x along the orbit of x, i.e., $[x, x_{k+1}]_\phi \cup (x_{k+1}, x_k)_\Sigma \cup [x_k, x]_\phi$. Clearly J'_k is homotopic to J_k. Hence the J'_{k_i} are all homotopic. Let $\tilde{J}'_{k_i}$ be the lift of J'_{k_i} starting at $\tilde{x}$. So there exists a single $S \in \Gamma$ such that $S\tilde{x}$ is the end point of $\tilde{J}'_{k_i}$ for all i. Hence $\tilde{x}_{k_{i+1}} \in S\tilde{\Sigma}_{k_i}$, $S\tilde{\Sigma}_{k_i} = \tilde{\Sigma}_{k_{i+1}}$ and the lift of J_{k_i} starting at $\tilde{x}_{k_i}$ ends at $S\tilde{x}_{k_i}$.

Now we can form the control curve $\tilde{J}_{k_i} = \bigcup_{n\in\mathbb{Z}} S^n([\tilde{x}_{k_i}, \tilde{x}_{k_i+1}]_{\tilde{\phi}} \cup (\tilde{x}_{k_i}, S^{-1}\tilde{x}_{k_i+1})_{\tilde{\Sigma}_{k_i}}$, which projects on J_{k_i} in M.

Because $\sigma_{k_i} < \sigma_{k_i+1}$, we have that $S \neq T^n$ for all n. Since $\tilde{x}t \to a$, which is not on the axis B of S, $d_h([\tilde{x}_{k_i}, \tilde{x}_{k_i+1}]_{\tilde{\phi}}, B) \to \infty$ and $d_h(\tilde{\Sigma}_{k_i}, B) \to \infty$ as $i \to \infty$. Since S is an isometry, $d_h(\tilde{J}_{k_i}, B) \to \infty$ as $i \to \infty$. It follows that for large i the projection of $\tilde{J}_{k_i}$ on M is not simple. But that projection is J_{k_i} which, by construction, is simple, a contradiction eliminating the first case.

Case 2: For the second case we suppose $\Sigma \cap \omega_{\mathrm{R}}(x) = C$, a Cantor set. Then there exist two consecutive crossings, say xt_1 and xt_2, of Σ by $\mathcal{O}^+(x)$ such that $(xt_1, xt_2)_\Sigma \cap \omega_{\mathrm{R}}(x) \neq \emptyset$ and therefore is also a Cantor set because $\mathcal{O}^+(x)$ can cross Σ only in the complementary intervals of C. It follows that $J = [xt_1, xt_2]_\phi \cup (xt_1, xt_2)_\Sigma$ is a simple closed curve on M. Note that J cannot be not null homotopic because $\mathcal{O}^+(x)$ must cross $[xt_1, xt_2]_\Sigma$ infinitely often.

Let $\tilde{J}$ be a component of $\pi^{-1}(J)$ in $\tilde{M}$. So $\pi|_{\tilde{J}}$ is a universal cover of J and $\tilde{J}$ is a control curve. The construction of J guarantees that $\mathcal{O}^+(\tilde{x})$ crosses infinitely many distinct copies $S_i\tilde{J}$, $S_i \in \Gamma$, with $\tilde{x}t_i \in S_i\tilde{J}$ and $t_i \nearrow \infty$. Because J is a simple closed curve the $S_i\tilde{J}$ are nested and converge in the Hausdorff metric to a. In particular, there exists an $S \in \Gamma$ such that the endpoints of $S\tilde{J}$ are on opposite sides of A, the axis of T.

Let $\tilde{\Sigma}$ be the lift of Σ on $S\tilde{J}$ which $\mathcal{O}^+(\tilde{x})$ crosses, say at $\tilde{x}t_3$. Let $\tilde{x}_1t_1$ and $\tilde{x}_2t_2$ be the lifts of xt_1 and xt_2, respectively, in $\tilde{\Sigma}$, and let a_1 and a_2 be the limits as $t \to \infty$ of $\tilde{x}_it$, $i = 1, 2$. Clearly a_1 and a_2 are on opposite sides of A. Now form the region R bounded by $\mathcal{O}^+(\tilde{x}_1t_1), \mathcal{O}^+(\tilde{x}_2t_2), (\tilde{x}_1t_1, \tilde{x}_2t_2)_{\tilde{\Sigma}}$, and the arc of $\mathbb{K}$ from a_1 to a_2 containing a. Note R is positively invariant.

For large k, $T^k\tilde{\Sigma} \subset R$. It follows that $T^kR \subset R$ for large k. Since $\{T^ka_1\}$ and $\{T^ka_2\}$ approach a from opposite sides this is only possible if $\mathcal{O}^+(\tilde{x}t_3)$ crosses $T^k\tilde{\Sigma}$ for large k. This is impossible since $\omega_{\mathrm{B}}(x) \subset F$, as we pointed out early in the proof. This is the second contradiction and completes the proof. □

The following corollary guarantees the existence of a sequence of control curves when $\omega(x)$ contains a moving point.

Corollary 9. *Suppose $\tilde{x}t \to a \in \mathbb{K}$ as $t \to \infty$ and a is rational. Let $T \in \Gamma$ be the primitive transformation that has a as its attracting fixed point. If $\omega(x) \not\subset F$, then there exists a sequence of T-invariant control curves $\{J_n\}_{n=1}^{\infty}$ constructed from $\mathcal{O}^+(\tilde{x})$ and a lift of a section Σ in M. Moreover, these control curves have the following properties*:

(a) $\overline{J_{n+1}^+} \subset \overline{J_n^+}$ *and* $\overline{J_n^-} \subset \overline{J_{n+1}^-}$

(b) $\overline{J_{n+2}^+} \subset J_n^+$ *and* $\overline{J_n^-} \subset J_{n+2}^-$.

Proof. By Theorem 8 there exists a moving point $y \in \omega_{\mathrm{B}}(x)$. Let Σ be a section at y. For any lift $\tilde{\Sigma}$ of Σ, all of the sections $T^n\tilde{\Sigma}$ are equidistant to A, the axis of T. Since

$y \in \omega_B(x)$ it follows that there exists a lift $\tilde{\Sigma}$ of Σ and a subsequence $\{T^{n_k}\tilde{\Sigma}\}$ of $\{T^n\tilde{\Sigma}\}_{n=0}^{\infty}$ which intersects $\mathcal{O}^+(\tilde{x})$.

In particular, we may assume, without loss of generality, that there exists $\{\tau_k\}$, $\tau_k \to \infty$ as $k \to \infty$, such that $\tilde{x}\tau_k \in T^{n_k}\tilde{\Sigma}$, $x\tau_k \to y$, and $x\tau_{k+1} \in (x\tau_k, y)_\Sigma$. Without loss of generality, $\tau_0 = 0$, $n_0 = 0$, and $\mathcal{O}^+(\tilde{x}) \cap \tilde{\Sigma} = \{\tilde{x}\}$. By Theorem 6, $\mathcal{O}^+(\tilde{x})$ consecutively crosses all of the sections $T^n\tilde{\Sigma}$, $n \geqslant 0$. Moreover, there exists $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that $\tilde{x}t_n \in T^n\tilde{\Sigma}, (\tilde{x}t_n, \tilde{x}t_{n+1})_{\tilde{\phi}} \cap T^m\tilde{\Sigma} = \emptyset$ for all $m \in \mathbb{Z}$, $xt_n \to y$, and $xt_{n+1} \in (xt_n, y)_\Sigma$.

For each $n \geqslant 0$ setting

$$J_n = \bigcup_{m \in \mathbb{Z}} T^m([T^{-n}\tilde{x}t_n, T^{-n}\tilde{x}t_{n+1}]_{\tilde{\phi}} \cup (T^{-n}\tilde{x}t_n, T^{-(n+1)}\tilde{x}t_{n+1})_{\tilde{\Sigma}})$$

produces the required sequence of T-invariant control curves. □

7. The dynamics of remote limit points

In [12] we constructed an example of a flow on a surface of genus 2 with points x such that $\tilde{x}t \to a$, a rational, as $t \to \infty$ and such that $\omega_R(x) \neq \emptyset$. If a is the fixed point of $T \in \Gamma$, then $\pi(A)$, where A is the axis of T, is a simple closed curve in M. For x to approach the points in $\omega_R(x)$ in our example, xt wrapped around a simple closed curve β not homotopic to $\pi(A)$ and then unwrapped. Consequently, the wrapping and unwrapping near β forced β to lie in the fixed points of the flow. This was no accident. Although $\omega_R(x)$ can be more complicated than a simple closed curve and the wrapping and unwrapping is not visible in the proof, we show in this section that the points in $\omega_R(x)$ are all fixed when a is rational, and the dynamics of the remote limit points is uninteresting. An earlier version of this result with the assumption that $\omega_B(x)$ contains a moving point appears in [15].

Theorem 10. *Let ϕ be a continuous flow on M. If there exists $x \in M$ with lift $\tilde{x}$ such that $\tilde{x}t \to a \in \mathbb{K}$, where a is rational, then $\omega_R(x) \subset F$.*

Proof. If $\omega(x) \subset F$, there is nothing to prove. If $\omega(x) \not\subset F$, then $\omega_B(x) \not\subset F$ by Theorem 8. Let T be the primitive covering transformation in Γ that has a as its attracting fixed point. Let Σ be a section at a moving point $y \in \omega_B(x)$. By Corollary 9 there exists a sequence of T-invariant control curves. In particular, there exists a lift $\tilde{\Sigma}$ of Σ and times $t_0 < t_1$ such that $\tilde{x}t_0 \in \tilde{\Sigma}$, $\tilde{x}t_1 \in T\tilde{\Sigma}$, $xt_1 \in (xt_0, y)_\Sigma$, and $J = \bigcup_{m \in \mathbb{Z}} T^m([\tilde{x}t_0, \tilde{x}t_1]_{\tilde{\phi}} \cup (\tilde{x}t_0, T^{-1}\tilde{x}t_1)_{\tilde{\Sigma}})$ is a control curve.

Now form the region R bounded by $\mathcal{O}^+(\tilde{x}t_1)$ and J', the piece of J from $\mathcal{O}^+(\tilde{x}t_1)$ to a, i.e., $J' = \bigcup_{m>0} T^m([\tilde{x}t_0, \tilde{x}t_1]_{\tilde{\phi}} \cup (\tilde{x}t_0, T^{-1}\tilde{x}t_1)_{\tilde{\Sigma}})$.

Now suppose $q \in \omega_R(x)$. We will first show that if $\tilde{q}$ is a lift of q, then $\tilde{q} \notin R$. We will proceed by contradiction. Suppose $\tilde{q} \in R$. Let V be a compact covering neighborhood of q in M such that the lift $\tilde{V}$ containing $\tilde{q}$ is contained itself in the interior of R. There exists $s > 0$ such that $xs \in V$ and

$$d_h(\tilde{x}s, A) > 2(d_h(\tilde{q}, A) + \delta), \tag{1}$$

where δ is the hyperbolic diameter of a lift of V. There exists $\tilde{q}_1 \in \pi^{-1}(q)$ such that $\tilde{x}s$ and $\tilde{q}_1$ are contained in the same lift $\tilde{V}_1$ of V. Let H be the element of Γ such that $H\tilde{q}_1 = \tilde{q}$. Clearly $H\tilde{V}_1 = \tilde{V}$ and $H\tilde{x}s \in R$. By the Rational Boundary Point Principle, $H = T^n$ for some n. Hence $d_h(\tilde{q}_1, A) = d_h(\tilde{q}, A)$ which is impossible by (1).

Now we will obtain a similar contradiction by assuming $q \notin F$. Hence there exists a section Σ' at q. Without loss of generality, we can assume $\Sigma' \cap \pi(J) = \emptyset$. There exist positive s_1 and s_2 such that xs_1 and xs_2 lie on the same side of q in Σ' and

$$d_h(\tilde{x}s_2, A) > 2(d_h(\tilde{x}s_1, A) + \delta), \tag{2}$$

where δ is the hyperbolic diameter of a lift of Σ'. Then $\tilde{x}s_i \in \tilde{\Sigma}'_i$, where $\tilde{\Sigma}'_i$ are the lifts of Σ' such that $\tilde{q}_i \in \tilde{\Sigma}'_i$. If $H\tilde{q}_1 = \tilde{q}_2$, then $H \neq T^n$ for all n by (2). Without loss of generality, xs_2 lies between xs_1 and q on Σ'. It now follows that $H\tilde{x}s_1 \in R$ because $\tilde{q}_1$ and $\tilde{q}_2 \notin R$ and $\tilde{\Sigma}'$ can intersect the simple closed curve bounding R at most once. By the Rational Boundary Point Principle, we have that $H = T^n$, the final contradiction. □

8. Geometric limit sets

Suppose $\omega(x)$ is a periodic orbit which is not null homotopic, that is, $\omega(x)$ is a limit cycle. Let $\tilde{x}$ be a lift of x. It is easy to see that $\tilde{x}t \to a$, a rational, as $t \to \infty$. Then a is the attracting fixed point of a primitive $T \in \Gamma$. Further, it can be shown that $\{T^{-n}\overline{\mathcal{O}^+(\tilde{x})}\}_{n=0}^{\infty}$ converges in the Hausdorff metric on $\mathcal{U}$ to a set L and $L \cap \tilde{M}$ is a lift of the periodic orbit equaling $\omega(x) = \omega_B(x)$ because $\omega_R(x) = \emptyset$.

In this section we will show that the Hausdorff convergence of $T^{-n}\overline{\mathcal{O}^+(\tilde{x})}$ to a set L is quite general, requiring only that $\omega(x)$ is not contained in F, the set of fixed points. (This result appeared in [15] under the more restrictive condition that $\omega_B(x)$ contains a moving point.) It follows that $TL = L$ and $\pi(L \cap \tilde{M}) = \omega_B(x)$. Thus $\omega_B(x)$ is a generalization of a limit cycle. In particular, the second author has shown [16] that when $\omega(x)$ is locally connected, $\omega_R(x) = \emptyset$ and $\omega(x) = \omega_B(x)$ contains a simple closed invariant curve which is not null homotopic. In contrast, $\omega_B(x)$ in the previously cited example [12] is a single orbit, $\mathcal{O}(y)$, such that $\omega(y) = \alpha(y) = \omega_R(x)$ is a simple closed curve of fixed points which is not null homotopic.

A *geometric limit set* of $\tilde{x}$ is a Hausdorff limit of a sequence of copies of $\overline{\mathcal{O}^+(\tilde{x})}$ by distinct elements of Γ. Note that the Hausdorff limit of $S_k\mathcal{O}^+(\tilde{x})$ is the same as that of $S_k\overline{\mathcal{O}^+(\tilde{x})}$.

Throughout this section we will assume that $\omega(x)\not\subset F$ (and hence $\omega_{\mathrm{B}}(x)\not\subset F$), $\tilde{x}t\to a\in\mathbb{K}$ as $t\to\infty$, and that a is rational, i.e., a is the attracting fixed point of some primitive $T\in\Gamma$. By Corollary 9, a sequence of T-invariant control curves J_n can be constructed from $\mathcal{O}^+(\tilde{x})$ and a lift $\tilde{\Sigma}$ of a section at a moving point in $\omega_{\mathrm{B}}(x)$ in M. In particular, there exists a sequence $\{t_n\}$, $t_n\to\infty$ as $n\to\infty$, such that $\tilde{x}t_n\in T^n\tilde{\Sigma}$, $(\tilde{x}t_n,\tilde{x}t_{n+1})_{\tilde{\phi}}\cap T^m\tilde{\Sigma}=\emptyset$ for all $m\in\mathbb{Z}$, $xt_n\to y$, and $xt_{n+1}\in(xt_n,y)_{\Sigma}$. For each $n\geqslant 0$,

$$J_m=\bigcup_{m\in\mathbb{Z}}T^m([\tilde{x}t_n,\tilde{x}t_{n+1}]_{\tilde{\phi}}\cup(\tilde{x}t_n,T^{-1}\tilde{x}t_{n+1})_{T^n\tilde{\Sigma}}).$$

We define the following two subsets of $\mathcal{U}$:

$$A=\bigcap\overline{J_n^+}=\bigcap J_n^+$$

and

$$B=\bigcup J_n^-=\bigcup\overline{J_n^-}.$$

Note that $\partial A=A\cap\bar{B}$ and $\partial A=\partial B$ by de Morgan.

Lemma 11. *The sequence of control curves J_n converges to ∂A in the Hausdorff metric on $\mathcal{U}$.*

Proof. Let $\{J_{n_k}\}$ be a convergent subsequence of $\{J_n\}$, say $J_{n_k}\to J$. From Part(a) of Corollary 9 it follows that $J\subset\partial A$.

Let $z\in\partial A$. Given $\varepsilon>0$ there exists M_ε such that $J_m^-\cap B_\varepsilon(z)\neq\emptyset$ for $m\geqslant M_\varepsilon$ because $z\in\bar{B}$ and $J_n^-\subset J_{n+1}^-$. Since $z\in A$, we also have $J_m^+\cap B_\varepsilon(z)\neq\emptyset$ for $m\geqslant M_\varepsilon$. Now, by connectivity of $B_\varepsilon(z)$, we have that $J_m\cap B_\varepsilon(z)\neq\emptyset$ for $m\geqslant M_\varepsilon$. In particular, $J_{n_k}\cap B_\varepsilon(z)\neq\emptyset$ for large k. It follows that $J\cap B_\varepsilon(z)\neq\emptyset$ for all $\varepsilon>0$ and $z\in J$.

Having shown that ∂A is the only possible limit of a convergent subsequence of J_n, it follows that J_n converges in the Hausdorff metric to ∂A. □

Theorem 12. *Suppose $\tilde{x}t\to a\in\mathbb{K}$ as $t\to\infty$ and a is rational, and the attracting fixed point of some primitive $T\in\Gamma$. If $\omega(x)$ contains a moving point, then $\{T^{-n}\mathcal{O}^+(\tilde{x})\}_{n=0}^{\infty}$ converges in the Hausdorff metric on $\mathcal{U}$.*

Proof. By Theorem 8 there exists a moving point $y\in\omega_{\mathrm{B}}(x)$. Let Σ be a local section at y. By Corollary 9 we can construct the sequence of T-invariant control curves J_n to which Lemma 11 applies.

Let L be the limit in the Hausdorff metric of a convergent subsequence of $\{T^{-n}\mathcal{O}^+(\tilde{x})\}$, say $T^{-n_j}\mathcal{O}^+(\tilde{x})\to L$. As in the proof of Lemma 11 it suffices to show that $L=\partial A$.

Let $z\in L$. There exists $\{s_j\}$, $s_j\to\infty$ as $j\to\infty$, such that $T^{-n_j}\tilde{x}s_j\to z$. Each $T^{-n_j}\tilde{x}s_j$ lies on some control curve in the sequence $\{J_n\}$. The specific curve depends on s_j. For each s_j there exists $t_{m_j}\in\{t_m\}$ such that $t_{m_j}<s_j<t_{m_j+1}$ and so $T^{-n_j}\tilde{x}s_j\in J_{m_j}$. Note that $m_j\to\infty$ as $j\to\infty$. By Lemma 11, $J_{m_j}\to\partial A$. Hence $z\in\partial A$ and $L\subset\partial A$.

To show that $\partial A\subset L$, let $w\in\partial A$. If $w\in T^n\tilde{\Sigma}$ for some n, then $w\in L$. If $w\notin T^n\tilde{\Sigma}$, then choose $\varepsilon>0$ such that $B_\varepsilon(w)\cap T^n\tilde{\Sigma}=\emptyset$ for all n and $B_\varepsilon(w)\cap J_1=\emptyset$. There exists m such that $J_m\cap B_\varepsilon(w)\neq\emptyset$ and $T^k(\tilde{x}t_m,\tilde{x}t_{m+1})_{\tilde{\phi}}\cap B_\varepsilon(w)\neq\emptyset$ for some k.

For $n>m$ construct the simple closed curve

$$G_n=T^{k+m}\{[\tilde{x}t_0,\tilde{x}t_1]_{\tilde{\phi}}\cup[T^{-n}\tilde{x}t_n,T^{-n}\tilde{x}t_{n+1}]_{\tilde{\phi}}$$
$$\cup[\tilde{x}t_0,T^{-n}\tilde{x}t_n]_{\tilde{\Sigma}}\cup[\tilde{x}t_1,T^{-n}\tilde{x}t_{n+1}]_{T\tilde{\Sigma}}\}.$$

Note that $T^k(\tilde{x}t_m,\tilde{x}t_{m+1})_{\tilde{\phi}}$ is in the interior of G_n for $n>m$, and hence $B_\varepsilon(w)$ also intersects the interior of G_n. Since the interior of G_n is contained in J_n^-, it does not contain $B_\varepsilon(w)$ because $w\in\partial A$. By the choice of ε, $B_\varepsilon(w)\cap T^{k+m}(T^{-n}\tilde{x}t_n,T^{-n}\tilde{x}t_{n+1})_{\tilde{\phi}}\neq\emptyset$ for all $n>m$ because $B_\varepsilon(w)$ is connected.

Using the fact that $k+m$ is fixed and n ($n>m$) is arbitrary, for $n_j>k+m$ we can find a sequence of times $\{s_j\}$, $s_j\to\infty$ as $j\to\infty$, such that $T^{-n_j}\tilde{x}s_j\to w$ and so $w\in L$. Hence $\partial A\subset L$ and thus $L=\partial A$. Since the argument was independent of choice of convergent subsequence of $\{T^{-n}\mathcal{O}^+(\tilde{x})\}$, we have that $T^{-n}\mathcal{O}^+(\tilde{x})$ converges in the Hausdorff metric to ∂A. □

We now set $L=\lim_{n\to\infty}T^{-n}\mathcal{O}^+(\tilde{x})$ when $\omega(x)\not\subset F$. It follows that $TL=L$ and that for $S\in\Gamma$,

$$\begin{aligned}SL&=S\lim_{n\to\infty}T^{-n}\mathcal{O}^+(\tilde{x})\\&=\lim_{n\to\infty}ST^{-n}\mathcal{O}^+(\tilde{x})\\&=\lim_{n\to\infty}ST^{-n}S^{-1}S\mathcal{O}^+(\tilde{x})\\&=\lim_{n\to\infty}(STS^{-1})^n\mathcal{O}^+(S\tilde{x}).\end{aligned}$$

So SL is a geometric limit set for all $S\in\Gamma$.

Remark 13. *Let S_k be a sequence of distinct elements of Γ, and let C be a compact subset of $\tilde{M}$, and let $\{w_k\}$ be a sequence of points in C. If S_kw_k converges to z, then $z\in\mathbb{K}$.*

Proof. C is contained in a finite number of compact fundamental domains as is any compact subset of $\tilde{M}$. Hence, given $0<r<1$, $S_k C$ must be outside of $\{z\colon |z| \leqslant r\}$ for large k. In particular, $r \leqslant z \leqslant 1$ for any r, $0<r<1$. □

Remark 14. *Let $K = \lim_{n\to\infty} S_k \mathcal{O}^+(\tilde{x})$ be a geometric limit set of $\mathcal{O}^+(\tilde{x})$. Then K is invariant under the flow $\tilde{\phi}$ and $\pi(K \cap \tilde{M}) \subset \omega(x)$.*

Proof. Let $w \in K \cap \tilde{M}$. Then there exists $\{\tau_k\}$ such that $\lim_{k\to\infty} S_k \tilde{x}\tau_k = w$. By Remark 13, $\tau_k \to \infty$ as $k \to \infty$. Hence given t, $\tau_k + t \to \infty$ and $wt \in K$. Clearly, $\pi(w) \in \omega(x)$. □

Proposition 15. $\pi(L \cap \tilde{M}) = \pi(SL \cap \tilde{M}) = \omega_{\mathrm{B}}(x)$ *for all* $S \in \Gamma$.

Proof. If $y \in \omega_{\mathrm{B}}(x)$, then there exists $s_k \nearrow \infty$ such that $xs_k \to y$ and $d_h(\tilde{x}s_k, A) \leqslant \beta$ for some $\beta > 0$ where A denotes the axis of T. Let $w \in A$ and let C_1 and C_2 be hyperbolic lines perpendicular to A at w and Tw. Then there exists n_k such that $T^{-n_k}\tilde{x}s_k$ is between C_1 and C_2. Without loss of generality, $n_k \nearrow \infty$ and $T^{-n_k}\tilde{x}s_k \to \tilde{y}$. ($d_h(T^{-n_k}\tilde{x}s_k, A) = d_h(\tilde{x}s_k, A) \leqslant \beta$ guarantees a convergent subsequence which must converge to a point in $\pi^{-1}(y)$ because $xs_k \to y$.) Since $T^{-n_k}\mathcal{O}^+(\tilde{x})$ converges to L, $\tilde{y} \in L$.

Now suppose $\tilde{y} \in L \cap \tilde{M}$. Then $d_h(\tilde{y}, A) < \beta$ for some $\beta > 0$. There exists $\tilde{x}s_n$ such that $T^{-n}\tilde{x}s_n \to \tilde{y}$. By Remarks 13 and 14 $s_n \to \infty$ and $\pi(\tilde{y}) \in \omega(x)$. Since $d_h(\tilde{x}s_n, A) = d_h(T^{-n}\tilde{x}s_n, A) < \beta$ for large n, it follows that $y \in \omega_{\mathrm{B}}(x)$. □

Theorem 16. *If $K = \lim_{n\to\infty} S_k \mathcal{O}^+(\tilde{x})$ is a geometric limit set, then either*

(1) $K = SL$ *for some* $S \in \Gamma$ *or*
(2) $\pi(K \cap \tilde{M}) \subset \omega_{\mathrm{R}}(x)$.

Proof. We know that $\pi(K \cap \tilde{M}) \subset \omega(x)$. Suppose $\pi(K \cap \tilde{M}) \not\subset \omega_{\mathrm{R}}(x)$, so there exists $\tilde{y} \in K$ such that $\pi(\tilde{y}) \notin \omega_{\mathrm{R}}(x)$, $\tilde{y} = \lim_{k\to\infty} S_k \tilde{x}\tau_k$. It follows that $d_h(\tilde{x}\tau_k, A) < \beta$ and $d_h(S_k^{-1}\tilde{y}, \tilde{x}\tau_k) \to 0$. Choose n_k so that $T^{n_k}\tilde{x}\tau_k$ is between C_1 and C_2 as before. Then $d_h(T^{-n_k}S_k^{-1}\tilde{y}, T^{-n_k}\tilde{x}\tau_k) \to 0$ and, without loss of generality, because $d_h(\tilde{x}\tau_k, A) < \beta$ for all k, $T^{-n_k}\tilde{x}\tau_k \to \tilde{y}_1$ such that $\pi(\tilde{y}_1) = \pi(\tilde{y})$. It follows that $S\tilde{y}_1 = \tilde{y}$ for some $S \in \Gamma$ and $T^{-n_k}S_k^{-1} = S^{-1}$ for large n because $T^{-n_k}S_k^{-1}\tilde{y} \to \tilde{y}_1$. Thus $S_k = ST^{-n_k}$ and $K = SL$.

If $K = SL$, then $\pi(K \cap \tilde{M}) = \omega_{\mathrm{B}}(x) \not\subset F$ and $\pi(K \cap \tilde{M}) \not\subset \omega_{\mathrm{R}}(x)$ because, by Theorem 10, $\omega_{\mathrm{R}}(x) \subset F$. □

Proposition 17. *If $y \in \omega_{\mathrm{R}}(x)$ and $y \notin \omega_{\mathrm{B}}(x)$, then there exists a geometric limit set K such that $y \in \pi(K \cap \tilde{M}) \subset \omega_{\mathrm{R}}(x)$.*

Proof. Let $y = \lim_{k\to\infty} x\tau_k$, $\tau_k \to \infty$, and $d_h(\tilde{x}\tau_k, A) \to \infty$. Choose n_k such that $T^{-n_k}\tilde{x}\tau_k$ is between C_1 and C_2, constructed as in the proof of Proposition 15 and $T^{-n_k}\tilde{x}\tau_k \to z$. Note $z \in L$ and $|z| = 1$.

Let $\tilde{y} \in \pi^{-1}(y)$ and choose $S_k \in \Gamma$ so that $d_h(S_k T^{-n_k} \tilde{x}\tau_k, \tilde{y})$ is minimal. By taking a subsequence we can assume $S_k T^{-n_k} \mathcal{O}^+(\tilde{x}) \to K$. Hence $\tilde{y}$ is in the geometric limit set and $\pi(K \cap \tilde{M}) \neq \omega_{\mathrm{B}}(x)$ because $y \notin \omega_{\mathrm{B}}(x)$. Therefore by Theorem 16 we have $\pi(K \cap \tilde{M}) \subset \omega_{\mathrm{R}}(x)$. $\square$

The following theorem summarizes the results of this section.

Theorem 18. *Suppose $\tilde{x}t \to a \in \mathbb{K}$ as $t \to \infty$ and a is the attractive fixed point of $T \in \Gamma$. If $\omega(x) \not\subset F$, then the sequence $\{T^{-n}\mathcal{O}^+(\tilde{x})\}_{n=0}^{\infty}$ converges in the Hausdorff metric on $\mathcal{U}$ to a set L with the following properties*:

(a) $TL = L$,
(b) *L is an invariant set of the flow $\tilde{\phi}$*,
(c) $\pi(L \cap \tilde{M}) = \omega_{\mathrm{B}}(x)$,
(d) *If K is a geometric limit set of $\mathcal{O}^+(\tilde{x})$ and $(K \cap \tilde{M}) \not\subset \tilde{F}$, then $K = SL$ for some $S \in \Gamma$.*

9. Asymptotic results for L and $\omega_{\mathbf{B}}(x)$

In this section, we describe the asymptotic behavior of points in both the geometric limit set L and $\omega_{\mathrm{B}}(x)$ under the usual assumptions. Specifically, $\tilde{x}$ is a lift of a point x in M such that $\tilde{x}t \to a$, where a is a rational point of $\mathbb{K}$, as $t \to \infty$, and T will denote a primitive element of Γ with attractive fixed point a.

Lemma 19. *Suppose $\omega(x) \not\subset F$, $J = \bigcup_{n \in \mathbb{Z}} T^n[\tilde{x}, \tilde{x}s]_{\tilde{\phi}} \cup (\tilde{x}, T^{-1}\tilde{x}s)_{\Sigma'}$ is a control curve in $\tilde{M}$, and Σ is another local section of $\tilde{\phi}$ such that $\Sigma \cap J = \emptyset$. Then the following hold*:

(a) *$T^n\mathcal{O}^+(\tilde{x}) \cap \Sigma$ contains at most one point, and*
(b) *$L \cap \Sigma$ contains at most one point.*

Proof. Either $\Sigma \subset J^-$ or $\Sigma \subset J^+$. If $\Sigma \subset J^-$, then both intersections are empty because $T^n\mathcal{O}^+(\tilde{x}) \subset \overline{J^+}$ and $L \subset J^+$. Hence we can assume $\Sigma \subset J^+$.

Part (a). We can assume $n = 0$ without loss of generality. If the conclusion is false, then there exist σ and τ, $s < \sigma < \tau$, such that $\tilde{x}\sigma$, $\tilde{x}\tau \in \Sigma$. We can also assume without loss of generality that $(\tilde{x}\sigma, \tilde{x}\tau)_{\tilde{\phi}} \cap \Sigma = \emptyset$. Let G be the Jordan curve

$$G = [\tilde{x}\sigma, \tilde{x}\tau]_{\tilde{\phi}} \cup (\tilde{x}\sigma, \tilde{x}\tau)_{\Sigma}.$$

Note that G and its interior are in J^+ and thus neither $\tilde{x}$ nor a is in the interior of G. But either $\mathcal{O}^+(\tilde{x}\tau)$ or $\mathcal{O}^-(\tilde{x}\sigma)$ is trapped in the interior of G, which is impossible.

Part (b). Now suppose y_1 and y_2 are distinct points in $\Sigma \cap L$. Then there exist sequences σ_n and τ_n going to infinity such that $T^{-n}\tilde{x}\sigma_n \to y_1$ and $T^{-n}\tilde{x}\tau_n \to y_2$. Then for large n both $T^{-n}\tilde{x}\sigma_n$ and $T^{-n}\tilde{x}\tau_n$ are in $\Sigma(-\varepsilon, \varepsilon) = \{zt\colon z \in \Sigma \text{ and } |t| < \varepsilon\}$ for small positive ε. Because $y_1 \neq y_2$ it is now apparent that, for large n, $\mathcal{O}^+(T^{-n}\tilde{x})$ crosses Σ twice, contradicting Part (a). □

It follows from the lemma that if $z \in L$, then $\alpha(z)$ and $\omega(z)$ can only contain fixed points including points in $\mathbb{K}$, which are fixed in the extended flow. Two natural questions arise that we will answer. First, since $\pi(z)$ is a typical point in $\omega_{\mathrm{B}}(x)$ when $z \in L$, are the limit points of $\pi(z)$ also only fixed points? Second, if $\omega_{\mathrm{B}}(x) \cap F = \emptyset$, does zt approach points of $\mathbb{K}$ as $t \to \pm\infty$, when $z \in L$?

The first question is answered in the main theorem of this section:

Theorem 20. *Let x be a point in M and let $y \in \omega(x)$. Suppose $\tilde{x}t \to a$ as $t \to \infty$ and a is a rational point of $\mathbb{K}$. If $v \in \omega(y) \cup \alpha(y)$ and $v \notin \mathcal{O}(y)$, then $v \in F$. More generally, y is either a periodic point or $\alpha(y) \cup \omega(y) \subset F$.*

Proof. Of course we need only consider $y \notin F$, so $\omega(x) \not\subset F$. Thus $y \in \omega_{\mathrm{B}}(x)$ and there exists $\tilde{y} \in L \cap \pi^{-1}(y)$. There exists $\tau_k \to \pm\infty$ such that $y\tau_k \to v$.

First, if $\tilde{y}\tau_k$ has a bounded subsequence then there exists $\tilde{v} \in \omega(\tilde{y}) \cup \alpha(\tilde{y})$. By the previous remarks $\tilde{v} \in \tilde{F}$ and $\pi(\tilde{v}) = v \in F$.

Next we consider the case where $\tilde{y}\tau_k$ does not have a bounded subsequence. Suppose $v \notin F$ and let Σ be a local section at v, $\tilde{\Sigma}$ a lift of Σ, and $\tilde{v} \in \tilde{\Sigma} \cap \pi^{-1}(v)$ to establish a reference point in $\tilde{M}$. Choose Σ small enough so that $\Sigma \cap \pi(J_1) = \emptyset$, for some J_1 constructed at $\tilde{y}$ using $\mathcal{O}^+(\tilde{x})$. Then $S\tilde{\Sigma} \cap J_1 = \emptyset$ for all $S \in \Gamma$ and Lemma 19 can be applied. There exists a sequence S_k of distinct elements of Γ such that $\tilde{y}\tau_k \in S_k\tilde{\Sigma}$.

Because $\tilde{y} \in L$, $\tilde{y} = \lim_{m\to\infty} T^{-m}\tilde{x}\sigma_m$ for some $\sigma_m \to \infty$. Hence there exists a subsequence of integers m_k (by continuity in initial conditions) such that $T^{-m_k}\tilde{x}(\sigma_{m_k} + \tau_k) \in S_k\tilde{\Sigma}$ and $\sigma_{m_k} + \tau_k \to \infty$. Clearly $x(\sigma_{m_k} + \tau_k) \to v$ and since $v \notin F$, $v \in \omega_{\mathrm{B}}(x)$ and $d_h(T^{-m_k}\tilde{x}(\sigma_{m_k} + \tau_k), A) < \beta$, where, as usual, A is the axis of T.

Construct C_1 and C_2 as in the proof of Proposition 15 and choose n_k such that $T^{n_k - m_k}\tilde{x}(\sigma_{m_k} + \tau_k) \in T^{n_k}S_k\tilde{\Sigma}$ lies between C_1 and C_2. Because these points are also within β of A, we can assume this sequence converges to $\tilde{v}_1 \in S\tilde{\Sigma}$, for some $S \in \Gamma$. Hence, for large k,

$$T^{n_k}S_k\tilde{\Sigma} = S\tilde{\Sigma}$$

and

$$T^{n_k}\tilde{y}\tau_k \in S\tilde{\Sigma}.$$

This contradicts Lemma 19 because $TL = L$ and $v \notin \mathcal{O}(y)$, proving the first part.

Finally, $\omega(y)$ must contain a minimal set X. If X is not a fixed point, then no point in X can be fixed and $X \subset \mathcal{O}(y)$ by the first part of the theorem. It follows that

$X = \mathcal{O}(y)$ is a periodic orbit. The same argument applies to $\alpha(y)$ to complete the proof. $\square$

Suppose y is a periodic point in $\omega(x)$. Then from Lemma 19 we know $\tilde{y}$ is not periodic. So the orbit of y is not null homotopic and $\tilde{y}t \to w^{\pm} \in \mathbb{K}$ as $t \to \pm\infty$. If $T(w^{\pm}) \neq w^{\pm}$, then $y \in \omega_{\mathrm{R}}(x)$ which is impossible by Theorem 10. Hence $\tilde{y}t$ converges to the fixed points a and b of T as $t \to \pm\infty$. The technique used to prove Lemma 1 applies to prove that $T\mathcal{O}(\tilde{y}) = \mathcal{O}(\tilde{y})$. From Lemma 11 and Theorem 12 we know that L can be obtained as a limit of control curves J_n constructed using a local section Σ at y. Using the notation for and the structure of these control curves as described preceding Lemma 11, it follows that the periodicity of y forces $t_{n+1} - t_n$ to converge to the period of y. It subsequently follows that given $\varepsilon > 0$, J_n is within ε of $\mathcal{O}^+(\tilde{y})$ for large n. Hence $L \subset \mathcal{O}(\tilde{y})$ and $L = \mathcal{O}(\tilde{y})$ because L is invariant. It follows from more general comments in the next paragraph that $\pi(L) = \omega(x)$ when y is periodic in $\omega(x)$.

Clearly, $\{a, b\}$, the fixed points of T, are always contained in L. It is easy to verify that $L \cap \mathbb{K} = \{a, b\}$ if and only if $\omega_{\mathrm{R}}(x) = \emptyset$ if and only if $\mathcal{O}^+(\tilde{x})$ is a bounded hyperbolic distance from A. Orbits of this type are studied in [15,16]. In particular, the significance of $\omega(x)$ being locally connected is exploited.

Returning to the more general situation, suppose $\tilde{y} \in L$ and $|\tilde{y}t| \to 1$ as $t \to \infty$. If $\tilde{y}t$ does not converge to a point in $\mathbb{K}$, then $\omega(\tilde{y})$ in the extended flow on $\mathbb{K}$ contains an open interval $I \subset \mathbb{K}$. Since $\tilde{y}$ is clearly not fixed, we can construct a control curve J using a local section at $\tilde{y}$ and $\mathcal{O}^+(\tilde{x})$. Then there exists a copy SJ, $S \in \Gamma$, such that both endpoints of SJ are in I. Since $\mathcal{O}(\tilde{y})$ can cross SJ at most once, all of I cannot be in $\omega(\tilde{y})$. Therefore, when $|\tilde{y}t| \to 1$ as $t \to \infty$, $\tilde{y}t$ must converge to some w in $\mathbb{K}$ as $t \to \infty$. Similar remarks apply when $|\tilde{y}t| \to 1$ as $t \to -\infty$.

Now consider any $\tilde{y} \in L$ such that $\tilde{y}t \to w \in \mathbb{K}$ as $t \to \infty$. If $Tw \neq w$, then it is easy to see that $\omega(y) \subset \omega_{\mathrm{R}}(x) \subset F$ where $y = \pi(\tilde{y})$. If, on the other hand, $Tw = w$, then by Theorem 20 either y is periodic, in which case $L = \mathcal{O}(\tilde{y})$ and $\mathcal{O}(y) = \omega(x)$, or $\alpha(y) \cup \omega(y) \subset F$.

It is possible to construct examples such that for $\tilde{y} \in L$, $\tilde{y}t \to a$ as $t \to \infty$ and $\omega(y)$ is a simple closed curve of fixed points homotopic to $\pi(A)$, where as usual A is the axis of T. (Attach tori to teach end of the cylinder flow on page 239 of [12].) In this example $\omega_{\mathrm{R}}(x) = \emptyset$ and $\omega(x)$ is not locally connected at each of the fixed points.

We will conclude by applying our results to two different restrictions on the fixed points in $\omega(x)$, one new and one classical.

First, suppose that $\omega_{\mathrm{B}}(x) \cap F = \emptyset$. It follows from Lemma 19 that $|\tilde{y}t| \to 1$ as $t \to \pm\infty$ and then $\tilde{y}t \to w^{\pm} \in \mathbb{K}$ as $t \to \pm\infty$. It is not difficult to verify that either both w^+ and w^- are fixed by T or neither is fixed by T (we are not assuming that $w^+ \neq w^-$) and that the sequence of orbits $T^{-n}\mathcal{O}^+(\tilde{x})$ all lie on the same side of $\mathcal{O}(\tilde{y})$, $\tilde{y} \in L$. It follows that L and hence $\omega_{\mathrm{B}}(x)$ contain at most a countable set of orbits.

In the example [12], L contains a countable number of orbits with rational limit points in $\mathbb{K}$ and $\omega_{\mathrm{B}}(x)$ consists of exactly one orbit. Suppose there are no fixed points in $\omega_{\mathrm{B}}(x)$ and let $y \in L$. Then yt tends to a point $b \in \mathbb{K}$ as $t \to \infty$. If b is rational and the attractive fixed point of $S \in \Gamma$, then $S^{-n}\mathcal{O}(\tilde{y})$ converges to K in the Hausdorff metric, by the same arguments in Section 7 used to show that the J_n's converge. Clearly, K is contained in the fixed points and $SK = K$. Although K need not be a simple closed curve as in the example, it is a more general analogous phenomenon.

The results in the previous section show that x approaches the orbits in $\pi(L \cap \tilde{M})$ very similarly to the way an orbit wraps around a limit cycle. In fact, the theory for $\omega_{\mathrm{R}}(x)$, L, and $\omega_{\mathrm{B}}(x)$ incorporates the basic limit cycle facts.

Second, suppose $\omega(x) \cap F = \emptyset$, then $\omega_{\mathrm{R}}(x) = \emptyset$ by Theorem 10 and $\omega(x) = \omega_{\mathrm{B}}(x)$. Let $y \in \omega_{\mathrm{B}}(x)$. By Theorem 20, y is periodic and in the subsequent discussion we showed that $L = \mathcal{O}(\tilde{y})$ for some $\tilde{y} \in \pi^{-1}(y)$ and $\omega(x) = \omega_{\mathrm{B}}(x) = \pi(L \cap \tilde{M}) = \pi(\mathcal{O}(\tilde{y})) = \mathcal{O}(y)$. Thus $\omega(x)$ is a limit cycle when $\omega(x) \cap F = \emptyset$ and $\tilde{x}t \to a$, a rational, as $t \to \infty$.

References

[1] D.V. Anosov, On the behavior in the Euclidean or Lobachevsky plane of trajectories that cover trajectories of flows on closed surfaces. I, Math. USSR Izv. 30 (1) (1988) 15–38.
[2] D.V. Anosov, On the behavior in the Euclidean or Lobachevsky plane of trajectories that cover trajectories of flows on closed surfaces. II, Math. USSR Izv. 32 (3) (1989) 449–474.
[3] D.V. Anosov, On infinite curves on the torus and closed surfaces with negative Euler characteristic, Proc. Steklov Inst. Math. 1990, no. 2, 205–219.
[4] D.V. Anosov, Flows on closed surfaces and behavior of trajectories lifted to the universal covering plane, J. Dyn. Control Syst. 1 (1) (1995) 125–138.
[5] S.H. Aranson, The absence of nonclosed Poisson-stable semitrajectories and trajectories doubly asymptotic to a double limit cycle for dynamical systems of the first degree of structural instability on orientable two-dimensional manifolds, Math. USSR-Sb. 5 (2) (1968) 205–219.
[6] S.H. Aranson, V.Z. Grines, On some invariants of dynamical systems on two-dimensional manifolds (necessary and sufficient conditions for the topological equivalence of transitive dynamical systems), Math. USSR-Sb. 19 (3) (1973) 365–393.
[7] S.H. Aranson, V.Z. Grines, E. Zhuzhoma, On Anosov–Weil problem, Preprint, 1996.
[8] S. Katok, Fuchsian Groups, The University of Chicago Press, Chicago, 1992.
[9] E. Lima, Common singularities of commuting vector fields on 2-manifolds, Comment. Math. Helv. 39 (1964) 97–110.
[10] N.G. Markley, The structure of flows on two-dimensional manifolds, Ph.D. Thesis, Yale University, 1966.
[11] N.G. Markley, Invariant simple closed curves on the torus, Michigan Math. J. 25 (1) (1978) 45–52.
[12] N.G. Markley, M.H. Vanderschoot, An exotic flow on a compact surface, Colloq. Math. 84/85 (2000) 235–243.
[13] V.V. Nemytskii, V.V. Stepanov, Qualitative Theory of Ordinary Differential Equations, Princeton University Press, Princeton, NJ, 1960.
[14] I. Nikolaev, E. Zhuzhoma, Flows on 2-Dimensional Manifolds: An Overview, Springer, Berlin, 1999.
[15] M.H. Vanderschoot, Limit sets for continuous flows on surfaces, Ph.D. Thesis, University of Maryland, College Park, 1999.
[16] M.H. Vanderschoot, Local connectivity of limit sets, Rocky Mountain J. Math., to appear.
[17] H. Whitney, Regular families of curves, Proc. Nat. Acad. Sci. 18 (1932) 275–278, 340–342.