On the Spectrum of Elliptic Operators with Respect to Indefinite Weights

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ABSTRACT

Recent results on linear elliptic eigenvalue problems with respect to indefinite weight functions are recalled, and some applications are given.

INTRODUCTION

In this note we give a survey of recent results concerning both the linear eigenvalue problem

\[ \mathcal{L}u = \lambda mu \quad \text{in } \Omega, \quad \mathcal{B}u = 0 \quad \text{on } \partial \Omega \quad (1) \]

and the closely connected problem

\[ (\mathcal{L} - \lambda m)u = \mu u \quad \text{in } \Omega, \quad \mathcal{B}u = 0 \quad \text{on } \partial \Omega \quad (2_{\lambda}) \]

(\lambda \text{ fixed}), and give some immediate applications to semilinear eigenvalue and boundary-value problems. Here \( \Omega \subset \mathbb{R}^N \) \((N \geq 1)\) is a bounded domain with smooth boundary \( \partial \Omega \), \( \mathcal{L} = \mathcal{L}(x, \partial / \partial x) \) a uniformly elliptic linear differential expression of second order of the form

\[ \mathcal{L}u = - \sum_{j,k=1}^{N} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{N} a_{j} \frac{\partial u}{\partial x_j} + a_{\partial} u \]


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with real-valued coefficient functions $a_{jk} = a_{kj}$, $a_j, a_0 \geq 0$ belonging to $C^\theta(\Omega) \ (0 < \theta \leq 1)$, and $\mathcal{B} = \mathcal{B}(x, \partial / \partial x)$ a linear boundary operator implying either Dirichlet boundary conditions ($\mathcal{B}u = u$), or Neumann or regular oblique derivative boundary conditions ($\mathcal{B}u = \partial u / \partial \beta + bu$, where $\beta$ is a smooth, outward-pointing, and nowhere tangent vector field on $\partial \Omega$ and $b \geq 0$ a smooth function). Further, $m \in C(\Omega)$ is a given real-valued weight function, $m \neq 0$. The eigenvalue parameters in (1) and (2), are denoted by $\lambda$ and $\mu$, respectively.

We assume throughout that for $(\mathcal{L}, \mathcal{B})$ the maximum principle holds: if $u \in C^2(\Omega) \cap C^1(\Omega)$ satisfies $\mathcal{L}u \geq 0$ in $\Omega$, $\mathcal{B}u \geq 0$ on $\partial \Omega$, then $u \geq 0$, and $u > 0$ in $\Omega$ as well as $\partial u / \partial \beta(x) < 0$ at points $x \in \partial \Omega$ where $u(x) = 0$, unless $\mathcal{B}u = 0$.

I. MOTIVATION

In many applications, e.g. to biomathematical problems and models in reactor theory, one searches for positive equilibrium solutions to semilinear problems of the form

$$\frac{\partial w}{\partial t} + \mathcal{L}(x, \frac{\partial}{\partial x})w = f(x, w) \quad \text{in } \Omega \times \mathbb{R}^+$$

$$\mathcal{B}(x, \frac{\partial}{\partial x})w = 0 \quad \text{on } \partial \Omega \times \mathbb{R}^+$$

$$w|_{t=0} = w_0 \quad \text{on } \Omega,$$

i.e. for positive solutions of

$$\mathcal{L}w = f(x, w) \quad \text{in } \Omega, \quad \mathcal{B}w = 0 \quad \text{on } \partial \Omega. \quad (1.2)$$

Further, the stability of these solutions is of interest. For the proof of existence two tools have turned out to be very fruitful: (i) the bifurcation theory and (ii) the method of sub- and supersolutions. In order to apply (i) one has to consider the linearization of (1.2) at a (trivial) solution $w$:

$$\mathcal{L}u = m u \quad \text{in } \Omega, \quad \mathcal{B}u = 0 \quad \text{on } \partial \Omega \quad (1.3)$$

$$[m(x) := f_u(x, w(x))]; \text{ the sign of the "principal eigenvalue" } \mu_1 \text{ of the}$$
associated eigenvalue problem

\[(\mathcal{L} - m)u = \mu u \quad \text{in } \Omega, \quad \mathcal{B}u = 0 \quad \text{on } \partial \Omega \quad (1.4)\]

is crucial for the determination of stability of \(w\). For the construction of positive sub- and supersolutions "principal eigenfunctions" are often used.

II. PRINCIPAL EIGENVALUES

It is convenient to work in the real ordered Banach space \(E := C(\Omega)\). Let \(L : E \supset D(L) \to E\) be the realization of \((\mathcal{L}, \mathcal{B})\) in \(E\), i.e. the maximal linear operator induced in \(E\) by \(\mathcal{L}\) and the boundary conditions. Then \(L\) is a closed operator that is invertible (by the maximum principle) and has a compact inverse (by embedding theorems). Further, let \(M \in \mathcal{L}(E)\) denote the multiplication operator by the function \(m\). The problems (1) and (2) take the form

\[Lu = \lambda Mu \quad (\Leftrightarrow \quad \frac{1}{\lambda} u = L^{-1}Mu) \quad (2.1)\]

and

\[(L - \lambda M)u = \mu u \quad (2.2)\]

in \(E\).

**Question.** Does (1) admit an eigenvalue \(\lambda_1 = \lambda_1(m) > 0\) having a positive eigenfunction ("principal eigenvalue")?

**Theorem 1** [17]. There exists a principal eigenvalue \(\lambda_1 > 0\) of (2.1) if and only if \(m \not< 0\). If \(m\) is positive somewhere in \(\Omega\), then \(\lambda_1\) is uniquely defined and

1. \(1/\lambda_1\) is an algebraically simple eigenvalue of \(L^{-1}M\),
2. if \(\lambda \in \mathbb{C}\) is an eigenvalue (of the complexified problem) with \(\text{Re } \lambda > 0\), then \(\lambda_1 > 0\) exists and \(\text{Re } \lambda \geq \lambda_1\).

Similarly, if \(m(x) < 0\) at some \(x \in \Omega\), there exists a (unique) negative principal eigenvalue \(\lambda_{-1} = \lambda_{-1}(m)\) [just look at the equation \(Lu = (\lambda)(-M)u\)]. The couple \((\mathcal{L} - \lambda m, \mathcal{B})\) satisfies the maximum principle if \(\lambda_{-1}(m) < \lambda < \lambda_1(m)\) [set \(\lambda_{-1}(m) = -\infty\) in case \(m \geq 0\)].
**Sketch of the proof.** That $m \neq 0$ is a necessary condition for the existence of a positive principal eigenvalue follows immediately by the maximum principle. A proof of the sufficiency of this condition can be led by looking at the associated problem $(2.2, \lambda)$ (this is not the original proof of [17]; cf. [20, 18]). For fixed $\lambda \in \mathbb{R}$ and sufficiently large $c > 0$, $(L - \lambda M + c)^{-1}$ is a positive compact operator in $E$ having positive spectral radius $\gamma_{\lambda, c}$. By the Krein-Rutman theorem [23] $\gamma_{\lambda, c}$ is the unique eigenvalue of $(L - \lambda M + c)^{-1}$ with a positive eigenfunction. We conclude that for each $\lambda \in \mathbb{R}$ there is a unique principal eigenvalue $\mu_1(\lambda) \in \mathbb{R}$ and a (up to multiplicative constants) unique positive eigenfunction $u_1(\lambda)$ of $(2.2, \lambda)$:

$$(L - \lambda M)u_1(\lambda) = \mu_1(\lambda)u_1(\lambda)$$

[In fact $\mu_1(\lambda)$ is the spectral bound of $L - \lambda M$. It is clear that $\lambda$ is a principal eigenvalue of $(2.1)$ iff $\mu_1(\lambda) = 0$. We list the properties of the function $\mu_1$:

1. $\mu_1(0) > 0$.
2. As a consequence of the implicit-function theorem, $\mu_1$ is an analytic function of $\lambda$, and also $u_1(\lambda)$ can be chosen to depend analytically on $\lambda$ (cf. [5]).
3. $\mu_1$ is a concave function of $\lambda$ (cf. [20]; a more direct approach is worked out in [4] for the periodic-parabolic case and goes back to an observation of H. Berestycki and P. L. Lions).
4. $\mu_1(\lambda) \to -\infty (\lambda \to +\infty)$ if $m(x) > 0$ at some $x \in \Omega$.

Thus a principal eigenvalue $\lambda_1 > 0$ exists if $m \neq 0$, and is unique. If $\lambda_1 > 0$ exists, we have $\mu_1'(\lambda_1) \neq 0$, which guarantees that $\lambda_1$ is an "M-simple eigenvalue of $L"$ [i.e., $\dim N(L - \lambda_1 M) = \text{codim } R(L - \lambda_1 M) = 1$ and $Mu_1 \in R(L - \lambda_1 M)$, where $u_1 = u_1(\lambda_1)$ spans $N(L - \lambda_1 M)$]. Therefore, since $L$ is invertible, $1/\lambda_1$ is a simple eigenvalue of $L^{-1}M$.

The assertion of Theorem 1(ii) and its sharpening [11, 12] "$\lambda_1$ is the only eigenvalue $\lambda \in \mathbb{C}$ of $(2.1)$ with $\text{Re } \lambda = \lambda_1$" are consequences of a variant of the so called "Kato inequality" $-\Delta |u| \leq \text{Re } [\text{sgn}(u)(-\Delta u)]$, where $\text{sgn}(u) = \bar{u}/|u|$ if $u \neq 0$, = 0 if $u = 0$ (cf. [19, 17, 12]).

### III. APPLICATIONS TO SEMILINEAR PROBLEMS

#### (a) Bifurcation Problems

Consider the nonlinear eigenvalue problem

$$\mathcal{L}w = \lambda f(x, w) \quad \text{in } \Omega, \quad \mathcal{B}w = 0 \quad \text{on } \partial \Omega, \quad (3.1)$$
and assume \( f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) is a smooth function such that \( f(\cdot, 0) = 0 \). Then (3.1) admits the line \( \mathbb{R} \times \{0\} \subseteq \mathbb{R} \times E \) of trivial solutions \((\lambda, 0)\). Set \( m := f_w(\cdot, 0) \). A necessary condition for \((\lambda, 0)\) to be a bifurcation point for positive solutions \((\lambda > 0, w > 0)\) of (3.1), from the line of trivial solutions, is that \( \lambda \) is the positive principal eigenvalue of the problem (1). An immediate consequence of Rabinowitz’s global bifurcation theorem [25] and Theorem 1 is

**Theorem 2 [17].** There exists a point \((\lambda, 0)\) of (global) bifurcation of positive solutions of (3.1) from the line of trivial solutions if and only if \( m \not< 0 \) in \( \Omega \). If \( m(x) > 0 \) at some \( x \in \Omega \), then \((\lambda, m(x), 0)\) is the only such bifurcation point.

(b) **Linearized Stability**

Let \( w \) be an equilibrium solution of the parabolic boundary-value problem (1.1), i.e., a solution of (1.2). By means of sub- and supersolution techniques one proves the following principle of linearized stability [set \( m := f_w(\cdot, w) \)]: \( w \) is locally asymptotically exponentially stable if the principal eigenvalue \( \mu_1 \) of (1.4) is positive, and unstable if \( \mu_1 \) is negative. Thus, in the situation of Theorem 2, the trivial solution \( u = 0 \) is for example stable for \( \lambda_{-1}(m) < \lambda < \lambda_1(m) \) and unstable for \( \lambda < \lambda_{-1}(m) \) and \( \lambda > \lambda_1(m) \). Moreover, the stability of the bifurcating positive solutions can often be determined; cf. [13], where it is shown that the “principle of exchange of stability” holds under very general assumptions.

(c) **Sub- and Supersolutions**

Let \( u \leq \overline{w} \) be a sub- and a supersolution of the problem (1.2), respectively. It is well known (e.g. [3, 26]) that there exists at least one solution \( w \) of (1.2) with \( w \leq u \leq \overline{w} \) (in fact there exist a minimal and a maximal solution in \([u, \overline{w}]\) provided \( f \) is sufficiently smooth). Positive sub- and supersolutions are often constructed with help of principal eigenfunctions. We have for example (see [14] for proofs in the periodic-parabolic case):

**Proposition 3.** Suppose for some \( s_0 > 0 \) that \( f(x, s) \geq m_0(x)s \) for all \( 0 \leq s \leq s_0 \) and all \( x \in \Omega \), where \( m_0 \in C(\overline{\Omega}) \) and \( m_0 \not< 0 \) in \( \Omega \). If \( \lambda_1(m_0) \leq 1 \), then (1.2) admits small positive subsolutions.

**Proposition 4.** Suppose \( f(x, s) \leq m_\infty(x)s + c \) for all \( s > 0 \) and all \( x \in \overline{\Omega} \), where \( m_\infty \in C(\overline{\Omega}) \), \( m_\infty \not< 0 \) in \( \Omega \), and \( c > 0 \). If \( \lambda_1(m_\infty) > 1 \), then (1.2) admits large positive supersolutions.
IV. THE WHOLE SPECTRUM

We return to the linear eigenvalue problem (1) and assume now $a_{jk} \in C^1(\Omega)$ for all $j, k$. On the other hand it suffices if the real-valued functions $a_j, a_0 \geq 0$ and $m$ belong only to $L^\infty(\Omega)$. We introduce the differential expression $\mathcal{L}_0$:

$$\mathcal{L}_0 u = -\sum_{j,k=1}^N \frac{\partial}{\partial x_j} \left( a_{jk} \frac{\partial u}{\partial x_k} \right) + a_0 u$$

(4.1)

and assume that $(\mathcal{L}_0, \mathcal{B})$ determines a self-adjoint boundary-value problem.

We choose to work in the complex Hilbert space $\mathcal{H} = L^2(\Omega)$. Let $L$ and $M$ be the differential and the multiplication operator in $\mathcal{H}$ induced by $(\mathcal{L}, \mathcal{B})$ and $m$, respectively. Note that

$$Lu = \lambda Mu \quad \text{in } \mathcal{H}$$

(4.2)

is not a variational eigenvalue problem ($L$ is not necessarily self-adjoint).

**Theorem 5 [15].**

(i) The eigenvalue problem (4.2) has a discrete spectrum, and for arbitrary $0 < \varepsilon < \pi/2$ all the eigenvalues $\lambda$, except possibly a finite number of them, lie in the two sectors

$$G_\varepsilon^+ = \{ \zeta \in \mathbb{C} : -\varepsilon < \arg \zeta < \varepsilon \}, \quad G_\varepsilon^- = \{ \zeta \in \mathbb{C} : \pi - \varepsilon < \arg \zeta < \pi + \varepsilon \}.$$

(ii) The system of generalized eigenvectors of $L^{-1}M$ is complete in $R(L^{-1}M)$.

(iii) (4.2) has infinitely many eigenvalues $\lambda$ in $G_\varepsilon^+$ if and only if the set $\{ x \in \Omega : m(x) > 0 \}$ has positive measure; a similar statement holds for the eigenvalues $\lambda \in G_\varepsilon^-$.  

Note that (ii) is the best we can hope for; thus if $M$ is not injective (i.e., if $m$ vanishes on a set of positive measure), the system of generalized eigenfunctions is not complete in $L^2(\Omega)$.

Theorem 5(i)–(ii) is well known if $m = 1$ and goes back to Keldyš [21]. Also for $m = 1$, Agmon [1] has obtained this result for a wider class of boundary conditions. In the presence of a possibly indefinite weight $m$, Agmon's approach, however, does not seem to work. We therefore go back
basically to the ideas of Keldyš and consider (4.2) as a (weak) perturbation of a self-adjoint problem. We first generalize an abstract perturbation result of Keldyš.

Thus let $\mathcal{H}$ for the moment be an (abstract) separable Hilbert space over $\mathbb{C}$. Following von Neumann and Schatten [7, Section XI.9], we say that the compact linear operator $A$ in $\mathcal{H}$ belongs to the two-sided ideal $C_p \subset \mathcal{L}(\mathcal{H})$ ($0 < p \leq \infty$) if the eigenvalues of $(A^*A)^{1/2}$, arranged in decreasing order and repeated according to multiplicity, form an $l^p$-sequence.

**Theorem 6** [15]. Let $A = H + S$, where $H \in \mathcal{L}(\mathcal{H})$ is compact and self-adjoint and belongs to $C_p$ for some $p < \infty$, while $S \in \mathcal{L}(\mathcal{H})$ is compact and $I + S$ invertible. Let $\mathcal{H}$ be orthogonally decomposed as $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, with $\mathcal{H}_1 = R(H)$, $\mathcal{H}_2 = N(H)$, and suppose

$$\mathcal{H}_1 \cap (I + S)^{-1} \mathcal{H}_2 = \{0\}. \quad (*)$$

Then

(i) For arbitrary $0 < \varepsilon < \pi/2$ all the characteristic values of $A$, except possibly a finite number of them, lie in the two sectors $G^- \cup G^+$.

(ii) The system of generalized eigenvectors of $A$ (to characteristic values) is complete in $\mathcal{H}_1$.

(iii) $A$ has infinitely many characteristic values in $G^+$ if and only if $H$ has infinitely many positive characteristic values.

($\lambda \in \mathbb{C}$ is a characteristic value of $A$ if there is an $x \in \mathcal{H}$, $x \neq 0$, such that $x = \lambda Ax$; of course $\lambda \neq 0$ and $x$ is an eigenfunction of $A$ corresponding to the eigenvalue $\lambda^{-1}$.)

Keldyš (for a proof see [10, Theorem V.8.1]) considered the case where $H$ is injective ($\mathcal{H}_2 = \{0\}$) and obtained Theorem 6(i), (ii) and the "only if" part of (iii); clearly the condition (*) is vacuous in that case. The proof is based on resolvent estimates and the Phragmen-Lindelöf principle.

We indicate how Theorem 5 follows from the abstract Theorem 6. Let $L$ be decomposed as $L = L_0 - L_1$, where $L_0$ is the self-adjoint positive operator induced in $\mathcal{H} = L^2(\Omega)$ by $(\mathcal{L}_0, \mathcal{B})$, and $L_1$ is a relatively compact perturbation. The eigenvalue problem (4.2) in $\mathcal{H}$ can then be written as

$$(L_0 - L_1)u = \lambda Mu, \quad (4.3)$$

which is equivalent to

$$L_0^{1/2}((I - L_0^{-1/2}L_1L_0^{-1/2})L_0^{1/2}u = \lambda L_0^{1/2}(L_0^{-1/2}ML_0^{-1/2})L_0^{1/2}u. \quad (4.4)$$
Cancel $L_0^{1/2}$ on both sides of (4.4), set $L_0^{1/2}u := v$, and introduce the compact operators $L_0^{-1/2}L_1L_0^{-1/2} =: T$, $L_0^{-1/2}ML_0^{-1/2} =: H$. Note that $I - T$ is invertible and $H$ is self-adjoint. Define the compact operator $S$ by $I + S = (I - T)^{-1}$, and set $(I - T)v =: w$. Then (4.4) is equivalent to

$$w = \lambda H(I + S)w. \quad (4.5)$$

The eigenvalues of (4.2) are thus the characteristic values of $A = H(I + S)$. By a result of Agmon [1], $L_0^{-1} \in C_p$ for $p > N/2$; hence $L_0^{-1/2} \in C_{2p}$, and consequently, by [7, Lemma XI.9.9, p. 1093] or [10, p. 92], $H \in C_p$. One further shows that condition (*) of Theorem 6 holds. Thus Theorem 6 applies. In order to prove Theorem 5(iii), one notes that the characteristic values of $H$ coincide with the eigenvalues $\gamma$ of the variational problem

$$L_0u = \gamma Mu \quad (4.6)$$

in $\mathcal{H}$; by the results of [24, 6], (4.6) has infinitely many positive eigenvalues if and only if $m > 0$ on a set of positive measure.

V. ASYMPTOTIC DISTRIBUTION OF EIGENVALUES

In addition to the hypotheses made in Section IV, we assume $m \in C(\overline{\Omega})$ and $m \geq 0$, i.e., consider the semidefinite eigenvalue problem (1). By Theorem 5 the eigenvalues condense along the positive axis; they can thus be ordered by their real part. For $t > 0$ let $n(t)$ denote the number of eigenvalues $\lambda$ of (1) with $\text{Re} \lambda \leq t$.

THEOREM 7 [16]. $n(t) \sim ct^{N/2}$ as $t \to + \infty$, where

$$c = \int_{\Omega} m(x)^{N/2} \mu_{\mathcal{L}_0}(x) \, dx \quad \text{and} \quad \mu_{\mathcal{L}_0}(x) - (2\pi)^{-N} \int_{\{\xi \in \mathbb{R}^N : \Sigma a_jk(x)\xi_j\xi_k < 1\}} \, d\xi.$$ 

Theorem 7 is again well known in the standard situation $m = 1$ (e.g. [2]). It says that the asymptotic distribution of eigenvalues is the same for the problem (4.2) and the variational problem (4.6). It is proved by a perturbation argument similar to that employed in the previous section: we apply the results of e.g. [8] to (4.6) together with the subsequent generalization of another abstract result of Keldyš (cf. [10, Theorem V.11.1]):
Theorem 8 [16]. Consider the situation of Theorem 6, and assume in addition that $H$ is nonnegative. Suppose there exists a nondecreasing function $\phi$ on $\mathbb{R}^+$ with $\phi(t) \to +\infty$ as $t \to +\infty$, satisfying

$$\frac{\phi(s)}{\phi(t)} \leq \left(\frac{s}{t}\right)^\gamma$$

for all sufficiently large $t < s$ and some constant $0 < \gamma < p$, such that $\lim_{t \to +\infty} n(t, H)/\phi(t) = 1$. Then also

$$\lim_{t \to +\infty} \frac{n(t, A)}{n(t, H)} = 1.$$

Here $n(t, A)$ and $n(t, H)$ denote the distribution functions of the characteristic values of $A$ and $H$, respectively [i.e., $n(t, A)$ = number of characteristic values $\lambda$ of $A$ with $\text{Re}\lambda < t$]. The condition (5.1) is trivially satisfied for the function $\phi(t) = ct^{\gamma}$ ($c > 0$), which usually determines the asymptotic behaviour. We remark that it is not clear whether Theorems 7 and 8 can be obtained from the known results by a limiting procedure, looking first at definite problems, since in the nonvariational case no monotonicity arguments for the eigenvalues are available (compare with [8, proof of Theorem 3.1]).

VI. OPEN PROBLEMS

(a) In order to prove the existence of the positive principal eigenvalue $\lambda_1(m)$ of (1), we have assumed in Section II that $m$ is continuous and positive somewhere; hence $m$ is positive on a set with nonempty interior. The existence of $\lambda_1(m)$ follows also (by Theorem 5) when $m$ is only in $L^\infty(\Omega)$ and positive on a set of positive measure, but at the cost of some extra smoothness of the coefficient functions: $a_{jk} \in C^1(\overline{\Omega})$ (and only for suitable boundary operators $\mathcal{B}$). It seems nobody has been able so far to prove the existence of $\lambda_1(m)$ if only $a_{jk} \in C^0(\overline{\Omega})$ and $m \in L^\infty(\Omega)$ is positive on a set of positive measure.

(b) It would be of interest to obtain results analogous to Theorem 5 for more general boundary conditions (as they have been derived by Agmon [1] for $m = 1$).

(c) In [29, 18] the problem (1) is considered with $a_0 = 0$ and $\mathcal{B} = \partial/\partial\beta$; this case is of particular importance in some biomathematical applications.
(e.g. [9, 27, 28]). Here 0 is a principal eigenvalue of (1) (with eigenfunction \( u_1 = \mathbf{1} \)), and \( (L, \mathcal{D}) \) no longer satisfies the maximum principle. If \( m \) changes sign in \( \Omega \), then (1) may have another (nontrivial) principal eigenvalue. To this problem, Theorem 6 is not applicable; another way has to be found to investigate the whole spectrum.

(d) The results of [8] on the asymptotic distribution of eigenvalues of the variational problems (4.6) hold for weight functions \( m \) that may change sign. We have not, however, been able to prove that the spectrum of (4.2) and (4.6) have the same asymptotic behavior also in case \( m \) is indefinite rather than semidefinite. The difficulty lies in establishing a result of Tauberian type due to Korenbljum [22] for an integral kernel that becomes negative somewhere in \( \mathbb{R}^+ \).

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