Lacunary tangential approximation

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ABSTRACT

Our aim is to give lacunary versions with upper density equal to the value one of Arakelian’s approximation theorem for special geometries of the domain and the closed set.

By use of the theorems, we are able to construct functions which have so-called maximal cluster sets along arbitrary curves and which have a lacunary structure in addition. Upper density equal to the value one will turn out to be best possible for this result.

1. INTRODUCTION

In this note we abbreviate the one-point-compactification of an arbitrary domain $\Omega$ to $\Omega^* := \Omega \cup \{\infty\}$. A set $E \subset \Omega^*$ is open in $\Omega^*$ if and only if $E$ is an open subset of $\Omega$ or $E = \Omega^* \setminus K$ for a compact set $K \subset \Omega$.

As usual, if $\Omega \subset \mathbb{C}$ is open, we denote by $H(\Omega)$ the set of all holomorphic functions in $\Omega$. If $M$ is any subset of $\mathbb{C}$, then $A(M)$ denotes the set of all continuous functions in $M$ which are holomorphic in $M^\circ$, the interior of $M$. The closure of $M$ is denoted by $\bar{M}$.

With these notations, the famous theorem of Arakelian [1] on uniform approximation on closed sets by holomorphic functions states as follows:

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Theorem A (1968). Suppose $\Omega$ any domain in $\mathbb{C}$ and $F$ a closed subset of $\Omega$ with $\Omega^* \setminus F$ connected and locally connected at $\infty$. Such a set $F$ is also called an Arakelian set in $\Omega$.

Then every function $f \in A(F)$ can be approximated uniformly on $F$ by functions $g \in H(\Omega)$.

Our aim is to give lacunary versions of this theorem for special geometries of the domain $\Omega$ and the closed set $F$.

For this, suppose $\Lambda \subset \mathbb{N}_0$. We recall some standard definitions:

With $n_\Lambda(r) := \# \{ n \in \Lambda: n \leq r \}$ we call $d(\Lambda) := \limsup_{r \to \infty} n_\Lambda(r)/r$ the upper density and $d(\Lambda) := \liminf_{r \to \infty} n_\Lambda(r)/r$ the lower density of $\Lambda$. If $d(\Lambda) = d(\Lambda)$, the value $d(\Lambda) := \lim_{r \to \infty} n_\Lambda(r)/r$ is the density of $\Lambda$.

Moreover, we define for the case that $0 \in M^\circ$

$$A_\Lambda(M) := \left\{ f \in A(M): f(z) = \sum_{n \in \Lambda} a_n z^n \text{ around } 0 \right\}$$

and for $\Omega$ with $0 \in \Omega$

$$H_\Lambda(\Omega) := \left\{ f \in H(\Omega): f(z) = \sum_{n \in \Lambda} a_n z^n \text{ around } 0 \right\},$$

that is, functions in $A_\Lambda(M)$ and $H_\Lambda(\Omega)$ satisfy $f^{(n)}(0) = 0$ for every $n \notin \Lambda$.

Finally, for a compact set $K$ in $\mathbb{C}$, let $P_{\Lambda}(K)$ be the closure of the linear span of the monomials $z^v, v \in \Lambda$, with respect to the sup-norm.

Our investigations are based on the following results of Dixon and Korevaar [3], as well as Gharibyan, Luh and Müller [9]. But before, we recall some definitions. A bounded domain with boundary equal to a closed Jordan curve is called a Jordan domain. Such a Jordan domain $S$ is called strongly starlike with respect to the origin, if every ray starting at the origin meets the boundary of $S$ exactly in one point.

Theorem DK1 (1977). If $K = \bar{S}$, where $S$ is a Jordan domain with $0 \in S$ which is strongly starlike with respect to the origin, then we have for all $\Lambda \subset \mathbb{N}_0$

$$P_{\Lambda}(K) = A_{\Lambda}(K).$$

Theorem DK2 (1977). If $d(\Lambda) \in \{0, 1\}$ and $K = \bar{S}$, where $S$ is a Jordan domain of class $C^1$ with $0 \in S$, then $P_{\Lambda}(K) = A_{\Lambda}(K)$.

Theorem GLM (2006). If $d(\Lambda) = 1$, $K$ is a compact subset of $\mathbb{C}$ with connected complement and $f \in A(K)$, then $f \in P_{\Lambda}(K)$ if and only if $f|_E \in P_{\Lambda}(E)$, where $E$ is the component of $K$ containing 0 (defined as no condition if $E = \emptyset$, i.e. $0 \notin K$).
We start immediately with our first result in this context supposing that the origin is an interior point of the considered closed set.

**Theorem 2.1.** Suppose $\Lambda \subset \mathbb{N}_0$ with $d(\Lambda) = 1$, $\Omega$ a simply connected domain in $\mathbb{C}$ and let $F := \bigcup_{n \in \mathbb{N}_0} F_n$ be a closed set in $\Omega$, whereas all the sets $F_n$ are pairwise disjoint compact subsets of $\Omega$ and have connected complement. Furthermore, suppose that $0$ is an interior point of $F_0$.

If there exists an exhaustion $\{S_n\}_{n \in \mathbb{N}}$ of $\Omega$ consisting of bounded domains such that

- $\bigcup_{i=0}^{n} F_i \subset K_n := S_n$ ($n \in \mathbb{N}$),
- $F_i \cap K_n = \emptyset$ for all $i > n$,
- $P(\Lambda(K_n)) = A(\Lambda(K_n))$ ($n \in \mathbb{N}$),

and if $f \in A(F)$ satisfies $f|_{F_0} \in P(\Lambda(F_0))$ additionally, then for every sequence $\{\varepsilon_n\}_{n \in \mathbb{N}_0}$ of positive real numbers we have a function $g \in H(\Lambda(\Omega))$ with

$$|f(z) - g(z)| < \varepsilon_n \quad (z \in F_n, n \in \mathbb{N}_0).$$

Before proving this result, we point out the following.

**Remark 2.2.** The existence of an exhaustion $\{S_n\}$ satisfying the first two bullets of Theorem 2.1 is equivalent to $F_n$ “escaping to $\infty$”, that is, for all compact subsets $K$ of $\Omega$ the set of indices $n$ for which $K \cap F_n \neq \emptyset$ is finite. The third bullet puts a non-trivial additional restriction on the geometry of the sets $F_n$.

**Proof of Theorem 2.1.**

1. We construct a telescoping series $\sum_{n=0}^{\infty} (P_n - P_{n-1})$ via the following sequence of polynomials $\{P_n\}_{n \in \mathbb{N}_0}$:
   
   (a) $P_{-1}(z) \equiv 0$.
   
   (b) Since $f|_{F_0} \in P(\Lambda(F_0))$, there exists a polynomial $P_0$ of the structure

   $$P_0(z) = \sum_{j=0}^{j_0} a_j^{(0)} z^j$$

   and

   $$|P_0(z) - f(z)| < \frac{\varepsilon_0}{2} \quad (z \in F_0).$$

   (c) We define $\varepsilon^*_j := \min\{\varepsilon_0, \ldots, \varepsilon_j\}$ for $j \in \mathbb{N}_0$. Further, we presume that $P_0, \ldots, P_{n-1}$ are already defined for some $n \in \mathbb{N}$ and that they are all of the structure as $P_0$. Then by Theorem GLM, applied to $K = K_{n-1} \cup F_n$
and \( f \) replaced by the function \( P_{n-1} \) on \( K_{n-1} \) and \( f \) on \( F_n \), there exists a polynomial \( P_n \) of the structure

\[
P_n(z) = \sum_{j=0}^{j_n} a_j^{(n)} z^j \quad (j \in \Lambda)
\]

with

\[
|P_n(z) - P_{n-1}(z)| < \frac{\varepsilon_n^*}{2n+2} \quad (z \in K_{n-1}) \tag{2}
\]

and

\[
|P_n(z) - f(z)| < \frac{\varepsilon_n}{2} \quad (z \in F_n).
\]

2. Now, we define

\[
g(z) := \sum_{n=0}^{\infty} \left( P_n(z) - P_{n-1}(z) \right) = P_N(z) + \sum_{n=N+1}^{\infty} \left( P_n(z) - P_{n-1}(z) \right).
\]

Because of (2), the series in the second description converges uniformly on \( S_N \). All the functions involved are entire, so \( g \) is holomorphic in \( S_N \) for every \( N \in \mathbb{N} \); hence \( g \) is holomorphic in \( \bigcup S_N = \Omega \).

Moreover, \( g \in H_\Lambda(\Omega) \), because the uniform convergence of the series implies the convergence of all derivatives at the origin, and the derivatives at the origin of the terms of order \( n \notin \Lambda \) are equal to zero.

3. Finally, we show that \( g \) approximates \( f \) for all \( z \in F_n, n \in \mathbb{N} \) with error less than \( \varepsilon_n \). Therefore, we fix \( N \in \mathbb{N} \) with \( z \in F_N \) and consequently we have

\[
|P_N(z) - f(z)| < \frac{\varepsilon_N}{2}.
\]

Furthermore, by (2) we obtain

\[
|g(z) - P_N(z)| = \left| \sum_{n=N+1}^{\infty} \left( P_n(z) - P_{n-1}(z) \right) \right| \leq \sum_{n=N+1}^{\infty} \max_{k_N} |P_n(z) - P_{n-1}(z)| \leq \sum_{n=N+1}^{\infty} \max_{k_{n-1}} |P_n(z) - P_{n-1}(z)| \leq \sum_{n=N+1}^{\infty} \frac{\varepsilon_n^*}{2n+2} \leq \sum_{n=N+1}^{\infty} \frac{\varepsilon_n}{2n+2} \leq \frac{\varepsilon_N}{4} \sum_{n=0}^{\infty} \frac{1}{2n} = \frac{\varepsilon_N}{2}.
\]
It follows for this arbitrary $z \in F_N$

$$|g(z) - f(z)| \leq |g(z) - P_N(z)| + |P_N(z) - f(z)| < \varepsilon_N,$$

as asserted. □

For closed sets not containing the origin we obtain the following theorem.

**Theorem 2.3.** Suppose $\Lambda \subset \mathbb{N}_0$, $\Omega$, $\{F_n\}$ and $\{S_n\}$ as in Theorem 2.1. Moreover, let $F := \bigcup_{n \in \mathbb{N}} F_n$ (so that now $0 \notin F$ and $F_0 = \emptyset$ respectively).

Then for every sequence $\{\varepsilon_n\}$ of positive numbers and every function $f \in A(F)$, we have a function $g \in H_{\Lambda}(\Omega)$ with

$$|f(z) - g(z)| < \varepsilon_n \quad (z \in F_n, n \in \mathbb{N}).$$

**Proof.** We define $\tilde{F} := F \cup \{z: |z| \leq r\}$ where $r > 0$ is so small that the union is disjoint. Then every function $f \in A(F)$ can be extended to a function $\tilde{f} \in A_{\Lambda}(\tilde{F})$ (e.g. by setting $\tilde{f} = 0$ on $\{z: |z| \leq r\}$). Now, the result follows from Theorem 2.1. □

**Remark 2.4.** If the sets $\Omega$ and $F$ in the above Theorems are of a special structure, which allows us to choose the sequence $\{S_n\}_{n \in \mathbb{N}}$ in such a way that the $S_n$ are strongly starlike with respect to the origin for all $n \in \mathbb{N}$, then, according to Theorem DK1, the conditions of Theorem 2.3 are satisfied. If, in addition, $F_0$ is also the closure of a Jordan domain $S$, which is strongly starlike with respect to the origin, then the same is true for Theorem 2.1.

For instance, if we are interested in the construction of so-called T-universal functions as for example in [7], the considered sets $K_n$ may be chosen as closed circular discs, so in this case $F$ has the above-mentioned property.

If the $S_n$ are strongly starlike, then necessarily $\Omega$ has to be starlike.

Unfortunately, even if $\Omega$ is starlike, an exhaustion of $\Omega$ by Jordan domains, that are strongly starlike with respect to the origin, does not exist for all $F$ as above, as the following example shows, see also Fig. 1. Suppose $\Omega = \mathbb{C}$ and $F_n$ the compact and piecewise linear curve in $\mathbb{C}$ through the points

$$n + i, \quad n - 2i, \quad (n + 1) - 2i \quad \text{and} \quad (n + 1) + i,$$

if $n \in \mathbb{N}$ is odd. And let $F_n$ be the compact and piecewise linear curve in $\mathbb{C}$ through the points

$$\left(n - \frac{1}{3}\right) - i, \quad \left(n - \frac{1}{3}\right) + 2i, \quad \left(n + \frac{4}{3}\right) + 2i \quad \text{and} \quad \left(n + \frac{4}{3}\right) - i,$$

if $n \in \mathbb{N}$ is even.

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Remark 2.5. If \( d(\Lambda) = 1 \), then the topological conditions can be weakened considerably. In this case, we can find – for all simply connected domains \( \Omega \) – a sequence \( S_n \) of Jordan domains with \( C^1 \)-boundary as required if only the \( F_n \) escape to \( \infty \), cf. Remark 2.2. Thus, according to Theorem DK2, under this condition Theorem 2.3 holds. If also \( F_0 \) is the closure of a \( C^1 \)-bounded Jordan domain, then the conditions of Theorem 2.1 are again satisfied.

For a more sophisticated result in the case \( d(\Lambda) = 1 \), where the components \( F_n \) of \( F \) may be unbounded, we refer to [5] and [8].

3. MAXIMAL CLUSTER SETS ALONG ARBITRARY CURVES

By use of the theorems in the second section, we are able to construct functions which have so-called maximal cluster sets along arbitrary curves and which have a lacunary structure in addition.

Assume that \( \Omega \) is a domain in \( \mathbb{C} \), that \( F : \Omega \rightarrow \mathbb{C} \) is a function defined on \( \Omega \) and that \( A \) is a subset of \( \Omega \). Denoting by \( \partial \Omega \) the boundary of \( \Omega \) with respect to the extended plane \( \mathbb{C}^* \), the cluster set of \( F \) along \( A \) is defined as the set

\[
C_A(F) := \{ w \in \mathbb{C} : \text{there exists a sequence } \{z_n\}_{n \in \mathbb{N}} \subset A \text{ tending to some point of } \partial \Omega \text{ such that } \lim_{n \to \infty} F(z_n) = w \}
\]

and the oscillation value set of \( A \) is the set

\[
\text{Osc}(A) = \{ t \in \partial \Omega : \text{there exists a sequence } \{z_n\}_{n \in \mathbb{N}} \subset A \text{ with } \lim_{n \to \infty} z_n = t \}.
\]

Bernal-González, Calderón-Moreno and Prado-Bassas made several investigations on this field. For details we refer the interested reader to [2]. For this note we quote only one result out of it:

**Theorem BCP (2004).** Let \( \Omega \) be a Jordan domain. Then there is a dense linear manifold \( D \) in \( H(\Omega) \) such that for every \( f \in D \setminus \{0\} \) and every curve \( \gamma \subset \Omega \) tending to the boundary with \( \text{Osc}(\gamma) \neq \partial \Omega \) we have \( C_\gamma(f) = \mathbb{C} \). In particular, \( f(\gamma) \) is dense in \( \mathbb{C} \) for each pair \( f, \gamma \) as before.
Now, we ask whether there exist holomorphic functions having maximal cluster sets along arbitrary curves and having a prescribed lacunary structure in addition. We give the following positive answer.

**Theorem 3.1.** Let $\Omega$ be a Jordan domain which is starlike with respect to the origin, and let $\Lambda \subset \mathbb{N}_0$ with $d(\Lambda) = 1$ be given. Then there exists a $\varphi \in H_{\Lambda}(\Omega)$ such that for every curve $\gamma \subset \Omega$ tending to the boundary with $\text{Osc}(\gamma) \neq \partial \Omega$ we have $C_\gamma(\varphi) = \mathbb{C}$. In particular, $\varphi(\gamma)$ is dense in $\mathbb{C}$ for each $\gamma$ as before.

**Proof.**

1. Let $\mathbb{D}$ denote the unit disk and let $\psi : \mathbb{D} \to \Omega$ be a bijective conformal mapping with $\psi'(0) = 0$, $\psi'(0) > 0$. We fix two sequences $\{r_n\}_{n \in \mathbb{N}}$ and $\{s_n\}_{n \in \mathbb{N}}$ of positive real numbers satisfying $r_1 < s_1 < r_2 < s_2 < \cdots < r_n < s_n \cdots$ and $\lim_{n \to \infty}r_n = \lim_{n \to \infty}s_n = 1$. Let us define $F_n := \psi(E_n)$ as the image under $\psi$ of the spiral compact sets $E_n := \{(r_n + \frac{s_n - r_n}{4\pi} \theta)e^{i\theta}: \theta \in [0, 4\pi]\} (n \in \mathbb{N})$.

Observe that each $F_n$ has connected complement and that the sequence $\{F_n\}_{n \in \mathbb{N}}$ tends to $\partial \Omega$ (that is, to $\infty$ with respect to the one-point compactification). Note also that $F := \bigcup_{n \in \mathbb{N}} F_n$ is an Arakelian set in $\Omega$.

2. We denote by $\{q_n\}_{n \in \mathbb{N}}$ any fixed sequence, dense in $\mathbb{C}$, and define the function $q : F \to \mathbb{C}$ by

\[ q(z) := q_n \quad (z \in F_n), \]

which is obviously continuous in $F$; hence we have $q \in A(F)$. Finally, we set $\varepsilon_n := \frac{1}{n}$ for each $n \in \mathbb{N}$.

3. The Jordan domains $S_n := \psi(\{w: |w| < s_n\})$ are strongly starlike with respect to the origin, which follows from the fact that for all $\theta$

\[ \frac{\partial}{\partial \theta}(\text{arg} \psi(s_ne^{i\theta})) = \text{Re} \left( s_ne^{i\theta} \psi'(s_ne^{i\theta})/\psi(s_ne^{i\theta}) \right) > 0 \]

(cf. [4], Theorem 2.10 and its proof). Moreover, the $S_n$ satisfy the conditions from the proof of Theorem 2.1. By Theorem 2.3 and Remark 2.4 we get a function $\varphi$ in $H_{\Lambda}(\Omega)$ such that

\[ |\varphi(z) - q_n| = |\varphi(z) - q(z)| < \varepsilon_n = \frac{1}{n} \quad (z \in F_n). \]

4. It remains to show that for every prescribed curve $\gamma \subset \Omega$ as in the hypothesis we have $C_\gamma(\varphi) = \mathbb{C}$. Since $\text{Osc}(\gamma) \neq \partial \Omega$ and $\gamma$ should escape towards $\partial \Omega$,
this curve must intersect all spirals $F_n$ except finitely many of them; indeed, otherwise the shape of the $F_n$’s together with the continuity of $\gamma$ would force $\gamma$ to make infinitely many windings around the origin while approaching $\partial \Omega$, which would contradict the hypothesis $\text{Osc} (\gamma) \neq \partial \Omega$. Therefore, there exist $N \in \mathbb{N}$ and $z_n \in (\gamma \cap F_n)$ for each $n \geq N$ with
\[ |\varphi(z_n) - q_n| < \frac{1}{n} \rightarrow 0 \quad (n \geq N, n \rightarrow \infty). \]

For an arbitrary $w \in \mathbb{C}$ there is a sequence \( \{n_k\}_{k \in \mathbb{N}} \) with $q_{n_k} \rightarrow w$ and $n_k \rightarrow \infty$ as $k \rightarrow \infty$, so finally we obtain
\[ |\varphi(z_{n_k}) - w| \rightarrow 0 \quad (n_k \geq N, k \rightarrow \infty), \]
i.e. $w \in C_\gamma (\varphi)$. □

**Remark 3.2.** The assertion of Theorem 3.1 is in general no longer true if $\Lambda$ has upper density $d(\Lambda) < 1$: Consider $\Omega = \Omega_\alpha$ to be of the form
\[ \Omega_\alpha := \mathbb{D} \cap \{z: \Re(z) > \alpha\} \]
for some $\alpha \in (-1, 0)$. Then $\Omega_\alpha$ is convex (and thus in particular starlike with respect to 0).

If $\Lambda$ consists of the even nonnegative integers (in which case $d(\Lambda) = 1/2$), then $\varphi$ is even and therefore $\varphi$ has a holomorphic extension to $\mathbb{D}$. This implies that $C_\gamma (\varphi)$ is a singleton for every $\gamma$ terminating at an arbitrary point in $\partial \Omega_\alpha \setminus \partial \mathbb{D}$.

If $\Lambda$ is an arbitrary sequence with $d(\Lambda) = d < 1$, then, according to the Fabry–Pólya gap theorem (see e.g. [6]), there is an $\alpha \in (-1, 0)$ such that every function $\varphi \in H_\Lambda (\Omega_\alpha)$ has a holomorphic extension to $\mathbb{D}$. Again, cluster sets for curves $\gamma$ terminating at a point in $\partial \Omega_\alpha \setminus \partial \mathbb{D}$ are singletons.

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