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The Number of Loopless 4-Regular Maps on the Projective Plane[1]

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In this paper rooted loopless (near) 4-regular maps on surfaces such as the sphere and the projective plane are counted and exact formulae with up to three or four parameters for such maps are given. Several classical results on regular maps and one-faced maps are deduced.

Key Words: (rooted) 4-regular map; enumerating function.

I. INTRODUCTION

The surfaces considered here are compact 2-manifolds. There are many ways to define an orientable surface or a nonorientable one. Surfaces with *g handles* denoted by S_g are considered *orientable* with genus g while those having g *crosscaps* (i.e., N_g) are called *nonorientable* of genus g. A *map* M on (or embedded on) S_g (*or* N_g) is a graph drawn on the surface so that each vertex is a point on the surface, each edge $\{x, y\}$, $x \neq y$, is a simple open curve whose endpoints are x and y, each loop incident to a vertex x is a simple closed curve containing x, no edge contains a vertex to which it is not incident, and each connected region of the complement of the graph in the surface is homeomorphic to a disc and is called a *face*. A circuit (simple closed curve) C on a surface Σ is called *essential* if $\Sigma - C$ has no connected

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region homeomorphic to a disc; otherwise it is *planar* (or *trivial* as some scholars named it). A loop is called *essential* if it is an essential circuit; otherwise it is called *planar* or *trivial*. A map is *rooted* if an edge, a direction along the edge, and a side of the edge are all distinguished. If the root is the oriented edge from u to v, then u is the root-vertex while the face on the oriented side of the edge is defined as the *root-face*. Two rooted maps on a surface Σ are considered the same if there is a self-homeomorphism of Σ which induces an isomorphism between them preserving the rooting.

A rooted near-4-regular map on a surface is one such that all its vertices are 4-valent except possibly the root vertex. One may see, of course, that such maps are Eulerian.

Enumeration of maps on surfaces, a theory based on the works by Tutte [24–27], has been developed and deepened by people such as D. Arquès [1], W. Brown [8, 9], Mullin *et al.* [21], Tutte [28], Walsh *et al.* [29], Bender *et al.* [2–7], Gao [11, 12], and Liu [14, 19]. Among them, Arquès, Brown, and Walsh *et al.* did some influential works on the (exact) enumeration of nonplanar maps. As for the case of toroidal maps, one of the initial steps towards nonplanar ones, both Arquès [1] and, independently, the team of Bender, Canfield, and Richmond have counted rooted maps on the torus as a function of the number of edges [5]. It is well known that any kind of nonplanar maps are very difficult to handle in an exact way, especially those of nonplanar 4-regular maps without loops (although Walsh and Lehman [30] did some work on rooted nonplanar Eulerian maps in a general way). Bender *et al.* and Gao did the most extended work in the field of asymptotic evaluation of nonplanar maps. For a survey one may see [2].

Among regular maps, those which are 4-regular (or quartic maps as some scholars called them) are very important in many fields. The usages of 4-regular maps can be seen for *rectilinear embedding* in *VLSI*, for the *Gaussian crossing problem* in graph theory, for the *knot problem* in topology, and for the enumeration of some other kinds of maps [14–19]. Rooted 4-regular planar maps (or their duals: quadrangulations) have been investigated by many scholars such as Tutte [27], Brown [10], Mullin and Schellenberg [21], and Liu [14, 19].

In this paper the number of rooted near-4-regular maps on the sphere and the projective plane are investigated with respect to the root-valency, the number of edges, the number of non-root-vertex loops and non-root faces: Formulae for rooted planar (projective planar) near-4-regular maps with given number of loops are given; in particular, formulae for rooted loopless planar (projective planar) 4-maps are presented as well. As special case(s), several known results are concluded (for instance, formulae for rooted trees by Tutte [28], one faced maps on the projective plane [23], and rooted near-4-regular maps [22] on such surfaces such as the sphere and the projective plane may be deduced directly with easy.)

Let $\mathcal{U}_{\tilde{1}}$ and $\mathcal{U}_0$ be respectively the sets of all the rooted near-4-regular maps on the projective plane and the sphere, and their *enumerating functions* be, respectively,

$$f_p(x, y, z, t) = \sum_{M \in \mathcal{U}_{\tilde{1}}} x^{2m(M)} y^{s(M)} z^{n(M)} t^{\alpha(M)}$$

$$f_0(x, y, z, t) = \sum_{M \in \mathcal{U}_0} x^{2m(M)} y^{s(M)} z^{n(M)} t^{\alpha(M)},$$

where $2mm(M)$, $n(M)$, $s(M)$, and $\alpha(M)$ are the root valency, the number of non-root faces, the number of edges, and the number of non-root-vertex loops of M respectively. The following theorem will be proved in Section II:

THEOREM A. *The enumerating function $f_0(x, y, z, t)$ of $\mathcal{U}_0$ satisfies the following equation*

$$x^4 yz f_0^2 + (2x^2y^2z(t-1) + y - x^2)\, f_0 + 2x^2y^2z(1-t) + (x^2 - y - x^2yh) = 0, \tag{1}$$

where h is the enumerating function of those in $\mathcal{U}_0$ having their root-valencies 2. Furthermore, the explicit solution of (1) *is*

$$f_0(x, y, z, t) = 1 + \sum_{\substack{m \geqslant 2 \\ n \geqslant m}} \sum_{k, l > 0} C(m, n, l)(t-1)^{\,l}\, y^{2(l+n)-m-1} z^{l+n-1} x^{2(m-1)}$$

where

$$C(m, n, l) = \frac{2^l 3^{n-m}(m-1)}{(2n-m)(2n-m-1)} \binom{2m-2}{m-1} \times \binom{2n+l-m-2}{l} \binom{2n-m}{n}. \tag{2}$$

By (2) one may determine some special types of maps on the plane. For instance, there are 2 such maps with 2 inner faces, 3 edges, 1 non-root-vertex loops and root-valency 2.

Remark. Equation (1) generalizes a result of Liu [14] (the case of $t=1$). If the distribution of t is considered, then some other results may be deduced.

THEOREM 1 [14, Theorem 5.4.1]. *The enumerating function of rooted planar near-4-regular maps with the root-valency, the number of edges and the number of inner faces as the parameters is*

$$1+\sum_{\substack{m\geqslant 2\\ n\geqslant m}} \frac{3^{n-m}(2m-2)!\,(2n-m-2)!}{(m-1)!\,(m-2)!\,(n-m)!\,n!}\,y^{2n-m-1}z^{n-1}x^{2(m-1)}.$$

Since maps on surfaces must obey the Euler formula, a classical formula on rooted trees (i.e., those having exactly one vertex) may be obtained:

THEOREM 2 [28]. *The enumerating function of rooted trees is*

$$\left.\frac{\partial(yf(x,y,y^{-1},1))}{\partial y}\right|_{y=0}=\sum_{m\geqslant 1}\frac{1}{m}\binom{2m-2}{m-1}x^{2m-2}.$$

In generality, we have

THEOREM 3. *The generating function of the maps in $\mathcal{U}_0$ with exactly k non-root-vertex loops is*

$$\left.\frac{1}{k!}\frac{\partial^k f_0(x,y,z,t)}{\partial t^k}\right|_{t=0}$$

In particular (in the case of $t=0$), we obtain a formula for loopless maps in $\mathcal{U}_0$:

THEOREM 4. *The enumerating function of rooted planar 4-regular maps without loops is*

$$\frac{(1-2y^2z)}{y}\left\{\frac{1}{2!}\left.\frac{\partial^2 f_0(x,y,z,0)}{\partial x^2}\right|_{x=0}-yz\right\}.$$

From this and Lagrangian inversion (for a reference one may see [13, pp. 17–18]) one may determine the number of rooted planar 4-regular maps without loops, i.e., the number of those having $2m$ edges is L_m-2L_{m-1}, where

$$L_m=\sum_{l+n=m}\frac{(-1)^l\,2^l3^n}{(n+1)(2n+1)}\binom{2n+2}{n}\binom{n+m}{l}.$$

Remark. Some other results may be further obtained if special values of x and t are taken. For instance,

$$\frac{1}{k!\,4!}\left.\frac{\partial^{k+4} f_0(x, y, z, t)}{\partial x^4\,\partial t^k}\right|_{x=t=0}$$

is the enumerating function of those 4-regular maps in $\mathscr{U}_0$ having exactly k non-root-vertex loops.

Now let us turn to the case of non-planar maps. Similar to Theorem A, we have a more general result on rooted near-4-regular maps on the projective plane as compared with a known formula [22]:

THEOREM B. *The enumerating function of $\mathscr{U}_{\tilde{1}}$ together with $f_0(x, y, z, t)$ satisfies the equation*

$$f_p = 2x^2yzf_0f_p + x^2y\frac{\partial(xf_0)}{\partial x} + 2y^2ztf_p + 3y^2t(f_0 - 1)$$
$$+\frac{y}{x^2}(f_p - x^2F_2 - 2x^2yzf_p - 3x^2y(f_0-1)) \tag{3}$$

and further, f_p may be expressed parametrically as

$$f_p = \frac{x^2\eta}{\sqrt{1-vx^2}}\left\{\frac{x^2(\partial(xf_0)/\partial x)|_x^{\sqrt{\eta}} + 3y(t-1)\,f_0|_x^{\sqrt{\eta}}}{\eta - x^2}\right\}, \tag{4}$$

where $f(X)|_b^a = f(a) - f(b)$, F_2 is the enumerating function of those in $\mathscr{U}_{\tilde{1}}$ with their root-valencies 2, *η is the double root of the discriminant of* (1) *in Theorem* A, *and*

$$v = \frac{2\theta}{y}, \qquad \theta = \frac{4y^2z}{2 - 4y^2z(t-1) - 3\theta}, \qquad \eta = \frac{y}{1 - \theta - 2y^2z(t-1)}, \tag{5}$$

$$F_2 = \left\{x^2\frac{\partial(xf_0)}{\partial x} + 3y(t-1)(f_0-1)\right\}\Bigg|_{x=\sqrt{\eta}}. \tag{6}$$

Remark. Based on (5) and Lagrangian inversion (for a reference one see may [13]), $f_p(x, y, z, t)$ may be expanded as power series of x, y, z, and t. As an application, some known results may be deduced.

THEOREM 5 [22]. *Let $\mathscr{U}_{\tilde{1}}$ be the set of all the rooted near-4-regular maps on the projective plane. Then*

(1) *The enumerating function of* $\mathscr{U}_{\tilde{1}}$ *respect to the root-valency, the number of edges and the number of inner faces is*

$$f_p(x, y, z, 1) = \sum_{\substack{m \geqslant 2 \\ n \geqslant m}} \sum_{k, l \geqslant 0} \sum_{q=0}^{m-1} \frac{(2m-1)\, C(m, n, 0)}{2^k} \binom{2k}{k} \times \binom{l+m-p-1}{l} y^{2n-k-p-1} z^{n-1} \theta^{k+l} x^{2(k+p+1)} + \sum_{k \geqslant 0} \frac{1}{2^k} \binom{2k}{k} \frac{\theta^k}{(1-\theta)} \frac{x^{2(k+1)}}{y^{k-1}},$$

where $C(m, n, 0)$ *is defined in Theorem* A *and for* $s \geqslant 1$ *(by Lagrangian inversion and* (5)*),*

(2) *In particular, the number of rooted* 4*-regular maps (which may have loops) on the projective plane with* $2p-2$ *edges is*

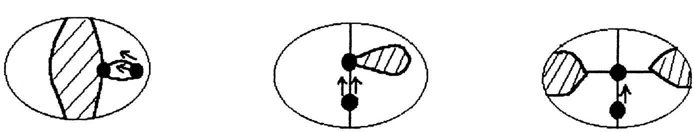
$$\sum_{\substack{m \geqslant 2 \\ n \geqslant 0 \\ m+n \leqslant p}} \frac{2^{n+1} 2^{p-m-n} (m+n)}{mp} \binom{2m-2}{m-1} \binom{m+n-2}{m-2} \times \binom{2p-m-n-1}{p-1} + \frac{3^{p-1}}{p} \binom{2p-2}{p-1}.$$

By the Euler formula, the exact formula for rooted one-faced maps (the duals of those with only one vertex) on the projective plane may be deduced.

THEOREM 6 [23]. *The enumerating function of rooted one faced-maps on the projective plane is*

$$\left. \frac{\partial (f_p(x, y, z, 0)|_{z=y^{-1}})}{\partial y} \right|_{y=0} = \frac{1}{\sqrt{1-4x^2}} \sum_{m \geqslant 1} \binom{2m-1}{m} x^{2m}.$$

Similar to Theorems 2 and 3, results on the number of loops may be deduced as follows:



M_1 M_2 M_3

FIGURE 1

THEOREM 7. *The enumerating function of the maps in $\mathcal{U}_{\tilde{1}}$ with exactly k non-root-vertex loops is*

$$\frac{1}{k!}\frac{\partial^k f_p(x, y, z, t)}{\partial t^k}\bigg|_{t=0}.$$

We now consider the case where $t=0$ (i.e., no loops will appear except possibly on the root-vertex). In order to eliminate the effects of loops on the root-vertex, 5 types of maps (as shown in Fig. 1) have to be excluded since a map of each of those types will contain a loop when the root-edge is contracted. These have the structure of M_1 consisting of a nonplanar map and a contractible cycle of length 2. Hence, their enumerating function is

$$2y^2(F_2-y),$$

where F_2 is defined in Theorem B. Similarly, the enumerating function of the maps having the structure of M_2 or M_3 is

$$3y^2(h-y),$$

where h is defined in Theorem A. Thus, we have the following theorem

THEOREM 8. *The enumerating function of rooted loopless* 4*-regular maps on the projective plane is*

$$\frac{(1-2y^2)(F_2-y)-3y^2(h-y)}{y},$$

i.e.,

$$\sum_{m\geqslant 2}(A_m+\tfrac{3}{2}Y_{m-1}-3X_{m-1}-X_m)(1-2y^2)\,y^{2m}-\sum_{m\geqslant 2}3B_{m-1}\,y^{2m},$$

where

$$A_m=\sum_{\substack{l,p\geqslant 0\\ k\geqslant 2}}\;\sum_{\substack{s+p=m+1\\ s\geqslant k+2}}\frac{(-1)^p\,2^{l+p+1}3^{s-k-l}(k+l)}{k\times s}\binom{2k-2}{k-1}$$
$$\times\binom{k+l-2}{l}\binom{2s-k-l-1}{s-1}\binom{2s+p-2}{p};$$

$$B_m=\sum_{l+n=m}\frac{(-1)^l\,2^l3^n}{(n+1)(2n+1)}\binom{2n+2}{n}\binom{n+m}{l};$$

$$X_m = \sum_{l+n=m} \frac{(-1)^l \, 2^{l+1} 2^n}{n+1} \binom{2n}{n} \binom{n+m}{l};$$

$$Y_m = \sum_{l+n=m} \frac{(-1)^l \, 2^{l+3} 3^n}{(n+2)} \binom{2n+1}{n} \binom{n+m}{l}.$$

Based on Theorem 8, one may calculate the number of rooted loopless 4-regular maps on the projective plane by the number of edges. For instance, there are 21 such maps having 6 edges as depicted in Fig. 2 together with Table I.

The contributions of M_i $(1 \leqslant i \leqslant 4)$ to rooted maps are listed in Table I.

II. PLANAR NEAR-4-REGULAR MAPS

In this section we shall prove Theorem A. Let us first define an operation on maps.

For two maps M_1, M_2 with their respective roots $r_1 = r(M_1)$, $r_2 = r(M_2)$, the map $M = M_1 \cup M_2$ such that $M_1 \cap M_2 = v$ with $v = v_{r_1} = v_{r_2}$, is defined to have its root, root-valency, and root-edge the same as those of M_1, but the root-face is the region bounded by the edges that used to bound the root-face of M_1 and the edges that used to bound the root-face of M_2. The operation for obtaining M from M_1 and M_2 is called 1*v-production* and M is denoted as

$$M = M_1 \odot M_2.$$

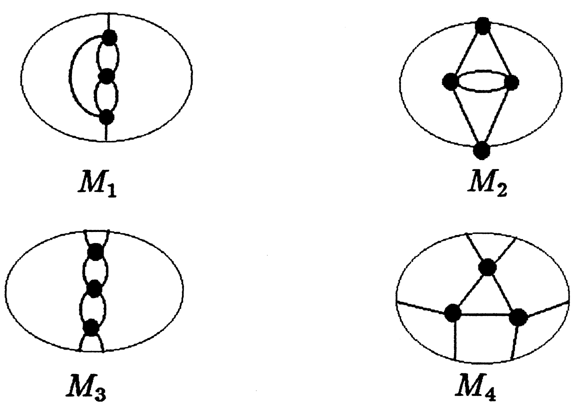


FIG. 2. The distinct embeddings of the double graph K_3^2 on the projective plane which will induce 21 rooted maps.

TABLE I

M_1	M_2	M_3	M_4
12	6	2	1

Further, for any two sets of maps $\mathscr{M}_1$ and $\mathscr{M}_2$, the set of maps

$$\mathscr{M}_1 \odot \mathscr{M}_2 = \{M_1 \odot M_2 \mid M_i \in \mathscr{M}_i, i = 1, 2\}.$$

is said to be the 1*v-production* of $\mathscr{M}_1$ and $\mathscr{M}_2$. If $\mathscr{M} = \mathscr{M}_1 = \mathscr{M}_2$, then we may write

$$\mathscr{M} \odot \mathscr{M} = \mathscr{M}^{\odot 2}$$

and further,

$$\mathscr{M}^{\odot k} = \mathscr{M} \odot \mathscr{M}^{\odot (k-1)}.$$

The set $\mathscr{U}_0$ may be partitioned into three parts as

$$\mathscr{U}_0 = \vartheta + \mathscr{U}_0^1 + \mathscr{U}_0^2,$$

where ϑ is the vertex map and

$$\mathscr{U}_0^1 = \{M \mid e_r(M) \text{ is a loop}\};$$

$$\mathscr{U}_0^2 = \{M \mid e_r(M) \text{ is a link}\}.$$

LEMMA 2.1. *Let* $\mathscr{U}_{\langle 0\rangle}^1 = \{M - e_r(M) \mid M \in \mathscr{U}_0^1\}$. *Then* $\mathscr{U}_{\langle 0\rangle}^1 = \mathscr{U}_0 \odot \mathscr{U}_0$.

Proof. For a map $M \in \mathscr{U}_0$, the root-edge $e_r(M)$ is a loop. The inner and outer regions determined by $e_r(M)$ are, respectively, two elements of $\mathscr{U}_0$. Since this procedure is reversible, $\mathscr{U}_{\langle 0\rangle}^1 = \mathscr{U}_0 \odot \mathscr{U}_0$. ∎

LEMMA 2.2. *Let* $\mathscr{U}_{(0)}^2 = \{M \bullet e_r(M) \mid M \in \mathscr{U}_0\}$. *Then* $\mathscr{U}_{(0)}^2 = \mathscr{U}_0 - \mathscr{U}_0(2) - \vartheta$, *where* $\mathscr{U}_0(2)$ *is the set of maps in* $\mathscr{U}_0$ *whose root-valency is* 2.

Proof. Notice that the shrinking of the root-edge of a map in $\mathscr{U}_0^2$ will result in an element in $\mathscr{U}_0$ whose root-valency not less than 4. Thus, $\mathscr{U}_{(0)}^2$ is a subset of $\mathscr{U}_0 - \mathscr{U}_0(2) - \vartheta$.

On the other hand, splitting the root-vertex of a map in $\mathscr{U}_0 - \mathscr{U}_0(2) - \vartheta$ will result in a map in $\mathscr{U}_2$. Hence, $\mathscr{U}_0 - \mathscr{U}_0(2) - \vartheta \subseteq \mathscr{U}_0$. ∎

As for the set $\mathscr{U}_{(0)}^2$, it may be further partitioned into another three parts as

$$\mathscr{U}_{(0)}^2 = \mathscr{U}_{(0)}^{21} + \mathscr{U}_{(0)}^{22} + \mathscr{U}_{(0)}^{23},$$

where

$$\mathscr{U}_{(0)}^{21} = \{M \mid e_r(M) \text{ is a loop with its inner face of valency } 1\};$$
$$\mathscr{U}_{(0)}^{22} = \{M \mid e_{P_r}(M) \text{ is a loop and it bounds a face of valency } 1\},$$

and P is the *cyclic permutation* of the edges incident with the root-vertex (as defined in [29]).

LEMMA 2.3. *The enumerating functions of* $\mathscr{U}_{(0)}^{21}$ *and* $\mathscr{U}_{(0)}^{22}$ *are equal to* $x^2yz(f_0-1)$.

Proof. By the definition of $\mathscr{U}_{(0)}^{21}$, $\mathscr{U}_{(0)}^{21} = \mathscr{L} \odot (\mathscr{U}_0 - \vartheta)$, where $\mathscr{L}$ is the loop map. Hence, the enumerating function of $\mathscr{U}_{(0)}^{21}$ is $f_{(0)}^{21} = x^2yz(f_0-1)$. In the same way, we have $f_{(0)}^{22} = x^2yz(f_0-1)$. (Here $f_{(0)}^{22}$ is the enumerating function of $\mathscr{U}_{(0)}^{22}$.) ▮

By Lemmas 2.2 and 2.3, we see that the enumerating function of $\mathscr{U}_{(0)}^{23}$ is thus

$$f_{(0)}^{23} = (f_0 - x^2h - 1) - 2x^2yz(f_0-1), \tag{7}$$

where h is the enumerating function of $\mathscr{U}_0(2)$.

Since only maps in $\mathscr{U}_{(0)}^{21}$ or $\mathscr{U}_{(0)}^{22}$ may, produce a non-root-vertex loop after splitting the root-vertex into two, we have (from Lemma 2.3 and (7))

$$f_0^2 = \frac{yt}{x^2}(2x^2yz(f_0-1)) + \frac{y}{x^2}(f_0 - x^2h - 1 - 2x^2yz(f_0-1)); \tag{8}$$

here, $f_0^2(x, y, z, t)$ is the enumerating function of $\mathscr{U}_0^2$. This together with Lemma 2.1 implies (1) of Theorem A.

Let Δ be the discriminant of (1) and $\Delta = (y+ux^2)^2(1-vx^2)$. Then by comparing the coefficients of x^2, x^4 and x^6 we obtain a system of equations:

$$u^2v = 8y^3z^2(1-t) + 4yz(1-yh);$$
$$u^2 - 2uvy = (1-2y^2z(t-1))^2 + 4y^2z;$$
$$2uy - vy^2 = -2y(1-2y^2z(t-1)).$$

Suppose $\theta = u + 1 - 2y^2z(t-1)$. Then

$$v = \frac{2\theta}{y}, \qquad \theta = \frac{4y^2z}{2-4y^2z(t-1)-3\theta}. \tag{9}$$

Now let us turn to the function $f_0(x, y, z, t)$. By using an expansion of $(1-x)^\alpha$ (where α is a real number) as a power series of x,

$$\begin{aligned} f_0 &= \frac{1}{2x^4yz}\{x^2-y-2x^2y^2z(t-1)+\sqrt{\Delta}\} \\ &= \frac{1}{2x^4yz}\left\{x^2-y-2x^2y^2z(t-1)+(y+ux^2)\left(1-\sum_{m\geqslant 1}\alpha_m v^m x^{2m}\right)\right\} \\ &= \frac{1}{2x^4yz}\sum_{m\geqslant 1}\alpha_m\left\{1-2y^2z(t-1)-\frac{3m\theta}{m+1}\right\}\frac{2^m\theta^m x^{2m+2}}{y^m}, \end{aligned}$$

here

$$\alpha_m = \frac{(2m-2)!}{2^{2m-1}m!\,(m-1)!}.$$

By applying the Lagrangian inversion for (9) and θ^m $(m \geqslant 1)$ and substituting this into $f_0(x, y, z, t)$, we obtain Theorem A.

III. NON-PLANAR MAPS

Here we prove Theorem B. As in Section II, of course, we first make a partition on $\mathscr{U}_{\tilde{1}}$, i.e.,

$$\mathscr{U}_{\tilde{1}} = \mathscr{U}_{\tilde{1}}^1 + \mathscr{U}_{\tilde{1}}^2,$$

where

$$\mathscr{U}_{\tilde{1}}^1 = \{M \mid e_r(M) \text{ is a loop}\};$$
$$\mathscr{U}_{\tilde{1}}^2 = \{M \mid e_r(M) \text{ is a link}\}.$$

We obtain the following lemma using the same reasoning as for Lemma 2.2.

LEMMA 3.1. *Let* $\mathscr{U}_{(\tilde{1})}^2 = \{M \bullet e_r(M) \mid M \in \mathscr{U}_{\tilde{1}}^2\}$. *Then* $\mathscr{U}_{(\tilde{1})}^2 = \mathscr{U}_{\tilde{1}} - \mathscr{U}_{\tilde{1}}(2)$, *where* $\mathscr{U}_{\tilde{1}}(2)$ *is the enumerating function of those maps in* $\mathscr{U}_{\tilde{1}}$ *whose root-valency is* 2.

Here, $M \bullet e_r(M)$ means the resulting map obtained by contracting the root-edge of M.

Furthermore, $\mathscr{U}_{\tilde{1}}^1$ may be partitioned into two parts

$$\mathscr{U}_{\tilde{1}}^1 = \mathscr{U}_{\tilde{1}}^{11} + \mathscr{U}_{\tilde{1}}^{12},$$

where

$$\mathcal{U}_{\tilde{1}}^{11} = \{M \mid e_r(M) \text{ is a planar loop}\};$$
$$\mathcal{U}_{\tilde{1}}^{12} = \{M \mid e_r(M) \text{ is an essential loop}\}.$$

LEMMA 3.2. *The contribution of* $\mathcal{U}_{\tilde{1}}^{11}$ *to the enumerating function of* $\mathcal{U}_{\tilde{1}}$ *is*

$$\mathcal{U}_{\tilde{1}}^{11} : f_{p_{11}} = 2x^2 yz f_0 f_p.$$

Proof. Let $\mathcal{U}_{\langle\tilde{1}\rangle}^{11} = \{M - e_r(M) \mid M \in \mathcal{U}_{\tilde{1}}^{11}\}$. For a map, say M, in $\mathcal{U}_{\tilde{1}}^{11}$, the root-edge $e_r(M)$ determines two maps such that one is planar and the other is non-planar. This shows that

$$\mathcal{U}_{\langle\tilde{1}\rangle}^{11} = \mathcal{U}_0 \odot \mathcal{U}_{\tilde{1}} + \odot\, \mathcal{U}_0,$$

and consequently, $f_{p11} = 2x^2 yz f_0 f_p$. ▮

LEMMA 3.3. *The contribution of* $\mathcal{U}_{\tilde{1}}^{12}$ *is*

$$\mathcal{U}_{\tilde{1}}^{12} : f_{p_{12}} = x^2 y \frac{\partial(x f_0)}{\partial x}. \tag{10}$$

Remark. By the definition of $\mathcal{U}_{\tilde{1}}^{12}$, the enumerating function of those maps in $\mathcal{U}_{\tilde{1}}$ whose root-edges are essential loops is (in the case of $y = z = t = 1$)

$$\sum_{m \geqslant 1} \binom{2m-1}{m} x^{2m}.$$

Proof. Let $\mathcal{U}_{2b}$ be the set of all the *rooted* 2-*boundary planar maps* (i.e., whose root-face valency is 2) with no vertices of odd degree except possibly those incident to the root-face. Then, one may readily see the following

Fact. Let

$$\begin{aligned}\mathcal{U}' &= \mathcal{U}_{(2b)} \\ &= \{M \bullet e_r(M) \mid M \in \mathcal{U}_{2b}\}; \\ \mathcal{U}'_{\langle r\rangle} &= \{M' - e'_r(M) \mid M' \in \mathcal{U}'\}.\end{aligned}$$

Then

(i) $\mathcal{U}_{(2b)}$ is the $1v$-production of loop map and $\mathcal{U}_0$;
(ii) $\mathcal{U}'_{\langle r\rangle} = \mathcal{U}_0$.

For any map M in $\mathcal{U}'$, there are $2m(M)-1$ maps in $\mathcal{U}_{2b}$ corresponding to M in such a way that

$$M: x^{2m(M)}y^{s(M)}z^{n(M)}t^{\alpha(M)} \mapsto x^i w^{2m(M)-i+2} y^{s(M)+1} z^{n(M)} t^{\alpha(M)},$$

for $(2 \leqslant i \leqslant 2m(M))$, where x, w denote the valencies of vertices on the root-face respectively and none of the possible non-root-vertex loops is on the boundary. Thus, $2m(M)+2$ is just the root-edge valency of the maps corresponding to M. Let $x = w$. Then

$$x^2 yz f_{p_{12}}(x, y, z, t) = \sum_{M \in \mathcal{U}'} (2m(M)-1)\, x^{2m(M)+2} y^{s(M)+1} z^{n(M)} t^{\alpha(M)},$$

i.e.,

$$f_{p_{12}}(x, y, z, t) = \frac{x^2}{z} \frac{\partial}{\partial x}\left(\frac{f_{\mathcal{U}'}(x, y, z, t)}{x}\right).$$

Since $\mathcal{U}'_{\langle r \rangle} = \mathcal{U}_0$ that implies $f_{\mathcal{U}'}(x, y, z, t) = x^2 yz f_0(x, y, z)$,

$$f_{p_{12}}(x, y, z, t) = x^2 y \frac{\partial}{\partial x}(x f_0(x, y, z, t)). \quad \blacksquare$$

Since maps in $\mathcal{U}^2_{(\tilde{1})}$ may produce non-root-vertex loops after splitting the root-vertex into two, the set $\mathcal{U}^2_{(\tilde{1})}$ has to be partitioned into several parts as

$$\mathcal{U}^2_{(\tilde{1})} = \sum_{i=1}^{6} \mathcal{U}^{2i}_{(\tilde{1})},$$

where the maps in $\mathcal{U}^{2i}_{(\tilde{1})}$ $(i = 1, 2, 3, 4, 5)$ have the structures as depicted in Fig. 3.

Hence, the contributions of $\mathcal{U}^{2i}_{(\tilde{1})}$ $(i = 1, 2, 3, 4, 5)$ to the enumerating function of $\mathcal{U}_{(\tilde{1})}$ are, respectively,

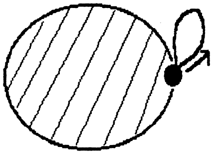
$$\mathcal{U}^{21}_{(\tilde{1})}: f^1_{p_2} = \frac{yt}{x^2}(x^2 yz f_p);$$

$$\mathcal{U}^{22}_{(\tilde{1})}: f^2_{p_2} = \frac{yt}{x^2}(x^2 yz f_p);$$

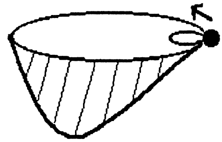
$$\mathcal{U}^{23}_{(\tilde{1})}: f^3_{p_2} = \frac{yt}{x^2}(x^2 y(f_0 - 1));$$

$$\mathcal{U}^{24}_{(\tilde{1})}: f^4_{p_2} = \frac{yt}{x^2}(x^2 y(f_0 - 1));$$

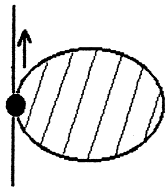
$$\mathcal{U}^{25}_{(\tilde{1})}: f^5_{p_2} = \frac{yt}{x^2}(x^2 y(f_0 - 1)),$$



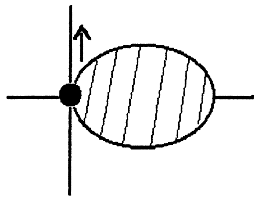
$M \in \mathcal{U}^{21}_{(\tilde{1})}$



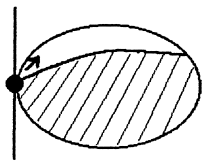
$M \in \mathcal{U}^{22}_{(\tilde{1})}$



$M \in \mathcal{U}^{23}_{(\tilde{1})}$



$M \in \mathcal{U}^{24}_{(\tilde{1})}$



$M \in \mathcal{U}^{25}_{(\tilde{1})}$

FIGURE 3.

and so (from this and Lemma 3.1), that of $\mathcal{U}^{26}_{(\tilde{1})}$ is

$$\mathcal{U}^{26}_{(\tilde{1})}: f^{6}_{p_2} = \frac{y}{x^2}(f_p - x^2F_2 - 2x^2yzf_p - 3x^2y(f_0 - 1)).$$

Those together with Lemma 3.2 imply Eq. (3) of Theorem B. After rearranging the items in (2), it may a rewritten as

$$\sqrt{\Delta}\, f_p = x^2y\left\{F_2 - x^2\frac{\partial(xf_0)}{\partial x} - 3y(t-1)(f_0-1)\right\}, \tag{11}$$

where Δ is the discriminant of (1) in Theorem A.

Suppose that $x^2 = \eta$ is the double root of Δ. Then we have

$$F_2 = \left.\left\{x^2\frac{\partial(xf_0)}{\partial x} + 3y(t-1)(f_0-1)\right\}\right|_{x=\sqrt{\eta}} \tag{12}$$

with

$$\eta = \frac{y}{1-\theta-2y^2z(t-1)}.$$

Substitute (12) into (11), and Theorem B immediately follows.

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