J. Math. Anal. Appl. 374 (2011) 481–498

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Stability to weak dissipative Bresse system

Fatiha Alabau Boussouira[a], Jaime E. Muñoz Rivera[b], Dilberto da S. Almeida Júnior[c,*]

[a] Universite Paul Verlaine-Metz, Laboratoire et Departement de Mathematiques, Bat. A, Ile du Saulcy, F-57045 Metz Cedex 01, France
[b] National Laboratory for Scientific Computation, Getúlio Vargas Street, Number 333, CEP 25651-075, RJ, Petrópolis, Brazil
[c] Department of Mathematics, Federal University of Pará, Augusto Corrêa Street, Number 01, CEP 66075-110, Belém, Pará, Brazil

ARTICLE INFO

*Article history:*
Received 19 January 2010
Available online 22 September 2010
Submitted by B. Straughan

*Keywords:*
Bresse system
Lack of exponential decay
Semigroups
Finite difference

ABSTRACT

In this paper we study the Bresse system with frictional dissipation working only on the angle displacement. Our main result is to prove that this dissipative mechanism is enough to stabilize exponentially the whole system provided the velocities of waves propagations are the same. This result is significative only from the mathematical point of view since in practice the velocities of waves propagations are always different. In that direction we show that when the velocities are not the same, the system is not exponentially stable and we prove that the solution in this case goes to zero polynomially, with rates that can be improved by taking more regular initial data. Finally, we give some numerical result to verify our analytical results.



## 1. Introduction

In this paper we consider the Bresse system with frictional damping effective only in one equation of the system. Our main result is to prove that in general the system is not exponentially stable. More precisely, we show the exponential stability if and only if the velocities of waves propagations are equals, which never happens in the practice. This means that to physical applications we never have exponential stability and when the velocities are different we prove that the solution decays polynomially to zero, with rates that depends on the initial data and some relationships between the coefficients.

The Bresse system is also known as the circular arch problem and is given by the following equations:

$$\rho_1\varphi_{tt} = Q_x + lN + F_1, \tag{1.1}$$

$$\rho_2\psi_{tt} = M_x - Q + F_2, \tag{1.2}$$

$$\rho_1 w_{tt} = N_x - lQ + F_3, \tag{1.3}$$

where

$$N = \kappa_0(w_x - l\varphi), \qquad Q = \kappa(\varphi_x + lw + \psi), \qquad M = b\psi_x. \tag{1.4}$$

We use $N$, $Q$ and $M$ to denote the axial force, the shear force and the bending moment. By $w$, $\varphi$ and $\psi$ we are denoting the longitudinal, vertical and shear angle displacements. See Fig. 1. Here $\rho_1 = \rho A$, $\rho_2 = \rho I$, $\kappa_0 = EA$, $\kappa = k'GA$, $b = EI$ and $l = R^{-1}$. To material properties, we use $\rho$ for density, $E$ for the modulus of elasticity, $G$ for the shear modulus, $k'$ for the shear factor, $A$ for the cross-sectional area, $I$ for the second moment of area of the cross-section and $R$ for the radius

* Corresponding author.
*E-mail address:* dilberto@ufpa.br (D.S. Almeida Júnior).

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2010.07.046

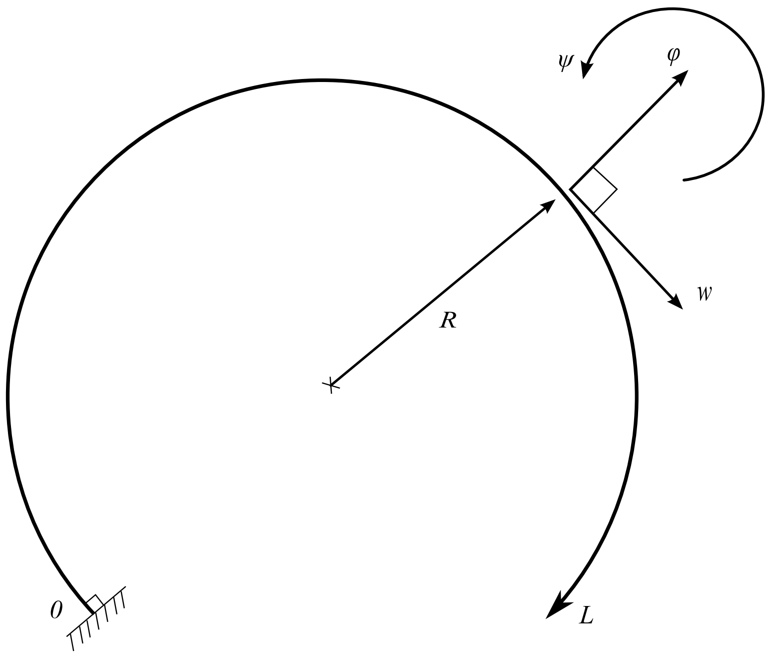


**Fig. 1.** The circular arch.

of curvature and we assume that all this quantities are positives. To more details on the system (1.1)–(1.4) see Lagnese et al. [3].

Finally by $F_i$ we are denoting external forces and therefore the motion equations are given by

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi + lw)_x - \kappa_0 l(w_x - l\varphi) = F_1, \quad \text{in } ]0,\infty[\times]0,L[, \tag{1.5}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi + lw) = F_2, \quad \text{in } ]0,\infty[\times]0,L[, \tag{1.6}$$

$$\rho_1 w_{tt} - \kappa_0(w_x - l\varphi)_x + \kappa l(\varphi_x + \psi + lw) = F_3, \quad \text{in } ]0,\infty[\times]0,L[. \tag{1.7}$$

We consider the following initial conditions

$$\varphi(0,\cdot) = \varphi_0, \qquad \varphi_t(0,\cdot) = \varphi_1, \qquad \psi(0,\cdot) = \psi_0, \qquad \psi_t(0,\cdot) = \psi_1, \qquad w(0,\cdot) = w_0, \qquad w_t(0,\cdot) = w_1$$

and Dirichlet boundary conditions

$$\varphi(t,0) = \varphi(t,L) = \psi(t,0) = \psi(t,L) = w(t,0) = w(t,L) = 0 \quad \text{in } ]0,\infty[, \tag{1.8}$$

or Dirichlet–Neumann boundary conditions

$$\varphi(t,0) = \varphi(t,L) = \psi_x(t,0) = \psi_x(t,L) = w_x(t,0) = w_x(t,L) = 0 \quad \text{in } ]0,\infty[. \tag{1.9}$$

The boundary conditions (1.9) make the calculations easier, because do not introduce pointwise terms when we apply the multiplicative techniques. Dirichlet boundary conditions (in all the equations) are more complicated because of the boundary terms, but by using observability result we can estimate them. The final remark about the boundary condition is that we use Dirichlet–Neumann–Neumann to prove the lack of exponential stability of the corresponding semigroup. The same result must be true for other boundaries conditions but as well as we know there is no a formal proof to this fact. In that follows we consider $F_1 = F_3 = 0$ and $F_2 = -\gamma\psi_t$ with $\gamma > 0$. For this case, making $l = R^{-1} \to 0$ we obtain that

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x = 0, \quad \text{in } ]0,\infty[\times]0,L[, \tag{1.10}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \gamma\psi_t = 0, \quad \text{in } ]0,\infty[\times]0,L[, \tag{1.11}$$

$$\rho_1 w_{tt} - \kappa_0 w_{xx} = 0, \quad \text{in } ]0,\infty[\times]0,L[ \tag{1.12}$$

where Eq. (1.12) can be negligible (see [9]) and the lack of exponential decay to Eqs. (1.10)–(1.11) was assured by Muñoz Rivera and Racke [6] using boundary conditions of type Dirichlet–Neumann.

Concerning the asymptotic behavior of the Bresse system (or circular arch problem) we have only a few results. The most important is due to Liu and Rao [4], where the authors proved to thermoelastic Bresse system (with two dissipative mechanisms) that the solutions decays exponentially to zero if and only if the velocities of waves propagations are the same.

Otherwise the solution decays polynomially to zero with rates $t^{-4+\epsilon}$ or $t^{-6+\epsilon}$ provided the boundary conditions is Dirichlet–Neumann–Neumann or Dirichlet–Dirichlet–Dirichlet type respectively.

In this paper we consider only one dissipative mechanism and we get a polynomial decay $t^{-6+\epsilon}$ for any boundary condition. Additionally we show some numerical results by using the finite difference method that, in particular, is locking free.

The methods we use to show the lack of exponential stability are based on *Gearhart–Herbst–Prüss–Huang* theorem to dissipative systems. See references classical [1,8].

**Theorem 1.1.** *Let $S(t) = e^{\mathcal{A}t}$ be a $C_0$-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta\colon\ \beta \in \mathbb{R}\} \equiv i\mathbb{R} \tag{1.13}$$

*and*

$$\overline{\lim_{|\beta|\to\infty}} \left\| (i\beta I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < \infty \tag{1.14}$$

*hold, where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$.*

On the other hand our result on the polynomial stability is based on the result of Z. Liu and B. Rao's theorem on stability. See [5].

**Theorem 1.2.** *Let $S(t) = e^{\mathcal{A}t}$ be a $C_0$-semigroup on a Hilbert space. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad \textit{and} \quad \sup_{|\beta|\geqslant 1} \frac{1}{\beta^l} \left\| (i\beta I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < M \tag{1.15}$$

*for some l, then there exist $C_k$ such that*

$$\left\| e^{t\mathcal{A}} U_0 \right\| \leqslant C_k \left( \frac{\ln t}{t} \right)^{\frac{k}{l}} \ln t \|U_0\|_{D(\mathcal{A}^k)}. \tag{1.16}$$

The remaining part of this paper is organized as follows. In Section 2 we show the well possedness of the Bresse system. In Section 3 we show the lack of exponential stability. In Section 4 we show the exponential stability since that the velocities of waves propagations are equals. In Section 5 we show the polynomial rate of decay for the general case. Finally, in Section 6 we show some numerical results.

## 2. The semigroup setting

We rewrite the initial–boundary value problem (1.5)–(1.8) or (1.5)–(1.9) as a first-order system for $U := (\varphi, \varphi_t, \psi, \psi_t, w, w_t)'$, where the prime is used to denote the transpose. Then $U$ satisfies

$$U_t = \mathcal{A}_i U, \qquad U(t=0) = U_0, \tag{2.1}$$

where $U_0 := (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1)'$ and $\mathcal{A}_i$ is the (formal) differential operator

$$\mathcal{A}_i := \begin{pmatrix} 0 & I_d & 0 & 0 & 0 & 0 \\ \kappa/\rho_1 \partial_x^2 - \kappa_0 l^2 I_d/\rho_1 & 0 & \kappa/\rho_1 \partial_x & 0 & (\kappa+\kappa_0) l \partial_x & 0 \\ 0 & 0 & 0 & I_d & 0 & 0 \\ -\kappa/\rho_2 \partial_x & 0 & b/\rho_2 \partial_x^2 - \kappa/\rho_2 I_d & -\gamma/\rho_2 I_d & -\kappa l/\rho_2 I_d & 0 \\ 0 & 0 & 0 & 0 & 0 & I_d \\ -(\kappa_0+\kappa) l/\rho_1 \partial_x & 0 & l\kappa I_d/\rho_1 & 0 & \kappa_0/\rho_1 \partial_x^2 - l^2 \kappa I_d & 0 \end{pmatrix} \tag{2.2}$$

where $I_d$ is the identity operator. Let

$$\mathcal{H}_1 := H_0^1(0,L) \times L^2(0,L) \times H_*^1(0,L) \times L_*^2(0,L) \times H_*^1(0,L) \times L_*^2(0,L), \tag{2.3}$$

$$\mathcal{H}_2 := H_0^1(0,L) \times L^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \tag{2.4}$$

be the Hilbert space with norm given by

$$\begin{aligned} \|U\|_{\mathcal{H}}^2 &= \left\| (\varphi, \widetilde{\varphi}, \psi, \widetilde{\psi}, w, \widetilde{w})' \right\|_{\mathcal{H}}^2 \\ &\equiv \int_0^L \left\{ \rho_1 |\widetilde{\varphi}|^2 + \rho_2 |\widetilde{\psi}|^2 + \rho_1 |\widetilde{w}|^2 + b|\psi_x|^2 + \kappa|\varphi_x + \psi + lw|^2 + \kappa_0 |w_x - l\varphi|^2 \right\} dx. \end{aligned} \tag{2.5}$$

Here we consider,

$$L^2_*(0,L)=\left\{f\in L^2(0,L);\ \int_0^L f(x)\,dx=0\right\},\qquad H^1_*(0,L)=H^1(0,L)\cap L^2_*(0,L). \tag{2.6}$$

The domain of $\mathcal{A}_i$ for $i=1,2$ is given by:

$$D(\mathcal{A}_1)=\left\{V\in\mathcal{H}_1\ \middle|\ \varphi,\psi,w\in H^2,\ \widetilde{\varphi}\in H^1_0,\ \widetilde{\psi},\widetilde{w}\in H^1_*,\ \psi_x,w_x\in H^1_0\right\}, \tag{2.7}$$

$$D(\mathcal{A}_2)=\left\{V\in\mathcal{H}_2\ \middle|\ \varphi,\psi,w\in H^2\cap H^1_0,\ \widetilde{\varphi},\widetilde{\psi},\widetilde{w}\in H^1_0\right\}. \tag{2.8}$$

It is not difficult to see that $\mathcal{A}_i$ is a dissipative operator in the space $\mathcal{H}_i$ for which $0\in\varrho(\mathcal{A}_i)$. More precisely we have

$$(\mathcal{A}_iU,U)_{\mathcal{H}_i}=-\gamma\int_0^L|\widetilde{\psi}|^2\,dx\leqslant 0,\quad i=1,2. \tag{2.9}$$

Therefore from Lummer Phillips theorem we have that $\mathcal{A}$ is the infinitesimal generator of a contraction $C_0$ semigroup.

In that follows we omit the index $i$ of the operator $\mathcal{A}_i$, and we consider only Dirichlet boundary condition. The exception is for Section 3, where the proof is valid only for $\mathcal{A}=\mathcal{A}_1$. For the operator $\mathcal{A}_2$, the lack of exponential stability must be true also, but our method cannot be applied. Finally, we remark that to prove the exponential and polynomial decay to $\mathcal{A}_1$ is simpler and we omit here.

## 3. The lack of exponential stability

Our starting point is to show that the semigroup associated to the Bresse system is not exponential stable. To show this we will consider Dirichlet–Neumann–Neumann boundary condition given by (1.9). We use Theorem 1.1 to prove the lack of exponential stability, that is we show that there exists a sequence of values $\lambda_\nu$ such that

$$\left\|(\lambda_\mu I-\mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})}\to\infty. \tag{3.1}$$

It is equivalent to prove that there exist a sequence of data $F_\mu\in\mathcal{H}$ and a sequence of complex numbers $\lambda_\mu\in i\mathbb{R}$, with $\|F_\mu\|_{\mathcal{H}}\leqslant 1$ such that

$$\|\underbrace{(\lambda_\mu I-\mathcal{A})^{-1}F_\mu}_{U_\mu}\|_{\mathcal{H}}\to\infty \tag{3.2}$$

where

$$\lambda_\mu U_\mu-\mathcal{A}U_\mu=F_\mu \tag{3.3}$$

with $U_\mu$ not bounded. Rewriting the spectral equation in term of its components we have

$$\lambda\varphi-\widetilde{\varphi}=f_1\in H^1_0, \tag{3.4}$$

$$\rho_1\lambda\widetilde{\varphi}-\kappa(\varphi_x+\psi+lw)_x-\kappa_0 l(w_x-l\varphi)=f_2\in L^2, \tag{3.5}$$

$$\lambda\psi-\widetilde{\psi}=f_3\in H^1_*, \tag{3.6}$$

$$\rho_2\lambda\widetilde{\psi}-b\psi_{xx}+\kappa(\varphi_x+\psi+lw)+\gamma\widetilde{\psi}=f_4\in L^2_*, \tag{3.7}$$

$$\lambda w-\widetilde{w}=f_5\in H^1_*, \tag{3.8}$$

$$\rho_1\lambda\widetilde{w}-\kappa_0(w_x-l\varphi)_x+\kappa l(\varphi_x+\psi+lw)=f_6\in L^2_*. \tag{3.9}$$

Under the above notations we established the main result of this section.

**Theorem 3.1.** *Let us suppose that*

$$\frac{\rho_1}{\rho_2}\neq\frac{\kappa}{b},\quad \text{or}\quad \kappa\neq\kappa_0, \tag{3.10}$$

*then the semigroup associated to system* (1.5)–(1.7) *with boundary condition* (1.9) *is not exponentially stable.*

**Proof.** We will prove that there exists a sequence of imaginary number $\lambda_\mu$ and functions $F_\mu \in \mathcal{H}$, with $\|F_\mu\|_{\mathcal{H}} \leqslant 1$ verifying (3.2). To do this, we take $f_1 = f_3 = f_5 = 0$. Using the equation to eliminate the terms $\widetilde{\varphi}$, $\widetilde{\psi}$ and $\widetilde{w}$ we get

$$\rho_1\lambda^2\varphi - \kappa(\varphi_x + \psi + lw)_x - \kappa_0 l(w_x - l\varphi) = f_2 \in L^2, \tag{3.11}$$

$$\rho_2\lambda^2\psi - b\psi_{xx} + \kappa(\varphi_x + \psi + lw) + \gamma\lambda\psi = f_4 \in L^2_*, \tag{3.12}$$

$$\rho_1\lambda^2 w - \kappa_0(w_x - l\varphi)_x + \kappa l(\varphi_x + \psi + lw) = f_6 \in L^2_*. \tag{3.13}$$

Taking $f_2$, $f_4$, $f_6$ as

$$f_2(x) = \sin\left(\frac{\mu\pi}{L}x\right), \qquad f_4(x) = \alpha\cos\left(\frac{\mu\pi}{L}x\right), \qquad f_6(x) = 0. \tag{3.14}$$

Because of the boundary conditions given in (1.9) we can suppose that

$$\varphi = A\sin\left(\frac{\mu\pi}{L}x\right), \qquad \psi = B\cos\left(\frac{\mu\pi}{L}x\right), \qquad w = C\cos\left(\frac{\mu\pi}{L}x\right). \tag{3.15}$$

Therefore, to find a solution of system (3.11)–(3.13) is equivalent to find the solution of the following system,

$$\left[\rho_1\lambda^2 + \kappa\left(\frac{\mu\pi}{L}\right)^2 + \kappa_0 l^2\right]A + \kappa\frac{\mu\pi}{L}B + l\frac{\mu\pi}{L}C(\kappa + \kappa_0) = 1, \tag{3.16}$$

$$\kappa\frac{\mu\pi}{L}A + \left[\rho_2\lambda^2 + b\left(\frac{\mu\pi}{L}\right)^2 + \gamma\lambda + \kappa\right]B + \kappa lC = \alpha, \tag{3.17}$$

$$l(\kappa + \kappa_0)\frac{\mu\pi}{L}A + l\kappa B + \left[\rho_1\lambda^2 + \kappa_0\left(\frac{\mu\pi}{L}\right)^2 + \kappa l^2\right]C = 0. \tag{3.18}$$

First, let us assume that

$$\frac{\rho_1}{\rho_2} \neq \frac{\kappa}{b}, \qquad \kappa = \kappa_0.$$

Now we take $\lambda = \lambda_\mu$ such that

$$\rho_1\lambda^2 + \kappa\left(\frac{\mu\pi}{L}\right)^2 + \kappa l^2 = 2\kappa l\frac{\mu\pi}{L}$$

therefore the above system can be written as

$$2\kappa l\frac{\mu\pi}{L}A + \kappa\frac{\mu\pi}{L}B + 2\kappa l\frac{\mu\pi}{L}C = 1, \tag{3.19}$$

$$\kappa\frac{\mu\pi}{L}A + \left[\rho_2\lambda^2 + b\left(\frac{\mu\pi}{L}\right)^2 + \gamma\lambda + \kappa\right]B + \kappa lC = 0, \tag{3.20}$$

$$2\kappa l\frac{\mu\pi}{L}A + l\kappa B + 2\kappa l\frac{\mu\pi}{L}C = 0 \tag{3.21}$$

where we consider $\alpha = 0$. Subtracting Eq. (3.19) to Eq. (3.21) we get

$$\left[\kappa\frac{\mu\pi}{L} - l\kappa\right]B = 1 \quad\Rightarrow\quad B = \frac{1}{\kappa(\frac{\mu\pi}{L} - l)}.$$

From (3.21) we get that

$$A = -C - \frac{l}{2\kappa l\frac{\mu\pi}{L}(\frac{\mu\pi}{L} - l)}.$$

Substitution into (3.20) yields

$$\kappa\left[\frac{\mu\pi}{L} - l\right]C = -\frac{l}{2l(\frac{\mu\pi}{L} - l)} + \left[\rho_2\lambda^2 + b\left(\frac{\mu\pi}{L}\right)^2 + \gamma\lambda + \kappa\right]\frac{1}{\kappa(\frac{\mu\pi}{L} - l)}.$$

Recalling the definition of $\lambda_\mu$ we get

$$\kappa\left[\frac{\mu\pi}{L} - l\right]C = -\frac{l}{2l(\frac{\mu\pi}{L} - l)} + \left[\left(b - \frac{\rho_2}{\rho_1}\right)\left(\frac{\mu\pi}{L}\right)^2 + \frac{\rho_2}{\rho_1}2\kappa l\frac{\mu\pi}{L} + \gamma\lambda + \kappa - \frac{\rho_2}{\rho_1}\kappa l^2\right]\frac{1}{\kappa(\frac{\mu\pi}{L} - l)}.$$

Therefore as $\mu \to \infty$ we get

$$C \to \frac{1}{\kappa^2}\left(b - \frac{\rho_2}{\rho_1}\right)$$

and also

$$A \to -\frac{1}{\kappa^2}\left(b - \frac{\rho_2}{\rho_1}\right),$$

as $\mu \to \infty$, which means that

$$\begin{aligned}
\|U_\mu\|^2_{\mathcal{H}} &\geqslant \kappa \int_0^L |w_x - l\varphi|^2\,dx \\
&\geqslant \kappa \int_0^L \left|\left[C\frac{\mu\pi}{L} - lA\right]\sin\left(\frac{\mu\pi}{L}x\right)\right|^2 dx \\
&\geqslant \kappa \left|C\frac{\mu\pi}{L} - lA\right|^2 \frac{L}{2} \to \infty
\end{aligned}$$

as $\mu \to \infty$. Therefore there is not exponential stability. Now let us assume that the coefficients satisfy

$$\frac{\rho_1}{\rho_2} = \frac{\kappa}{b}, \qquad \kappa \neq \kappa_0.$$

Here we will assume that $\alpha = 1$. Multiplying Eq. (3.18) by $\kappa/l(\kappa+\kappa_0)$ we get

$$\kappa\frac{\mu\pi}{L}A + \frac{\kappa^2}{\kappa+\kappa_0}B + \frac{\kappa}{l(\kappa+\kappa_0)}\left[\rho_1\lambda^2 + \kappa_0\left(\frac{\mu\pi}{L}\right)^2 + \kappa l^2\right]C = 0.$$

Now we will choose $\lambda$ such that

$$\frac{\kappa}{l(\kappa+\kappa_0)}\left[\rho_1\lambda^2 + \kappa_0\left(\frac{\mu\pi}{L}\right)^2 + \kappa l^2\right] = \kappa l$$

that is to say, $\lambda$ is such that

$$\rho_1\lambda^2 + \kappa_0\left(\frac{\mu\pi}{L}\right)^2 + \kappa l^2 = l^2(\kappa+\kappa_0) \quad \Rightarrow \quad \rho_1\lambda^2 + \kappa_0\left(\frac{\mu\pi}{L}\right)^2 - \kappa_0 l^2 = 0.$$

Therefore, system (3.16)–(3.18) can be rewriting as

$$\left[(\kappa-\kappa_0)\left(\frac{\mu\pi}{L}\right)^2 + 2\kappa_0 l^2\right]A + k\frac{\mu\pi}{L}B + l\frac{\mu\pi}{L}C(\kappa+\kappa_0) = 1, \tag{3.22}$$

$$\kappa\frac{\mu\pi}{L}A + \left[b\left(1-\frac{\kappa_0}{\kappa}\right)\left(\frac{\mu\pi}{L}\right)^2 + \gamma\lambda - \frac{\rho_2}{\rho_1}\kappa_0 l^2 + \kappa\right]B + \kappa l C = 0, \tag{3.23}$$

$$\kappa\frac{\mu\pi}{L}A + \frac{\kappa^2}{\kappa+\kappa_0}B + \kappa l C = 1. \tag{3.24}$$

Subtracting Eq. (3.24) to (3.23) we conclude that

$$\left[b\left(1-\frac{\kappa_0}{\kappa}\right)\left(\frac{\mu\pi}{L}\right)^2 + \gamma\lambda - \frac{\rho_2}{\rho_1}\kappa_0 l^2 + \frac{\kappa\kappa_0}{\kappa+\kappa_0}\right]B = -1$$

which implies that

$$b\left(1-\frac{\kappa_0}{\kappa}\right)\left(\frac{\mu\pi}{L}\right)^2 B \to -1$$

as $\mu \to \infty$. Substituting this expression into (3.22)–(3.23) we get

$$\left[(\kappa-\kappa_0)\left(\frac{\mu\pi}{L}\right)^2+2\kappa_0 l^2\right]A+l\frac{\mu\pi}{L}C(\kappa+\kappa_0)=1-o\left(\frac{1}{\mu}\right), \tag{3.25}$$

$$\kappa\frac{\mu\pi}{L}A+\kappa lC=1+o\left(\frac{1}{\mu^2}\right). \tag{3.26}$$

So we have

$$-2\kappa_0\left[\left(\frac{\mu\pi}{L}\right)^2-l^2\right]A=-\frac{\kappa+\kappa_0}{\kappa}\frac{\mu\pi}{L}+1-o\left(\frac{1}{\mu}\right). \tag{3.27}$$

From where it follows that

$$\frac{\mu\pi}{L}A\to\frac{\kappa+\kappa_0}{2\kappa\kappa_0}. \tag{3.28}$$

From (3.24) we conclude that

$$C\to\frac{1}{\kappa l}-\frac{\kappa+\kappa_0}{2\kappa_0}.$$

Using the same argument as above we conclude that

$$\|U_\mu\|_{\mathcal{H}}\to\infty.$$

So we have not exponential stability. □

## 4. Exponential stability

In this section we will prove the semigroup associates to system (1.5)–(1.7) with boundary conditions (1.8) is exponential stable provided

$$\frac{\rho_1}{\rho_2}=\frac{\kappa}{b},\quad\text{and}\quad\kappa=\kappa_0. \tag{4.1}$$

To simplify notation we will put

$$\chi_0=\left|\rho_2-\frac{\rho_1\kappa}{b}\right|,\qquad\nu_0=\left|1-\frac{\kappa}{\kappa_0}\right|. \tag{4.2}$$

Our starting point will be the resolvent equation

$$\lambda\varphi-\widetilde{\varphi}=f_1\in H_0^1, \tag{4.3}$$

$$\rho_1\lambda\widetilde{\varphi}-\kappa(\varphi_x+\psi+lw)_x-\kappa_0 l(w_x-l\varphi)=f_2\in L^2, \tag{4.4}$$

$$\lambda\psi-\widetilde{\psi}=f_3\in H^1, \tag{4.5}$$

$$\rho_2\lambda\widetilde{\psi}-b\psi_{xx}+\kappa(\varphi_x+\psi+lw)+\gamma\widetilde{\psi}=f_4\in L^2, \tag{4.6}$$

$$\lambda w-\widetilde{w}=f_5\in H^1, \tag{4.7}$$

$$\rho_1\lambda\widetilde{w}-\kappa_0(w_x-l\varphi)_x+\kappa l(\varphi_x+\psi+lw)=f_6\in L^2. \tag{4.8}$$

Multiplying (4.3), (4.5), (4.7) by $-\kappa\overline{(\varphi_x+\psi+lw)_x}$, $-b\overline{\psi_{xx}}$ and $-\kappa_0\overline{(w_x-l\varphi)_x}$ respectively and Eqs. (4.4), (4.6), (4.8) by $\overline{\widetilde{\varphi}}$, $\overline{\widetilde{\psi}}$ and $\overline{\widetilde{w}}$ respectively adding the product resulting and taking the real part we get

$$\operatorname{Re}\lambda\|U\|_{\mathcal{H}}^2+\gamma\|\widetilde{\psi}\|_{L^2}^2=\operatorname{Re}(F,U)_{\mathcal{H}}. \tag{4.9}$$

Taking $\lambda=i\beta$ we get

$$\gamma\|\widetilde{\psi}\|_{L^2}^2=\operatorname{Re}(F,U)_{\mathcal{H}}. \tag{4.10}$$

**Remark 4.1.** Note that $i\mathbb{R}\subset\varrho(\mathcal{A})$. In fact, if $\lambda=i\beta\in\sigma(\mathcal{A})$, from identity (4.10) we have that $\widetilde{\psi}=0$. Therefore we have that $\psi=0$. From Eq. (4.6) we get that $\varphi_x+lw=0$. Substituting this identity into (4.4) and (4.8) we conclude that

$$\rho_1\lambda\widetilde{\varphi}=\kappa_0 l(w_x-l\varphi),\qquad\rho_1\lambda\widetilde{w}=\kappa_0(w_x-l\varphi)_x.$$

The above relation implies that $\varphi_x-lw=0$. Since $\varphi_x+lw=0$ we have that $\varphi=0$ and $w=0$. Therefore there is not imaginary eigenvalues.

**Lemma 4.2.** *Under the above notations we have*

$$\frac{\rho_1}{2}\int_0^L |\widetilde{\varphi}|^2\,dx+\kappa l^2\int_0^L |w|^2\,dx\leqslant C\|\varphi_x+\psi+lw\|_{L^2}^2+C\|\psi\|_{L^2}^2+\frac{C}{\beta^2}\|F\|_{\mathcal{H}}^2, \tag{4.11}$$

$$\int_0^L |\psi_x|^2\,dx\leqslant c\|\psi\|_{L^2}\|\varphi_x+\psi+lw\|_{L^2}+c\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{4.12}$$

**Proof.** Multiplying Eq. (4.4) by $\overline{\varphi}$ we get that

$$\begin{aligned}
\rho_1\int_0^L |\widetilde{\varphi}|^2\,dx &= -\kappa\,\mathrm{Re}\int_0^L (\varphi_x+\psi+lw)_x\overline{\varphi}\,dx-\kappa_0 l\,\mathrm{Re}\int_0^L (w_x-l\varphi)\overline{\varphi}\,dx+\mathrm{Re}\int_0^L f_2\overline{\varphi}\,dx\\
&\leqslant \kappa\,\mathrm{Re}\int_0^L (\varphi_x+\psi+lw)\overline{\varphi_x}\,dx+\kappa_0 l\,\mathrm{Re}\int_0^L \big(w\overline{\varphi_x}+l|\varphi|^2\big)\,dx+\mathrm{Re}\int_0^L f_2\overline{\varphi}\,dx\\
&\leqslant \kappa\int_0^L |\varphi_x+\psi+lw|^2\,dx-\kappa\,\mathrm{Re}\int_0^L (\varphi_x+\psi+lw)(\overline{\psi+lw})\,dx+\kappa_0 l\,\mathrm{Re}\int_0^L w(\overline{\varphi_x+\psi+lw})\,dx\\
&\quad-\kappa_0 l\,\mathrm{Re}\int_0^L w(\overline{\psi+lw})\,dx+\kappa_0 l^2\,\mathrm{Re}\int_0^L |\varphi|^2\,dx+\mathrm{Re}\int_0^L f_2\overline{\varphi}\,dx\\
&\leqslant \kappa\,\mathrm{Re}\int_0^L |\varphi_x+\psi+lw|^2\,dx-\kappa\,\mathrm{Re}\int_0^L (\varphi_x+\psi+lw)(\overline{\psi+lw})\,dx+\kappa_0 l\,\mathrm{Re}\int_0^L w(\overline{\varphi_x+\psi+lw})\,dx\\
&\quad-\kappa_0 l\,\mathrm{Re}\int_0^L w\overline{\psi}\,dx+\kappa_0 l^2\,\mathrm{Re}\int_0^L |\varphi|^2\,dx+\mathrm{Re}\int_0^L f_2\overline{\varphi}\,dx,
\end{aligned}$$

which implies that

$$\begin{aligned}
\rho_1\int_0^L |\widetilde{\varphi}|^2\,dx+\kappa l^2\int_0^L |w|^2\,dx &\leqslant \kappa\int_0^L |\varphi_x+\psi+lw|^2\,dx-\kappa\,\mathrm{Re}\int_0^L (\varphi_x+\psi+lw)\overline{\psi}\,dx\\
&\quad+\kappa_0 l\,\mathrm{Re}\int_0^L w(\overline{\varphi_x+\psi+lw})\,dx-\kappa_0 l\,\mathrm{Re}\int_0^L w\overline{\psi}\,dx\\
&\quad+\kappa_0 l^2\int_0^L |\varphi|^2\,dx+\mathrm{Re}\int_0^L f_2\overline{\varphi}\,dx.
\end{aligned}$$

From where it follows that

$$\frac{\rho_1}{2}\int_0^L |\widetilde{\varphi}|^2\,dx+\kappa l^2\int_0^L |w|^2\,dx\leqslant C\|\varphi_x+\psi+lw\|_{L^2}^2+C\|\psi\|_{L^2}^2+\frac{C}{\beta^2}\|F\|_{\mathcal{H}}^2$$

for $\lambda=i\beta$ large enough and $C$ constant positive. So the first part of Lemma 4.2 follows. Now, multiplying (4.6) by $\psi$ we get

$$\rho_2\lambda\int_0^L \widetilde{\psi}\overline{\psi}\,dx+b\int_0^L |\psi_x|^2\,dx+\kappa\int_0^L (\varphi_x+\psi+lw)\overline{\psi}\,dx+\gamma\int_0^L \widetilde{\psi}\overline{\psi}\,dx=\int_0^L f_4\overline{\psi}\,dx$$

therefore,

$$\frac{b}{2}\int_0^L |\psi_x|^2\,dx \leqslant \rho_2\int_0^L |\widetilde{\psi}|^2\,dx + \kappa\,\mathrm{Re}\int_0^L (\varphi_x+\psi+lw)\overline{\psi}\,dx + c\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Using (4.10) and Eq. (4.5) to $\overline{\psi}$ for $|\lambda|\geqslant 1$ we get

$$\int_0^L |\psi_x|^2\,dx \leqslant c\|\psi\|_{L^2}\|\varphi_x+\psi+lw\|_{L^2} + c\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} \tag{4.13}$$

from where the second part of the lemma follows. □

Let us denote by

$$I_\psi := \left|\psi_x(L)\right|^2 + \left|\psi_x(0)\right|^2, \tag{4.14}$$

$$I_w := \left|w_x(L)\right|^2 + \left|w_x(0)\right|^2, \tag{4.15}$$

$$I_\varphi := \left|\varphi_x(L)\right|^2 + \left|\varphi_x(0)\right|^2. \tag{4.16}$$

**Lemma 4.3.** *Under the above notations there exists a positive constant $C$ such that*

$$I_\psi \leqslant C\|\varphi_x+\psi+lw\|_{L^2}\|\widetilde{\psi}\|_{L^2} + C\|U\|\|\psi\|_{L^2} + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}, \tag{4.17}$$

$$I_w \leqslant C\|U\|_{\mathcal{H}}^2 + C\|F\|_{\mathcal{H}}^2, \tag{4.18}$$

$$I_\varphi \leqslant C\|U\|_{\mathcal{H}}^2 + C\|F\|_{\mathcal{H}}^2. \tag{4.19}$$

*Additionally we have*

$$I_\varphi \leqslant C\mathcal{N} \tag{4.20}$$

*where*

$$\mathcal{N} := \|\varphi_x+\psi+lw\|_{L^2}^2 + \|\varphi_x+\psi+lw\|_{L^2}\|U\|_{\mathcal{H}} + \|\psi\|_{L^2}\|U\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + \frac{1}{\beta^2}\|F\|_{\mathcal{H}}^2.$$

**Proof.** Using Eqs. (4.5) and (4.6) we get

$$-\rho_2\beta^2\psi - b\psi_{xx} + \kappa(\varphi_x+\psi+lw) + \gamma\widetilde{\psi} = \rho_2 i\beta f_3 + f_4. \tag{4.21}$$

Multiplying the above expression by $q\overline{\psi_x}$ where $q = x - L/2$. Then

$$\begin{aligned}
\mathrm{Re}\int_0^L q\overline{\psi_x}\kappa(\varphi_x+\psi+lw)\,dx &= -\mathrm{Re}\,\kappa\int_0^L q\overline{\psi}(\varphi_x+\psi+lw)_x\,dx \\
&= -\mathrm{Re}\,\kappa\int_0^L q\overline{\psi}\left[\rho_1\lambda\widetilde{\varphi} - \kappa_0 l(w_x - l\varphi) - f_2\right]dx \\
&\leqslant C\int_0^L |\widetilde{\psi}\overline{\widetilde{\varphi}}|\,dx + \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|\psi\|_{L^2}.
\end{aligned}$$

Therefore we have that

$$I_\psi \leqslant C\int_0^L |\widetilde{\psi}\overline{\widetilde{\varphi}}|\,dx + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}\|\psi\|_{L^2} + c\|\psi_x\|_{L^2}^2.$$

Using Lemma 4.2 and Cauchy–Schwartz inequality the first inequality follows. To get the other inequality let us multiply Eq. (4.4) by $q\overline{(\varphi_x+\psi+lw)}$ and taking real part we get

$$\frac{\kappa}{2}\int_0^L q\frac{d}{dx}|\varphi_x+\psi+lw|^2\,dx=\rho_1\operatorname{Re}\lambda\int_0^L \widetilde{\varphi}q(\overline{\varphi_x+\psi+lw})\,dx-\kappa_0 l\operatorname{Re}\int_0^L (w_x-l\varphi)(\overline{\varphi_x+\psi+lw})\,dx$$
$$-\operatorname{Re}\int_0^L f_2(\overline{\varphi_x+\psi+lw})\,dx.$$

Note that

$$\rho_1\operatorname{Re}\lambda\int_0^L \widetilde{\varphi}q(\overline{\varphi_x+\psi+lw})\,dx=-\rho_1\operatorname{Re}\beta^2\int_0^L \varphi q(\overline{\varphi_x+\psi+lw})\,dx-\rho_1\operatorname{Re}i\beta\int_0^L qf_1(\overline{\varphi_x+\psi+lw})\,dx$$
$$\leqslant c\|\widetilde{\varphi}\|_{L^2}^2+c\|\widetilde{\varphi}\|_{L^2}\|U\|_{\mathcal{H}}+c\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Using Lemma 4.2, identity (4.20) follows. Finally, multiplying Eq. (4.8) by $\overline{q(w_x-l\varphi)}$, and using the same above argument our conclusion follows. □

**Lemma 4.4.** *For any $\epsilon>0$ there exists $c_\epsilon>0$, such that the solution of Bresse system satisfies*

$$|\psi_x\varphi_x|_{x=0}^{x=L}|\leqslant\epsilon^2\|\varphi_x+\psi+lw\|_{L^2}^2+\epsilon^2\big(1+\nu_0/\beta^2\big)\|U\|_{\mathcal{H}}^2+c_\epsilon^2\big(1+\nu_0\beta^{10}\big)\|F\|_{\mathcal{H}}^2.$$

**Proof.** From Lemma 4.3 we have that

$$I_\psi\leqslant c\|\varphi_x+\psi+lw\|_{L^2}\|\widetilde{\psi}\|_{L^2}+c\mathcal{R}$$

where

$$\mathcal{R}:=\|\psi\|_{L^2}\|U\|_{\mathcal{H}}+\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Therefore from (4.20) and the above inequality we get

$$\begin{aligned}|\psi_x\varphi_x|_{x=0}^{x=L}|&\leqslant cI_\psi^{1/2}\big(\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}\big)\\&\leqslant c\big[\|\varphi_x+\psi+lw\|_{L^2}^{1/2}\|\widetilde{\psi}\|_{L^2}^{1/2}+\mathcal{R}^{1/2}\big]\big(\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}\big)\\&\leqslant\underbrace{c\|\varphi_x+\psi+lw\|_{L^2}^{1/2}\|U\|_{\mathcal{H}}\|\widetilde{\psi}\|_{L^2}^{1/2}}_{:=J_0}+c\mathcal{R}^{1/2}\|U\|_{\mathcal{H}}+\underbrace{\|\varphi_x+\psi+lw\|_{L^2}^{1/2}\|\widetilde{\psi}\|_{L^2}^{1/2}\|F\|_{\mathcal{H}}}_{\leqslant c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}}\\&\quad+\underbrace{c\mathcal{R}^{1/2}\|F\|_{\mathcal{H}}}_{c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+c\|F\|_{\mathcal{H}}^2}.\end{aligned}$$

Using inequality (4.10) we get

$$\begin{aligned}J_0&\leqslant\epsilon\|\varphi_x+\psi+lw\|_{L^2}^2+\left(\frac{c}{|\beta|^{4/3}}\|U\|_{\mathcal{H}}^{4/3}\right)\big(|\beta|^{4/3}\|\widetilde{\psi}\|_{L^2}^{2/3}\big)\\&\leqslant\epsilon\|\varphi_x+\psi+lw\|_{L^2}^2+\frac{\epsilon^2}{2|\beta|^2}\|U\|_{\mathcal{H}}^2+c_\epsilon|\beta|^4\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}\\&\leqslant\epsilon\|\varphi_x+\psi+lw\|_{L^2}^2+\frac{\epsilon^2}{|\beta|^2}\|U\|_{\mathcal{H}}^2+c_\epsilon|\beta|^{10}\|F\|_{\mathcal{H}}^2,\end{aligned}$$

$$\begin{aligned}c\mathcal{R}^{1/2}\|U\|_{\mathcal{H}}&\leqslant c\|\psi\|_{L^2}^{1/2}\|U\|_{\mathcal{H}}^{3/2}+c\|U\|_{\mathcal{H}}^{3/2}\|F\|_{\mathcal{H}}^{1/2}\\&\leqslant\frac{c}{|\beta|}\|\widetilde{\psi}\|_{L^2}^{1/2}\|U\|_{\mathcal{H}}^{3/2}+\frac{c}{|\beta|}\|F\|_{\mathcal{H}}^{1/2}\|U\|_{\mathcal{H}}^{3/2}+c\|U\|_{\mathcal{H}}^{3/2}\|F\|_{\mathcal{H}}^{1/2}\\&\leqslant c\big(|\beta|^{1/2}\|\widetilde{\psi}\|_{L^2}^{1/2}\big)\left(\frac{1}{|\beta|^{3/2}}\|U\|_{\mathcal{H}}^{3/2}\right)+|\beta|^{1/2}\|F\|_{\mathcal{H}}^{1/2}\frac{1}{|\beta|^{3/2}}\|U\|_{\mathcal{H}}^{3/2}+c\frac{1}{|\beta|^{3/2}}\|U\|_{\mathcal{H}}^{3/2}|\beta|^{3/2}\|F\|_{\mathcal{H}}^{1/2}\\&\leqslant\frac{\epsilon^2}{|\beta|^2}\|U\|_{\mathcal{H}}^2+c_\epsilon\beta^2\|\widetilde{\psi}\|_{L^2}^2+c_\epsilon\beta^6\|F\|_{\mathcal{H}}^2.\end{aligned}$$

From where we get that

$$|\psi_x\varphi_x|_{x=0}^{x=L}| \leqslant \epsilon\|\varphi_x+\psi+lw\|_{L^2}^2+\frac{\epsilon^2}{|\beta|^2}\|U\|_{\mathcal{H}}^2+c_\epsilon|\beta|^{10}\|F\|_{\mathcal{H}}^2. \tag{4.22}$$

On the other hand, using similar arguments it is not difficult to see that

$$|\psi_x\varphi_x|_{x=0}^{x=L}| \leqslant \epsilon\|\varphi_x+\psi+lw\|_{L^2}^2+\epsilon^2\|U\|_{\mathcal{H}}^2+c_\epsilon\|F\|_{\mathcal{H}}^2. \tag{4.23}$$

Multiplying Eq. (4.22) by $\nu_0$ and summing up with (4.23) our conclusion follows. □

**Lemma 4.5.** *Under the above notations we have that the solution of the resolvent system satisfies*

$$\|\varphi_x+\psi+lw\|_{L^2}^2 \leqslant \epsilon^2\left(1+\frac{\nu_0+\chi}{\beta^2}\right)\|U\|_{\mathcal{H}}^2+c_\epsilon^2\big(1+\chi\beta^6+\nu_0\beta^{10}\big)\|F\|_{\mathcal{H}}^2.$$

**Proof.** Multiplying Eq. (4.6) by $\overline{\varphi_x+\psi+lw}$ we get

$$\kappa\|\varphi_x+\psi+lw\|_{L^2}^2 = -\rho_2\lambda\int_0^L \widetilde{\psi}\,\overline{(\varphi_x+\psi+lw)}\,dx - b\int_0^L \psi_x\overline{(\varphi_x+\psi+lw)_x}\,dx$$
$$-\gamma\int_0^L \widetilde{\psi}\,\overline{(\varphi_x+\psi+lw)}\,dx+\int_0^L f_4\overline{(\varphi_x+\psi+lw)}\,dx+b\operatorname{Re}\psi_x\overline{\varphi_x}|_{x=0}^{x=L}.$$

Using (4.4) we get

$$\begin{aligned}
\frac{\kappa}{2}\|\varphi_x+\psi+lw\|_{L^2}^2 &\leqslant -\operatorname{Re}\rho_2\lambda\int_0^L \widetilde{\psi}\,\overline{\varphi_x}\,dx+\operatorname{Re}\frac{b}{\kappa}\int_0^L \psi_x\big[\overline{\rho_1\lambda\widetilde{\varphi}-\kappa_0 l(w_x-l\varphi)-f_2}\big] \\
&\quad + l\operatorname{Re}\int_0^L \widetilde{\psi}\,\overline{\widetilde{w}}\,dx + C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+b\operatorname{Re}\psi_x\overline{\varphi_x}|_{x=0}^{x=L} \\
&\leqslant -\rho_2\operatorname{Re}\lambda\int_0^L \widetilde{\psi}\,\overline{\varphi_x}\,dx-\rho_1\frac{b}{\kappa}\operatorname{Re}\int_0^L \lambda\widetilde{\psi}\,\overline{\varphi}_x\,dx+C\|U\|_{\mathcal{H}}\|\widetilde{\psi}\|_{L^2} \\
&\quad \underbrace{-\kappa_0 l\frac{b}{\kappa}\operatorname{Re}\int_0^L \psi_x\overline{(w_x-l\varphi)}\,dx}_{:=J_1} + C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+b\operatorname{Re}\psi_x\overline{\varphi_x}|_{x=0}^{x=L} \\
&\leqslant -\left(\rho_2-\rho_1\frac{b}{\kappa}\right)\operatorname{Re}\lambda\int_0^L \widetilde{\psi}\,\overline{\varphi_x}\,dx+J_1+C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}+C\|U\|_{\mathcal{H}}\|\widetilde{\psi}\|_{L^2}+b\operatorname{Re}\psi_x\overline{\varphi_x}|_{x=0}^{x=L}.
\end{aligned} \tag{4.24}$$

Note that

$$\operatorname{Re}\lambda\int_0^L \widetilde{\psi}\,\overline{\varphi_x}\,dx = \operatorname{Re}\lambda\int_0^L \widetilde{\psi}\,\overline{(\varphi_x+\psi+lw)}\,dx-\operatorname{Re}\lambda\int_0^L \widetilde{\psi}\,\overline{(\psi+lw)}\,dx$$
$$\leqslant \frac{\kappa}{4}\|\varphi_x+\psi+lw\|_{L^2}^2+c\chi_0\beta^2\|\widetilde{\psi}\|_{L^2}^2+C\|U\|_{\mathcal{H}}\|\widetilde{\psi}\|_{L^2}+C\|U\|_{\mathcal{H}}\|F\|_{L^2}.$$

On the other hand, using Eq. (4.8) we get

$$\begin{aligned}\operatorname{Re} J_1 &= l\frac{b}{\kappa}\operatorname{Re}\int_0^L \psi\overline{\left[\rho_1\lambda\widetilde{w}+\kappa l(\varphi_x+\psi+lw)-f_6\right]}\,dx\\ &\leqslant c\int_0^L |\widetilde{\psi}||\widetilde{w}|\,dx + c\int_0^L |\psi||\varphi_x+\psi+lw|\,dx + C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}\\ &\leqslant C\|U\|_{\mathcal{H}}\|\widetilde{\psi}\|_{L^2} + C\|U\|_{\mathcal{H}}\|F\|_{L^2}.\end{aligned}$$

Inserting this inequality into Eq. (4.24) using (4.5) and Lemma 4.4 for $\epsilon < k/4$ and $|\beta| \geqslant 1$, our conclusion follows. □

**Remark 4.6.** The above lemma in particular implies that

$$\mathcal{N} \leqslant \delta\|U\|^2_{\mathcal{H}} + c_\delta\|\varphi_x+\psi+lw\|^2_{L^2} + c_\delta\|F\|^2_{\mathcal{H}}.$$

Therefore Lemma 4.3 we get that

$$I_\varphi \leqslant \delta c\|U\|^2_{\mathcal{H}} + c_\delta\|\varphi_x+\psi+lw\|^2_{L^2} + c_\delta\|F\|^2_{\mathcal{H}}.$$

**Lemma 4.7.** *There exists a positive constant c such that*

$$\int_0^L \left(\rho_1|\widetilde{\varphi}|^2 + \rho_2|\widetilde{\psi}|^2 + \rho_1|\widetilde{w}|^2\right)dx + \kappa_0 l\int_0^L |w_x - l\varphi|^2\,dx + \rho_1 l\int_0^L |\widetilde{w}|^2\,dx$$

$$\leqslant c_1\epsilon^2\|U\|^2_{\mathcal{H}} + c_\epsilon^2\left(1+\chi\beta^6+\nu_0\beta^{10}\right)\|F\|^2_{\mathcal{H}},$$

*for $|\beta| > 1$, large enough.*

**Proof.** Multiplying Eq. (4.4) by $\overline{(w_x - l\varphi)}$ we have

$$\begin{aligned}\kappa_0 l\int_0^L |w_x-l\varphi|^2\,dx &= \int_0^L \left[\rho_1\lambda\widetilde{\varphi} - \kappa(\varphi_x+\psi+lw)_x - f_2\right]\overline{(w_x-l\varphi)}\,dx\\ &= \int_0^L \rho_1\lambda\widetilde{\varphi}\overline{(w_x-l\varphi)}\,dx + \kappa\int_0^L (\varphi_x+\psi+lw)\overline{(w_x-l\varphi)}_x\,dx\\ &\quad - \int_0^L f_2\overline{(w_x-l\varphi)}\,dx - \operatorname{Re}\kappa\varphi_x\overline{w_x}\big|_{x=0}^{x=L}\\ &\leqslant \operatorname{Re}\int_0^L \rho_1\lambda\widetilde{\varphi}\overline{w_x}\,dx + l\rho_1\int_0^L |\widetilde{\varphi}|^2\,dx + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}\\ &\quad + \frac{\kappa}{\kappa_0}\operatorname{Re}\int_0^L (\varphi_x+\psi+lw)\overline{\left[\rho_1\lambda\widetilde{w}+\kappa l(\varphi_x+\psi+lw)-f_6\right]}\,dx - \operatorname{Re}\kappa\varphi_x\overline{w_x}\big|_{x=0}^{x=L}\\ &\leqslant \rho_1\left(1-\frac{\kappa}{\kappa_0}\right)\operatorname{Re}\int_0^L \lambda\widetilde{\varphi}\overline{w_x}\,dx + l\rho_1\int_0^L |\widetilde{\varphi}|^2\,dx + C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}\\ &\quad + \frac{\kappa^2}{\kappa_0}l\int_0^L |\varphi_x+\psi+lw|^2\,dx - \frac{\kappa}{\kappa_0}\rho_1\operatorname{Re}\int_0^L \widetilde{\psi}\overline{\widetilde{w}}\,dx - \frac{\kappa}{\kappa_0}\rho_1 l\int_0^L |\widetilde{w}|^2\,dx - \operatorname{Re}\kappa\varphi_x\overline{w_x}\big|_{x=0}^{x=L}.\end{aligned}$$

From where we get

$$\kappa_0 l\int_0^L |w_x-l\varphi|^2\,dx+\frac{\kappa}{\kappa_0}\rho_1 l\int_0^L |\widetilde{w}|^2\,dx\leqslant c\big(\nu_0\beta^2+1\big)\int_0^L |\widetilde{\varphi}|^2\,dx+C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}+\frac{\kappa^2}{\kappa_0}l\int_0^L |\varphi_x+\psi+lw|^2\,dx$$
$$-\frac{\kappa}{\kappa_0}\rho_1\int_0^L \widetilde{\psi}\,\widetilde{w}\,dx-\operatorname{Re}\kappa\varphi_x\overline{w_x}|_{x=0}^{x=L}.$$

Using Lemma 4.5 we conclude that there exists a positive constant $c$ such that

$$\int_0^L |w_x-l\varphi|^2\,dx+\int_0^L |\widetilde{w}|^2\,dx\leqslant c\big(\nu_0\beta^2+1\big)\int_0^L |\widetilde{\varphi}|^2\,dx+\epsilon^2\left(1+\frac{\nu_0+\chi}{\beta^2}\right)\|U\|_{\mathcal{H}}^2$$
$$+c_\epsilon^2\big(1+\chi\beta^6+\nu_0\beta^{10}\big)\|F\|_{\mathcal{H}}^2-\operatorname{Re}k\varphi_x\overline{w_x}|_{x=0}^{x=L} \tag{4.25}$$

for $|\beta|>1$ large enough. Using Lemma 4.2 and Lemma 4.5 we get

$$\rho_1\int_0^L |\widetilde{\varphi}|^2\,dx+\kappa l^2\int_0^L |w|^2\,dx\leqslant\epsilon^2\left(1+\frac{\nu_0+\chi}{\beta^2}\right)\|U\|_{\mathcal{H}}^2+c_\epsilon^2\big(1+\chi\beta^6+\nu_0\beta^{10}\big)\|F\|_{\mathcal{H}}^2.$$

Inserting this inequality into (4.25) we get

$$\kappa_0 l\int_0^L |w_x-l\varphi|^2\,dx+\rho_1 l\int_0^L |\widetilde{w}|^2\,dx\leqslant c_0\epsilon^2\|U\|_{\mathcal{H}}^2+c_\epsilon^2\big(1+\chi\beta^6+\nu_0^2\beta^{12}\big)\|F\|_{\mathcal{H}}^2-\operatorname{Re}\kappa\varphi_x\overline{w_x}|_{x=0}^{x=L}. \tag{4.26}$$

From Remark 4.6 we conclude that

$$\begin{aligned}\operatorname{Re}\kappa\varphi_x\overline{w_x}|_{x=0}^{x=L}&\leqslant I_\varphi^{1/2}I_w^{1/2}\\&\leqslant\big(c_0\epsilon^2\|U\|_{\mathcal{H}}^2+c_\epsilon^2\big(1+\chi\beta^6+\nu_0\beta^{10}\big)\|F\|_{\mathcal{H}}^2\big)^{1/2}\big(\|U\|_{\mathcal{H}}+\|F\|_{\mathcal{H}}\big)\\&\leqslant c_1\epsilon^2\|U\|_{\mathcal{H}}^2+c_\epsilon^2\big(1+\chi\beta^6+\nu_0^2\beta^{12}\big)\|F\|_{\mathcal{H}}^2.\end{aligned}$$

Multiplying Eqs. (4.4), (4.6), (4.8) by $\overline{\varphi}$, $\overline{\psi}$ and $\overline{w}$ respectively adding the product result and taking the real part we get

$$\int_0^L\big(\rho_1|\widetilde{\varphi}|^2+\rho_2|\widetilde{\psi}|^2+\rho_1|\widetilde{w}|^2\big)\,dx\leqslant b\int_0^L |\psi_x|^2\,dx+\kappa\int_0^L |\varphi_x+\psi+lw|^2\,dx+\kappa_0\int_0^L |w_x-l\varphi|^2\,dx$$
$$+\gamma\int\widetilde{\psi}\,\overline{\psi}\,dx+c\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Using (4.26), Lemma 4.5 and Lemma 4.2 we get

$$\int_0^L\big(\rho_1|\widetilde{\varphi}|^2+\rho_2|\widetilde{\psi}|^2+\rho_1|\widetilde{w}|^2\big)\,dx\leqslant c_1\epsilon^2\|U\|_{\mathcal{H}}^2+c_\epsilon^2\big(1+\chi\beta^6+\nu_0^2\beta^{12}\big)\|F\|_{\mathcal{H}}^2,$$

from where our conclusion follows. □

Now we are in conditions to show the main result of this paper.

**Theorem 4.8.** *The semigroup associates to system* (1.5)–(1.7) *more boundary conditions* (1.8) *is exponentially stable if and only if*

$$\frac{\rho_1}{\rho_2}=\frac{k}{b}\quad\text{and}\quad k=k_0.$$

**Proof.** Summing up the inequalities of Lemma 4.5 and Lemma 4.7 and since $\nu_0=0$ and $\chi_0=0$ we get that

$$\|U\|_{\mathcal{H}}^2\leqslant\epsilon c_2\|U\|_{\mathcal{H}}^2+c_\epsilon\|F\|_{\mathcal{H}}^2$$

and for $\epsilon$ small we get

$$\|U\|_{\mathcal{H}}^2 \leqslant c_\epsilon \|F\|_{\mathcal{H}}^2. \qquad \square$$

Therefore the system is exponentially stable.

## 5. Polynomial rate of decay

Here we will prove that the solution decays polynomially with rates that depends on the regularity of the initial data and some relationships between the coefficients.

**Theorem 5.1.** *The semigroup associates to system* (1.5)–(1.7) *with boundary conditions* (1.8) *satisfies the following polynomial decay*

$$\left\|e^{\mathcal{A}t}U_0\right\|_{\mathcal{H}} \leqslant \frac{C}{t^{1/6-\epsilon}}\|U\|_{D(\mathcal{A})}.$$

*Moreover, if*

$$\frac{\rho_1}{\rho_2} \neq \frac{\kappa}{b} \quad \textit{and} \quad \kappa = \kappa_0,$$

*then we have that*

$$\left\|e^{\mathcal{A}t}U_0\right\|_{\mathcal{H}} \leqslant \frac{C}{t^{1/3-\epsilon}}\|U\|_{D(\mathcal{A})}.$$

**Proof.** Summing up the inequalities of Lemma 4.5 and Lemma 4.7 we get that

$$\|U\|_{\mathcal{H}}^2 \leqslant c_1\epsilon^2\|U\|_{\mathcal{H}}^2 + c_\epsilon^2\left(1 + \chi\beta^6 + \nu_0^2\beta^{12}\right)\|F\|_{\mathcal{H}}^2.$$

For $\epsilon$ small we get

$$\|U\|_{\mathcal{H}}^2 \leqslant c\beta^{12}\|F\|_{\mathcal{H}}^2.$$

Using Liu's result we conclude that

$$\|U(t)\|_{\mathcal{H}} \leqslant \frac{C}{t^{1/6-\epsilon}}\|U\|_{D(A)}.$$

In the case of $\nu_0 = 0$, we have that

$$\|U\|_{\mathcal{H}}^2 \leqslant c\beta^6\|F\|_{\mathcal{H}}^2.$$

From where our conclusion follows. $\square$

## 6. The numerical schemes and results

To numerical certification of the ours analytical results we use the explicit time integration method in finite difference applied to system (1.5)–(1.8). Firstly, we point out a numerical anomaly know as *shear locking*. This numerical problem is common in finite element method by using the linear shape functions when applied to structures mechanics like plates and beams. For more details of this numerical problem see [2,7]. In our case the numerical scheme in finite difference is locking free.

Our numerical approach applied to system (1.5)–(1.8) is make on two discretizations in finite difference: the approach spatial semi-discrete and the totally discrete. In both cases we have a numerical energy and, by using of the totally discrete scheme in finite difference we proved numerically the issues concerning to lack of exponential stability of the sections earlier. We believe that this approach numerical to prove the lack of exponential decay is new for dynamic system of beams.

### 6.1. The semi-discrete scheme in finite difference

To our purposes, let us denote by $J$ a positive integer number, by $h = \frac{L}{J+1}$ we denote the spatial step size and a uniform partition given by

$$0 = x_0 < x_1 < \cdots < x_J < x_{J+1} = L, \tag{6.1}$$

with $x_j = jh$. The numerical equations applied to Eqs. (1.5)–(1.7) with $F_1 = F_3 = 0$ and $F_2 = -\gamma\psi_t$, is given by

$$\rho_1\varphi_j'' = \kappa\frac{\varphi_{j+1}-2\varphi_j+\varphi_{j-1}}{h^2} + \kappa\frac{\psi_{j+1}-\psi_{j-1}}{2h} + \bar{\kappa}\frac{w_{j+1}-w_{j-1}}{2h} - \kappa_0 l^2\frac{\varphi_{j+1}+2\varphi_j+\varphi_{j-1}}{4}, \tag{6.2}$$

$$\rho_2\psi_j'' = b\frac{\psi_{j+1}-2\psi_j+\psi_{j-1}}{h^2} - \kappa\frac{\varphi_{j+1}-\varphi_{j-1}}{2h} - \kappa\frac{\psi_{j+1}+2\psi_j+\psi_{j-1}}{4} - \kappa l\frac{w_{j+1}+2w_j+w_{j-1}}{4} - \gamma\psi_j', \tag{6.3}$$

$$\rho_1 w_j'' = \kappa_0\frac{w_{j+1}-2w_j+w_{j-1}}{h^2} - \bar{\kappa}\frac{\varphi_{j+1}-\varphi_{j-1}}{2h} - \kappa l\frac{\psi_{j+1}+2\psi_j+\psi_{j-1}}{4} - \kappa l^2\frac{w_{j+1}+2w_j+w_{j-1}}{4}. \tag{6.4}$$

Here $\bar{\kappa} = l(\kappa+\kappa_0)$, for $j = 1, 2, \ldots, J$. We denote $\varphi(jh,t) = \varphi_j(t)$, $\psi(jh,t) = \psi_j(t)$ and $w(jh,t) = w_j(t)$. For $'$ we denote the derivative with respect to the time and the discretizations with respect to space variable are all of second order in $h$.

Numerical approaches like (6.2)–(6.4) was used by Wright [10,11] to treat the issues related to numerical stability for Timoshenko beam problem in one-dimensional. In fact, making $\gamma = 0$ and $l \to 0$ we get

$$\rho_1\varphi_j'' = \kappa\frac{\varphi_{j+1}-2\varphi_j+\varphi_{j-1}}{h^2} + \kappa\frac{\psi_{j+1}-\psi_{j-1}}{2h}, \tag{6.5}$$

$$\rho_2\psi_j'' = b\frac{\psi_{j+1}-2\psi_j+\psi_{j-1}}{h^2} - \kappa\frac{\varphi_{j+1}-\varphi_{j-1}}{2h} - \kappa\frac{\psi_{j+1}+2\psi_j+\psi_{j-1}}{4}, \tag{6.6}$$

$$\rho_1 w_j'' = \kappa_0\frac{w_{j+1}-2w_j+w_{j-1}}{h^2}. \tag{6.7}$$

We point out that Eq. (6.7) to displacement longitudinal can be negligible for plane beams. See [9]. Thus, the numerical equations (6.5)–(6.6) are the same discretizations assumed by Wright in [10,11] to analyze the questions about numerical stability of the respective explicit time integration method and this numerical equations are locking free. The *shear locking problem* to some numerical methods is characterized by the following over-estimation about the coefficient $b$,

$$\mathbf{b}_{\mathbf{h}}^* = b\left(1 + \frac{\kappa}{12b}h^2\right).$$

It is clear that numerical alternatives to this problem was performed in the literature and to more details we indicate the classical references by Hughes et al. [2] and Prathap and Bhashyam [7]. In our case, the semi-discrete scheme (6.2)–(6.4) is *locking free*, because the respective energy is given by

$$E_h(t) := \frac{h}{2}\sum_{j=0}^{J}\Bigg[\rho_1\big|\varphi_j'(t)\big|^2 + \rho_2\big|\psi_j'(t)\big|^2 + \rho_1\big|w_j'(t)\big|^2 + b\left|\frac{\psi_{j+1}(t)-\psi_j(t)}{h}\right|^2 + \kappa_0\left|\frac{w_{j+1}(t)-w_j(t)}{h} - l\frac{\varphi_{j+1}(t)+\varphi_j(t)}{2}\right|^2 + \kappa\left|\frac{\varphi_{j+1}(t)-\varphi_j(t)}{h} + \frac{\psi_{j+1}(t)+\psi_j(t)}{2} + l\frac{w_{j+1}(t)+w_j(t)}{2}\right|^2\Bigg].$$

In the energy $E_h(t)$ the coefficients $b, \kappa$ and $\kappa_0$ are exactly those of the continuous case (1.5)–(1.7) for $F_i = 0$, $i = 1, 2, 3$. In fact, there exists a compatibility of $E_h(t)$ with the continuous energy given by

$$E(t) := \frac{\rho_1}{2}\int_0^L |\varphi_t|^2\,dx + \frac{\rho_2}{2}\int_0^L |\psi_t|^2\,dx + \frac{\rho_1}{2}\int_0^L |w_t|^2\,dx + \frac{b}{2}\int_0^L |\psi_x|^2\,dx + \frac{\kappa}{2}\int_0^L |\varphi_x+\psi+lw|^2\,dx + \frac{\kappa_0}{2}\int_0^L |w_x - l\varphi|^2\,dx. \tag{6.8}$$

Moreover, we have the following dissipation law in the numerical context analogous to continuous case:

$$\frac{d}{dt}E_h(t) = -h\gamma\sum_{j=1}^{J}\big|\psi_j'(t)\big|^2 \leqslant 0, \quad \forall t \geqslant 0. \tag{6.9}$$

Therefore

$$E_h(t) \leqslant E_h(0), \quad \forall t \geqslant 0. \tag{6.10}$$

As in the continuous case, the technical procedure for obtaining the energy $E_h(t)$ is analogous to the continuous case for discretizations appropriate of the boundary conditions and the proof we omitted in this work.

*6.2. The Discrete scheme in finite difference*

To numerical experiments we applied the temporal discretization in finite difference to numerical equations in (6.2)–(6.4). For the aim, we consider the discretization with respect to the time $t$ given by

$$t_0 = 0 < t_1 = \Delta t < \cdots < t_n < \cdots < t_N = N\Delta t < t_{N+1} = T, \tag{6.11}$$

where $t_n = n\Delta t$ for $n = 0, 1, 2, \ldots, N+1$ and we make

$$h = \Delta x = \frac{L}{J+1}, \qquad \Delta t = \frac{T}{N+1}, \quad J, N \in \mathbb{N} \tag{6.12}$$

with $x_j = j\Delta x$, $j = 0, 1, 2, \ldots, J+1$ and $t_n = n\Delta t$. The fully discrete system is then given by

$$\begin{aligned}
&\rho_1 \bar{\partial}_t \partial_t \varphi_j^n = \kappa \bar{\partial}_x \partial_x \varphi_j^n + \kappa \frac{\partial_x + \bar{\partial}_x}{2} \psi_j^n + \bar{\kappa} \frac{\partial_x + \bar{\partial}_x}{2} w_j^n - \frac{\kappa_0 l^2}{2} \left(\varphi_{j-1/2}^n + \varphi_{j+1/2}^n\right), \\
&\rho_2 \bar{\partial}_t \partial_t \psi_j^n = b \bar{\partial}_x \partial_x \psi_j^n - \kappa \frac{\partial_x + \bar{\partial}_x}{2} \varphi_j^n - \frac{\kappa}{2} \left(\psi_{j-1/2}^n + \psi_{j+1/2}^n\right) - \frac{\kappa l}{2} \left(w_{j-1/2}^n + w_{j+1/2}^n\right) - \gamma \frac{\partial_t + \bar{\partial}_t}{2} \psi_j^n, \\
&\rho_1 \bar{\partial}_t \partial_t w_j^n = \kappa_0 \bar{\partial}_x \partial_x w_j^n - \bar{\kappa} \frac{\partial_x + \bar{\partial}_x}{2} \varphi_j^n - \frac{\kappa l}{2} \left(\psi_{j-1/2}^n + \psi_{j+1/2}^n\right) - \frac{\kappa l^2}{2} \left(w_{j-1/2}^n + w_{j+1/2}^n\right)
\end{aligned}$$

where we assumed the following operators:

$$\frac{\partial_x + \bar{\partial}_x}{2} u_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}, \qquad \frac{\partial_t + \bar{\partial}_t}{2} u_j^n = \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t}, \tag{6.13}$$

$$\bar{\partial}_x \partial_x u_j^n = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}, \qquad \bar{\partial}_t \partial_t u_j^n = \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2}. \tag{6.14}$$

The approximations of the type $u_{j-1/2}^n$ and $u_{j+1/2}^n$ denote the average of $u$ on the points $(x_{j-1}, t_n), (x_j, t_n)$ and $(x_{j+1}, t_n), (x_j, t_n)$, respectively. The discrete energy for this case is given by

$$\begin{aligned}
E^n := \frac{\Delta x}{2} \sum_{j=0}^{J} \Bigg[ &\rho_1 \left(\frac{\varphi_j^{n+1} - \varphi_j^n}{\Delta t}\right)^2 + \rho_2 \left(\frac{\psi_j^{n+1} - \psi_j^n}{\Delta t}\right)^2 + \rho_1 \left(\frac{w_j^{n+1} - w_j^n}{\Delta t}\right)^2 \\
&+ b \left(\frac{\psi_{j+1}^{n+1} - \psi_j^{n+1}}{\Delta x}\right)\left(\frac{\psi_{j+1}^n - \psi_j^n}{\Delta x}\right) \\
&+ \kappa_0 \left(\frac{w_{j+1}^{n+1} - w_j^{n+1}}{\Delta x} - l \frac{\varphi_{j+1}^{n+1} + \varphi_j^{n+1}}{2}\right)\left(\frac{w_{j+1}^n - w_j^n}{\Delta x} - l \frac{\varphi_{j+1}^n + \varphi_j^n}{2}\right) \\
&+ \kappa \left(\frac{\varphi_{j+1}^n - \varphi_j^n}{\Delta x} + \frac{\psi_{j+1}^n + \psi_j^n}{2} + l \frac{w_{j+1}^n + w_j^n}{2}\right) \Bigg]
\end{aligned} \tag{6.15}$$

for $n \geqslant 1$ and it is *locking free*. The totally discrete equations are all consistent and of order $\mathcal{O}(\Delta x^2, \Delta t^2)$. Besides, they converge with $\Delta x, \Delta t \to 0$ if and only if they are stable. For issues concerning to numerical stability we need to make an analysis more elaborate with base in the references by Wright [10,11].

For numerical example, we consider $L = 3.14$ m, thickness $\epsilon = 0.015$ m, width 0.0048 m, $E = 21 \times 10^4$ N/m$^2$, $\rho = 7850$ kg/m$^3$, $k' = 5/6$, $r = 0.29$ (Poisson ratio) and the following initial conditions:

$$\begin{aligned}
&\varphi(x_j, 0) = \psi(x_j, 0) = w(x_j, 0) = 0, \\
&\varphi_t(x_j, 0) = \sin\left(\mu \frac{\pi x_j}{L}\right), \qquad \psi_t(x_j, 0) = w_t(x_j, 0) = 0, \quad \mu \in \mathbb{N}.
\end{aligned}$$

In respect to our mathematical analysis we established that the exponential stability holds if and only if $G = E/k'$. To consider the realistic case $G \neq E/k'$ we take the real value given by $G = E/(2 + 2r)$. We also use the following parameters:

$\Delta x = 0.0982$ and $\Delta t = 0.0017$. Firstly, we reproduce the conservative case to assure the accuracy of the method (see Fig. 2).

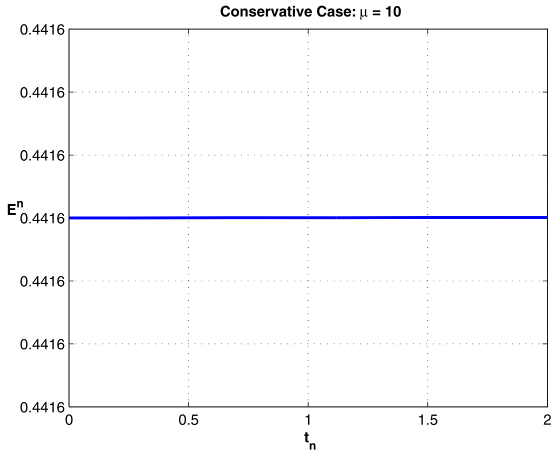


**Fig. 2.** $G = E/(2 + 2r)$.

Figs. 3–8 show the lack of exponential decay.

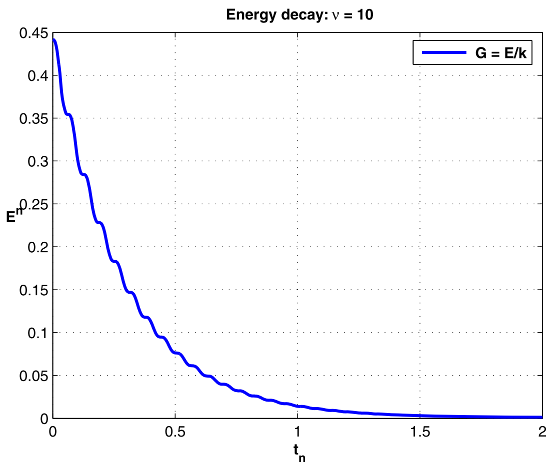


**Fig. 3.** $G = E/k'$.

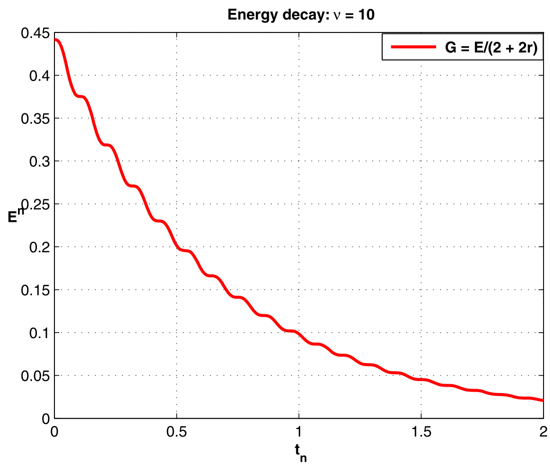


**Fig. 4.** $G = E/(2 + 2r)$.

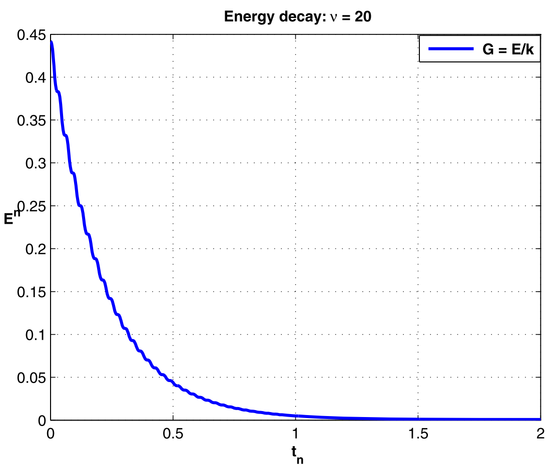


**Fig. 5.** $G = E/k'$.

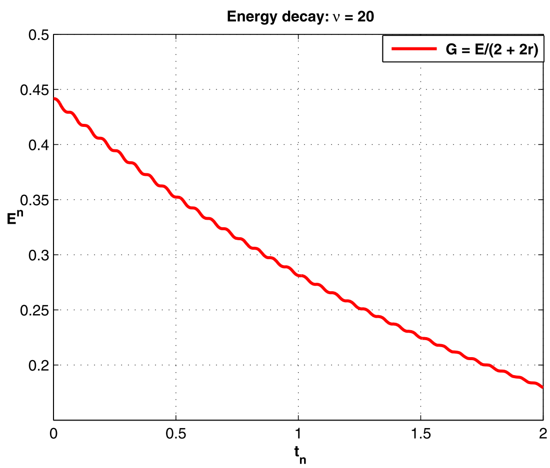


**Fig. 6.** $G = E/(2 + 2r)$.

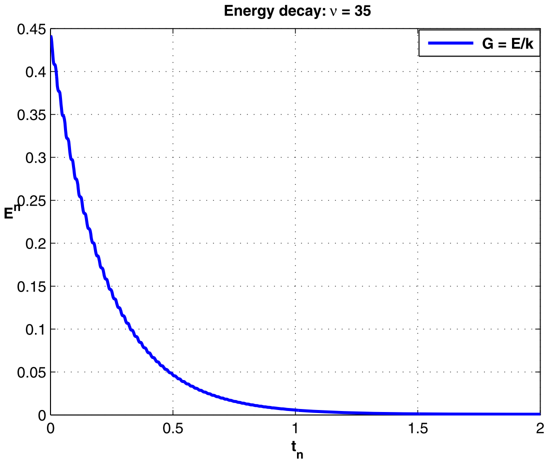


**Fig. 7.** $G = E/k'$.

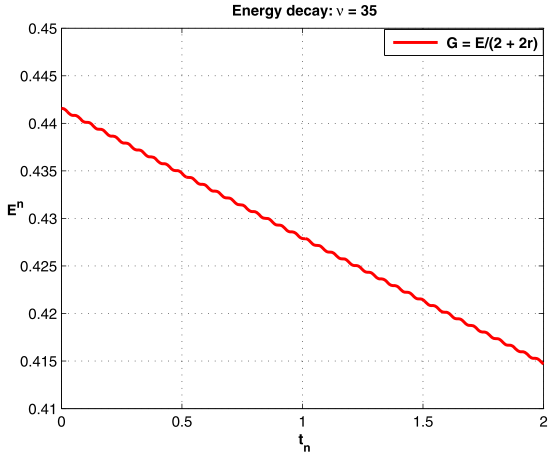


**Fig. 8.** $G = E/(2 + 2r)$.

In the numerical experiments we can to note the slow decay of the numerical energy $E^n$ in the case of different velocities of waves propagations.

## References


[1] F. Huang, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, Ann. Differential Equations 1 (1985) 43–56.
[2] T.J.R. Hughes, R.L. Taylor, W. Kanoknukulcahi, A simple and efficient finite element method for plate bending, Internat. J. Numer. Methods Engrg. 11 (1977) 1529–1543.
[3] J.E. Lagnese, G. Leugering, J.P.G. Schmidt, Modelling Analysis and Control of Dynamic Elastic Multi-Link Structures, Systems Control Found. Appl., 1994.
[4] Z. Liu, B. Rao, Energy decay rate of the thermoelastic Bresse system, Z. Angew. Math. Phys. 60 (2009) 54–69.
[5] Z. Liu, B. Rao, Characterization of polynomial decay rate for the solution of linear evolution equation, Z. Angew. Math. Phys. 56 (2005) 630–644.
[6] J.E. Muñoz Rivera, R. Racke, Global stability for damped Timoshenko systems, Discrete Contin. Dyn. Syst. Ser. B 9 (2003) 1625–1639.
[7] G. Prathap, G.R. Bhashyam, Reduced integration and the shear-flexible beam element, Internat. J. Numer. Methods Engrg. (1982) 195–210.
[8] J. Prüss, On the spectrum of $C_0$-semigroups, Trans. Amer. Math. Soc. 284 (1984) 847–857.
[9] S.P. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars, Philos. Magazine 6 (1921) 744–746.
[10] Joseph P. Wright, A mixed time integration method for Timoshenko and Mindlin type elements, Commun. Appl. Numer. Methods 3 (1987) 181–185.
[11] Joseph P. Wright, Numerical stability of a variable time step explicit method for Timoshenko and Mindlin type structures, Comm. Numer. Methods Engrg. 14 (1998) 81–86.