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Nonoscillation for second order sublinear dynamic equations on time scales

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ABSTRACT

Consider the Emden–Fowler sublinear dynamic equation

$$x^{\Delta\Delta}(t) + p(t)f(x(\sigma(t))) = 0, \tag{0.1}$$

where $p \in C(\mathbb{T}, \mathbb{R})$, $\mathbb{T}$ is a time scale, $f(x) = \sum_{i=1}^{m} a_i x^{\beta_i}$, where $a_i > 0$, $0 < \beta_i < 1$, with β_i the quotient of odd positive integers, $1 \le i \le m$. When $m = 1$, and $\mathbb{T} = [a, \infty) \subset \mathbb{R}$, (0.1) is the usual sublinear Emden–Fowler equation which has attracted the attention of many researchers. In this paper, we allow the coefficient function $p(t)$ to be negative for arbitrarily large values of t. We extend a nonoscillation result of Wong for the second order sublinear Emden–Fowler equation in the continuous case to the dynamic equation (0.1). As applications, we show that the sublinear difference equation

$$\Delta^2 x(n) + b(-1)^n n^{-c} x^{\alpha}(n+1) = 0, \quad 0 < \alpha < 1,$$

has a nonoscillatory solution, for $b > 0$, $c > \alpha$, and the sublinear q-difference equation

$$x^{\Delta\Delta}(t) + b(-1)^n t^{-c} x^{\alpha}(qt) = 0, \quad 0 < \alpha < 1,$$

has a nonoscillatory solution, for $t = q^n \in \mathbb{T} = q_0^{\mathbb{N}}$, $q > 1$, $b > 0$, $c > 1 + \alpha$.

1. Introduction

Consider the Emden–Fowler sublinear dynamic equation

$$x^{\Delta\Delta}(t) + p(t)f(x(\sigma(t))) = 0, \tag{1.1}$$

where $p \in C(\mathbb{T}, R)$, $\mathbb{T}$ is a time scale, $f(x) := \sum_{i=1}^{m} a_i x^{\beta_i}$, where $0 < \beta_i < 1$, β_i is the quotient of odd positive integers, $1 \le i \le m$.

When $\mathbb{T} = \mathbb{R}$, $m = 1$, the dynamic equation (1.1) is the second order sublinear differential equation

$$x''(t) + p(t)x^{\alpha}(t) = 0, \quad 0 < \alpha < 1. \tag{1.2}$$

In this case, the Emden–Fowler equation has several interesting physical applications in astrophysics (cf. Bellman [1] and Fowler [2]).

Kwong and Wong [3] proved the following theorem.

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Theorem 1.1. *Suppose that $P(t) = \int_t^\infty p(s)\mathrm{d}s$ exists for all $t \geq 0$. If there exists a function $F \in C^1[0,\infty)$ such that $|P(t)| \leq F(t)$ for all large t where $F(t) = O(t^{-\alpha})$ as $t \to \infty$ and*

$$\int_0^\infty t^\alpha |F'(t)|\mathrm{d}t = B_0 < \infty, \tag{1.3}$$

then (1.2) has a nonoscillatory solution.

In this paper, we extend Theorem 1.1 to dynamic equations on time scales. As applications, we show that the sublinear difference equation

$$\Delta^2 x(n) + b(-1)^n n^{-c} x^\alpha(n+1) = 0, \quad 0 < \alpha < 1, \tag{1.4}$$

has a nonoscillatory solution, for $b > 0$, $c > \alpha$, and the sublinear q-difference equation

$$x^{\Delta\Delta}(t) + b(-1)^n t^{-c} x^\alpha(qt) = 0, \quad 0 < \alpha < 1, \tag{1.5}$$

has a nonoscillatory solution, for $t = q^n \in \mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $b > 0$, $c > 1 + \alpha$. Eqs. (1.4) and (1.5) are discrete analogs of Eq. (1.2) with $p(t) = t^\lambda \sin t$.

Remark 1.2. In [4], we proved that (1.4) is oscillatory for $c < -1$. For Eq. (1.2), (in particular, $p(t) = t^\lambda \sin t$), oscillation and nonoscillation criteria have been established in [3,5–8]. In a few of these results, we make the following conjecture as an open problem.

Conjecture: For $0 < \alpha < 1$, $-1 \leq c \leq \alpha$, $b > 0$, Eq. (1.4) is oscillatory.

For completeness, (see [9,10] for elementary results for the time scale calculus), we recall some basic results for dynamic equations and the calculus on time scales. Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup \mathbb{T} = \infty$. The forward jump operator is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \emptyset = \sup \mathbb{T}$, where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$ we say t is right-dense, while if $\rho(t) = t$ and $t \neq \inf \mathbb{T}$ we say t is left-dense. Given a time scale interval $[c,d]_\mathbb{T} := \{t \in \mathbb{T} : c \leq t \leq d\}$ in $\mathbb{T}$ the notation $[c,d]_\mathbb{T}^\kappa$ denotes the interval $[c,d]_\mathbb{T}$ in case $\rho(d) = d$ and denotes the interval $[c,d)_\mathbb{T}$ in case $\rho(d) < d$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$. We say that $x : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}$ provided

$$x^\Delta(t) := \lim_{s \to t} \frac{x(t) - x(s)}{t - s},$$

exists when $\sigma(t) = t$ (here by $s \to t$ it is understood that s approaches t in the time scale) and when x is continuous at t and $\sigma(t) > t$

$$x^\Delta(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}.$$

Note that if $\mathbb{T} = \mathbb{R}$, then the delta derivative is just the standard derivative, and when $\mathbb{T} = \mathbb{Z}$ the delta derivative is just the forward difference operator. Hence our results contain the discrete and continuous cases as special cases and generalize these results to arbitrary time scales. For example, the q-difference time scale $\mathbb{T} = q^{\mathbb{N}_0} := \{1, q, q^2, q^3, \ldots\}$, $q > 1$, has important applications in quantum theory (see Kac and Cheung [11]).

2. Nonoscillation theorem

Throughout this paper we assume $t_0 \in \mathbb{T}$ and $t_0 > 0$. We now state our main theorem.

Theorem 2.1. *Suppose that $P(t) = \int_t^\infty p(s)\Delta s$ converges. Let $\alpha = \max\{\beta_i, i = 1, 2, \ldots, m\}$, $0 < \alpha < 1$. If there exists a nonnegative function $F \in C_{rd}^1[0,\infty)_\mathbb{T}$ such that $|P(t)| \leq F(\sigma(t))$ for all large $t \in \mathbb{T}$, where $F(t) = O(t^{-\alpha})$ as $t \to \infty$ and*

$$\int^\infty t^\alpha |F^\Delta(t)|\Delta t < \infty, \tag{2.1}$$

then (1.1) has a nonoscillatory solution.

Proof. Pick $T > \max\{t_0, 1\}$, sufficiently large, so that

$$|P(t)| \le F(\sigma(t)), \quad t \in [T, \infty)_{\mathbb{T}}. \tag{2.2}$$

Let $x_k(t)$ be a solution of (1.1) satisfying $x_k(T) = 0$, $x_k^{\Delta}(T) = k$, where k is a positive number. We claim that when k is large enough, $x^{\Delta}(t) > 0$ for all $t > T$ and so $x(t)$ is nonoscillatory. For the sake of brevity, we omit the subscript k in the following discussion. Recall (see [9, page 146]) that $h : \mathbb{T} \to \mathbb{R}$ is said to have a generalized zero at $\hat{t} \in \mathbb{T}$ if either $h(\hat{t}) = 0$ or $\hat{t}$ is left-scattered and $h(\rho(\hat{t}))h(\hat{t}) < 0$. Suppose now that $x^{\Delta}(t)$ has a generalized zero at some $t \in [T, \infty)_{\mathbb{T}}$. Let t_1 be the smallest such t. If $x^{\Delta}(t) < 2k$ for all $t \in [T, \infty)_{\mathbb{T}}$, let $t_2 := \infty$. Otherwise, let t_2 be the smallest generalized zero of $x^{\Delta}(t) - 2k$ in $(T, \infty)_{\mathbb{T}}$. Finally, let $\tau := \min\{t_1, t_2\}$. Then on $[T, \tau)$, $0 < x^{\Delta}(t) < 2k$. By the Mean Value Theorem on time scales (see [10, page 5]), there exist $\xi, \eta \in [T, \tau)$ such that for $t \in (T, \tau]$

$$x^{\Delta}(\xi)(t - T) \le x(t) - x(T) \le x^{\Delta}(\eta)(t - T).$$

So

$$0 < x(t) < 2kt, \quad t \in (T, \tau]. \tag{2.3}$$

At $t = \tau$, we have either

$$\tau \text{ is a generalized zero of } x^{\Delta}(t) \quad \text{if } \tau = t_1$$

or

$$\tau \text{ is a generalized zero of } x^{\Delta}(t) - 2k \quad \text{if } \tau = t_2.$$

That is, at $t = \tau$ either

$$x^{\Delta}(\tau) = 0 \quad \text{or} \quad x^{\Delta}(\rho(\tau)) > 0, \quad x^{\Delta}(\tau) < 0 \text{ when } \tau \text{ is left-scattered}, \text{ if } \tau = t_1, \tag{2.4}$$

or

$$x^{\Delta}(\tau) = 2k \quad \text{or} \quad x^{\Delta}(\rho(\tau)) < 2k, x^{\Delta}(\tau) > 2k \text{ when } \tau \text{ is left-scattered}, \text{ if } \tau = t_2. \tag{2.5}$$

Integrating (1.1) from T to t, for $t \in [T, \tau]$ we have

$$x^{\Delta}(t) = k - \int_T^t p(s) f(x(\sigma(s)))\Delta s. \tag{2.6}$$

Integrating by parts (see [9, Theorem 1.77, (v)]), we get

$$\left| \int_T^t p(s) f(x(\sigma(s)))\Delta s \right| \le \left| -P(t) f(x(t)) + \int_T^t P(s)(f(x(s)))^{\Delta} \Delta s \right|. \tag{2.7}$$

For $s \in [T, \tau]$, using the Pötzsche chain rule [9, Theorem 1.90]

$$\begin{aligned}(f(x(s)))^{\Delta} &= \int_0^1 f'(x(s) + h\mu(s)x^{\Delta}(s))\mathrm{d}h x^{\Delta}(s) \\ &= \int_0^1 f'((1-h)x(s) + hx(\sigma(s)))\mathrm{d}h x^{\Delta}(s) \\ &> 0.\end{aligned}$$

So from (2.7) and (2.2), integrating by parts, and using (2.3) we get that, for $t \in (T, \tau]$

$$\begin{aligned}\left| \int_T^t p(s) f(x(\sigma(s)))\Delta s \right| &\le F(\sigma(t)) f(x(t)) + \int_T^t F(\sigma(s))(f(x(s)))^{\Delta} \Delta s \\ &\le [F(\sigma(t)) + F(t)] f(x(t)) + \int_T^t |F^{\Delta}(s)| f(x(s))\Delta s \\ &\le (2k)^{\alpha} t^{\alpha} [F(\sigma(t)) + F(t)] \sum_{i=1}^m a_i (2kt)^{\beta_i - \alpha} \\ &\quad + (2k)^{\alpha} \int_T^t |F^{\Delta}(s)| \sum_{i=1}^m a_i (2ks)^{\beta_i - \alpha} s^{\alpha} \Delta s.\end{aligned} \tag{2.8}$$

Since

$$F(t) = O(t^{-\alpha}), \quad \text{as } t \to \infty$$

there exists a constant $B_1 > 0$ such that $|t^\alpha F(t)| \le B_1$ (without loss of generality) for $t \in (T, \infty)_{\mathbb{T}}$. Also, let $B_0 := \int_T^\infty t^\alpha |F^\Delta(t)| \Delta t$. Note that

$$F(\sigma(t))t^\alpha = F(\sigma(t))(\sigma(t))^\alpha \left(\frac{t}{\sigma(t)}\right)^\alpha \le B_1.$$

So from (2.8), we get

$$\left|\int_T^t p(s) f(x(\sigma(s)))\Delta s\right| \le (2k)^\alpha (2B_1 + B_0) \sum_{i=1}^m a_i = M(2k)^\alpha, \tag{2.9}$$

where $M = (2B_1 + B_0)\sum_{i=1}^m a_i$. Using (2.9) and (2.6), we obtain

$$k - M(2k)^\alpha \le x^\Delta(t) \le k + M(2k)^\alpha, \quad \text{for all } t \in [1, \tau].$$

For $k > (2^\alpha M)^{\frac{1}{1-\alpha}}$, we have in particular $0 < x^\Delta(\tau) < 2k$. This contradicts (2.4) and (2.5). □

3. Example 1

Consider the sublinear difference equation (here $\mathbb{T} = \mathbb{N}_0$)

$$\Delta^2 x(n) + p(n) x^\alpha(n+1) = 0, \quad 0 < \alpha < 1, \tag{3.1}$$

where $p(n) = b(-1)^n n^{-c}$, $b > 0$. For $c > \alpha$, we obtain

$$P(n) = \int_n^\infty p(s)\Delta s = b\sum_{i=n}^\infty (-1)^i i^{-c}. \tag{3.2}$$

Let $\epsilon > 0$ be given, then since

$$\lim_{x\to 0} \frac{(1+x)^c - 1}{x} = c,$$

there is a $\delta > 0$ such that

$$-c\epsilon < \frac{(1+x)^c - 1}{x} - c < c\epsilon$$

for $|x| < \delta$. It follows that there is a positive integer K such that for all $i \ge K$

$$-c\epsilon < \frac{(1+\frac{1}{2i})^c - 1}{\frac{1}{2i}} - c < c\epsilon.$$

Then we have that for all $i \ge K$ that

$$\frac{c(1-\epsilon)}{2i} < \left(1 + \frac{1}{2i}\right)^c - 1 < \frac{c(1+\epsilon)}{2i}.$$

It follows from this that for all $i \ge K$

$$\frac{c(1-\epsilon)}{2i(2i+1)^c} < \frac{1}{(2i)^c} - \frac{1}{(2i+1)^c} < \frac{c(1+\epsilon)}{2i(2i+1)^c}.$$

Summing all sides of this last set of inequalities from $i = k$ to infinity, where $k \ge K$, we get that

$$(1-\epsilon)\sum_{i=k}^\infty \frac{c}{2i(2i+1)^c} \le \sum_{i=k}^\infty \left[\frac{1}{(2i)^c} - \frac{1}{(2i+1)^c}\right] \le (1+\epsilon)\sum_{i=k}^\infty \frac{c}{2i(2i+1)^c}.$$

Hence we get that

$$\sum_{i=k}^\infty \left[\frac{1}{(2i)^c} - \frac{1}{(2i+1)^c}\right] \sim \sum_{i=k}^\infty \frac{c}{2i(2i+1)^c}. \tag{3.3}$$

From (3.3) we get

$$\begin{aligned} P(2k) &= b\sum_{i=2k}^{\infty}\frac{(-1)^i}{i^c} \\ &= b\sum_{i=k}^{\infty}\left[\frac{1}{(2i)^c}-\frac{1}{(2i+1)^c}\right] \\ &\sim b\sum_{i=k}^{\infty}\frac{c}{2i(2i+1)^c}. \end{aligned} \tag{3.4}$$

Next note that

$$\begin{aligned} \sum_{i=k}^{\infty}\frac{c}{2i(2i+1)^c} &\le \sum_{i=k}^{\infty}\frac{c}{(2i)^{c+1}} \\ &= \frac{c}{2^{c+1}}\sum_{i=k}^{\infty}\frac{1}{i^{c+1}} \\ &\le \frac{c}{2^{c+1}}\int_{k-1}^{\infty}\frac{1}{t^{c+1}}\mathrm{d}t \\ &= \frac{1}{2[2(k-1)]^c}. \end{aligned} \tag{3.5}$$

Also note that

$$\begin{aligned} \sum_{i=k}^{\infty}\frac{c}{2i(2i+1)^c} &\ge \sum_{i=k}^{\infty}\frac{c}{(2i+2)^{c+1}} \\ &= \frac{c}{2^{c+1}}\sum_{i=k}^{\infty}\frac{1}{(i+1)^{c+1}} \\ &\ge \frac{c}{2^{c+1}}\int_{k}^{\infty}\frac{1}{t^{c+1}}\mathrm{d}t \\ &= \frac{1}{2(2k)^c}. \end{aligned} \tag{3.6}$$

From (3.6) and (3.5) we get that

$$\sum_{i=k}^{\infty}\frac{c}{2i(2i+1)^c} \sim \frac{1}{2(2k)^c}.$$

Hence from (3.4) we have

$$P(2k) \sim b\frac{1}{2(2k)^c}. \tag{3.7}$$

Similar to the proof of (3.7)

$$P(2k+1) = b\sum_{i=2k+1}^{\infty}\frac{(-1)^i}{i^c} \sim -\frac{b}{2(2k+1)^c}. \tag{3.8}$$

From (3.7) and (3.8) we have

$$P(n) \sim b\frac{(-1)^n}{2n^c},$$

so it follows that given $\epsilon > 0$ for large n

$$|P(n)| = \left|b\sum_{i=n}^{\infty}(-1)^i i^{-c}\right| \le \frac{b(1+\epsilon)}{2(n+1)^c}.$$

Let

$$F(n) = \frac{b(1+\epsilon)}{2n^c} \le \frac{b(1+\epsilon)}{2n^\alpha}.$$

We have

$$|P(n)| \le F(\sigma(n)) \quad \text{and} \quad F(n) = O(n^{-\alpha}).$$

Note that

$$\begin{aligned} n^{\alpha}|(n+1)^{-c} - n^{-c}| &= n^{\alpha-c}|(1+n^{-1})^{-c} - 1| \\ &= cn^{\alpha-c-1}\left|1 - n \cdot o\left(\frac{1}{n}\right)\right| \\ &\sim \frac{|c|}{n^{1-\alpha+c}}. \end{aligned}$$

So when $c > \alpha$,

$$\int_1^{\infty} t^{\alpha}|F^{\Delta}(t)|\Delta t = 2b(1+\epsilon)\sum_{n=1}^{\infty} n^{\alpha}|(n+1)^{-c} - n^{-c}|$$

is convergent. Therefore $F(n)$ satisfies the hypotheses of Theorem 2.1 and so (3.1) has a nonoscillatory solution.

4. Example 2

Consider the q-difference equation

$$x^{\Delta\Delta}(t) + p(t)x^{\alpha}(qt) = 0, \quad 0 < \alpha < 1, \tag{4.1}$$

where $p(t) = b(-1)^n t^{-c}$, $t = q^n \in \mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $b > 0$, $c > 1 + \alpha$.

It is easy to get that, for $t = q^n$, when $c > 1$,

$$\begin{aligned} P(t) &= \int_t^{\infty} b(-1)^{\frac{\ln s}{\ln q}} s^{-c}\Delta s \\ &= b(-1)^n q^{(1-c)n}\frac{q-1}{1+q^{1-c}} = b(-1)^n t^{1-c}\frac{q-1}{1+q^{1-c}}. \end{aligned}$$

If we let $F(t) := b\frac{t^{1-c}(q-1)}{q^{1-c}(1+q^{1-c})}$, then we have $|P(t)| \le F(\sigma(t))$. If $c \ge 1 + \alpha$, we have $F(t) = O(t^{-\alpha})$, and if $c > 1 + \alpha$, we have that

$$\int_1^{\infty} |F^{\Delta}(t)|t^{\alpha}\Delta t = b\frac{|1 - q^{c-1}|}{1+q^{1-c}}\int_1^{\infty} t^{\alpha-c}\Delta t < \infty.$$

By Theorem 2.1, Eq. (4.1) has a nonoscillatory solution.

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