Convergence of rules based on nodal splines for the numerical evaluation of certain 2D Cauchy principal value integrals

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Abstract

We consider interpolatory type integration rules for the numerical approximation of certain 2D Cauchy Principal Value integrals, based on tensor product of nodal splines.

We present convergence results which generalize those known in one-dimensional case. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Recently, we have proposed and studied integration rules based on tensor product of quasi-interpolatory splines [2] for the numerical approximation of certain 2D Cauchy Principal Value integrals.

In this paper we shall consider the problem of evaluating the same kind of singular integrals by using integration rules based on tensor product of optimal nodal interpolatory splines. The univariate nodal splines [7–10], recently used in one-dimensional integration [4, 6, 15], have many of the desirable properties of quasi-interpolatory splines studied in [11]. However, they have the advantage of being interpolatory at the expense of a certain complexity in their definition.

In Section 2 we shall introduce a 2D interpolation operator, defined as tensor product of univariate nodal splines and in Section 3 we shall study its smoothness.
Section 4 we shall consider integration rules for certain 2D Cauchy Principal Value integrals based on the above operator and we present convergence results, which generalize those known in the one-dimensional case [4, 6, 15].

2. On tensor product of optimal nodal splines

Let \([a, b]\) be a finite interval on the real line. For a given integer \(m \geq 2\), let \(n\) satisfy \(n \geq m - 1\). We consider a partition of \([a, b]\)

\[
X_{m, n} := \{a = x_0 < x_1 < \cdots < x_{(m-1)n} = b\}
\]

and, by setting \(\tau_i = x_{(m-1)i}, 0 \leq i \leq n\), we define the partition

\[
\Pi_{m, n} := \{\tau_i, i = 0, \ldots, n\},
\]

so that \(\Pi_{m, n} \subseteq X_{m, n}\). The points of \(\Pi_{m, n}\) and \(X_{m, n} \setminus \Pi_{m, n}\) are denoted, respectively, primary and secondary knots corresponding to the partition \(X_{m, n}\).

Let \(P_m\) be the set of polynomials of order \(m\) (degree \(\leq m-1\)) and \(S_{m, n}\) the set of polynomial splines of order \(m\) with simple knots at the points \(x_i, i = 1, \ldots, (m-1)n - 1\), so that \(S_{m, n} \subset C^{m-2}([a, b])\) [5].

It has been proved [8, 9] that, for any order \(m\), there exists a local spline approximation operator \(W_n : B[a, b] \rightarrow S_{m, n}\), where \(B[a, b]\) denotes the set of bounded real-valued functions on \([a, b]\), with the following properties:

(i) \(W_n f(\tau_i) = f(\tau_i), i = 0, \ldots, n;\)
(ii) \(W_n p = p, \forall p \in P_m;\)
(iii) \(W_n\) is local, in the sense that, for a fixed \(x \in [a, b]\) and \(j\) such that \(x \in [\tau_j, \tau_{j+1}]\), the value of \(W_n f\) at \(x\) depends on the values of \(f\) in at most \((m + 1)\) neighbouring primary knots.

In the linear case \((m = 2)\) we obtain the trivial piecewise linear interpolant of \(f\). Assuming therefore \(m \geq 3\), the defining formula for the operator \(W_n\) on \([a, b]\) is given by

\[
(W_n f)(x) := \sum_{i=0}^{q_j} f(\tau_i) \omega_{q_j}(x), \quad x \in [\tau_j, \tau_{j+1}], \quad j = 0, \ldots, n - 1,
\]

where [4, 7]

\[
p_j := \begin{cases} 0 & \text{if } j = 0, \ldots, i_1 - 2, \\ j - i_1 + 1 & \text{if } j = i_1 - 1, \ldots, n - i_0, \\ n - (m - 1) & \text{if } j = n - i_0 + 1, \ldots, n - 1, \end{cases}
\]

\[
q_j := \begin{cases} m - 1 & \text{if } j = 0, \ldots, i_1 - 2, \\ j + i_0 & \text{if } j = i_1 - 1, \ldots, n - i_0, \\ n & \text{if } j = n - i_0 + 1, \ldots, n - 1, \end{cases}
\]

with \(i_0\) and \(i_1\) defined by

\[
i_0 := \left\lfloor \frac{m}{2} \right\rfloor + 1, \quad i_1 := m - \left\lfloor \frac{m}{2} \right\rfloor
\]

and where, for any \(t \in \mathbb{R}\), \([t] := \text{the maximum integer less than or equal to } t\).
The relevant values on \([a, b]\) of the functions \(\omega_{m,i}(x), \, i=0, \ldots, n\) can be evaluated by the formulas
\[
\omega_{m,i}(x) := \begin{cases} 
\prod_{k=0, k \neq i}^{m-1} \frac{x - \tau_k}{\tau_i - \tau_k}, & x \in [a, \tau_{i-1}] \quad (i \leq m - 1), \\
s_i(x), & x \in [\tau_{i-1}, \tau_{n-i_0+1}] \quad (n \geq m), \\
\prod_{k=0, k \neq n-i}^{m-1} \frac{x - \tau_{n-k}}{\tau_i - \tau_{n-k}}, & x \in [\tau_{n-i_0+1}, b] \quad (i \geq n - (m - 1)),
\end{cases}
\]
where
\[
s_i(x) = \sum_{r=0}^{m-2} \sum_{s=-i_0}^{(-i_0+m-1)} \alpha_{i,r,s} B_{(m-1)(i+s)+r}^m(x), \quad i = 0, \ldots, n
\]
and \(\{B_i^m(x): i = 1 - m, \ldots, (m-1)n-1\}\) are the normalized B-splines of order \(m\), forming a basis for the spline space \(S_{m,n}\) \([5, 14]\). A computational formula for the coefficients \(\alpha_{i,r,s}\) is given in \([8]\).

Let \(D\) the \(\mathbb{R}^2\) subset defined by \(D = J \times \tilde{J}\), with \(J = [a, b]\) and \(\tilde{J} = [\tilde{a}, \tilde{b}]\). We consider partitions (1) and (2) of \(J\) on which we construct the spline functions \(\{\omega_{m,i}(x), \, i=0, \ldots, n\}\) defined in (4). Then we consider similar partitions of \(\tilde{J}, \tilde{X}, \tilde{Y}, \tilde{a}, \tilde{b}, \tilde{n}\) and we construct the corresponding functions \(\{\tilde{\omega}_{m,i} (\tilde{x}), \, \tilde{i} = 0, \ldots, \tilde{n}\}\). Now we may generate a set of bivariate splines
\[
\omega_{m,\tilde{m},i} (x, \tilde{x}) := \omega_{m,i}(x) \tilde{\omega}_{m,i}(\tilde{x}),
\]
tensor product of the (4) ones. Let \(B(D)\) denote the set of bounded real-valued functions on \(D\). Then for any \(f \in B(D)\) we may define the following spline interpolating operator:
\[
W_{n,\tilde{n}}^* f(x, \tilde{x}) := \sum_{i=0}^{\tilde{n}} \sum_{j=0}^{\tilde{n}} \omega_{m,\tilde{m},i,j}(x, \tilde{x}) f(\tau_i, \tilde{\tau}_j), \quad (x, \tilde{x}) \in [\tau_j, \tau_{j+1}] \times [\tilde{\tau}_j, \tilde{\tau}_{j+1}],
\]
with \(j = 0, \ldots, n-1\) and \(\tilde{j} = 0, \ldots, \tilde{n}-1\).

In order to obtain the maximal order polynomial reproduction, we can assume \(m = \tilde{m}\), i.e. we use splines of the same order on both axes.

We remark that \(W_{n,\tilde{n}}^*\) is a spline operator with the following properties:
(i) \(W_{n,\tilde{n}}^*\) is local, in the sense that \(W_{n,\tilde{n}}^* f(x, \tilde{x})\) depends only on values of \(f\) in a small neighbourhood of \((x, \tilde{x})\);
(ii) \(W_{n,\tilde{n}}^*\) interpolates \(f\) at the primary knots, i.e. \(W_{n,\tilde{n}}^* f(\tau_i, \tilde{\tau}_j) = f(\tau_i, \tilde{\tau}_j)\);
(iii) \(W_{n,\tilde{n}}^*\) has the optimal order polynomial reproduction property, that means \(W_{n,\tilde{n}}^* p = p\), for all \(p \in \mathbb{P}_m^2\), where \(\mathbb{P}_m^2\) is the set of bivariate polynomials of total order \(m\).

3. Smoothness of the interpolation operator \(W_{n,\tilde{n}}^*\)

In order to study on how well our tensor product operator \(W_{n,\tilde{n}}^*\) approximates a function \(f \in C^{s-1}(D), \, 1 \leq s < m\), we introduce the following quantity:
\[
E_{v,\tilde{v},s}(t, \tilde{t}) := \begin{cases} 
D^{v, \tilde{v}} (f(t, \tilde{t}) - W_{n,\tilde{n}}^* f(t, \tilde{t})) & \text{if } 0 \leq v + \tilde{v} < s, \\
D^{v, \tilde{v}} W_{n,\tilde{n}}^* f(t, \tilde{t}) & \text{if } s \leq v + \tilde{v} < m,
\end{cases}
\]
where \( D^v \) is the usual partial derivative operator and we assume \( t \in [x_j, x_{j+1}) \subseteq [\tau_j, \tau_{j+1}] \), with \( l_j \in \{(m-1)j, \ldots, (m-1)(j+1)-1\}, j = 0, \ldots, n-1 \) and \( \tilde{t} \in [\tilde{x}_j, \tilde{x}_{j+1}) \subseteq [\tilde{\tau}_j, \tilde{\tau}_{j+1}], \) with \( l_j \in \{(m-1)\tilde{j}, \ldots, (m-1)(\tilde{j}+1)-1\}, \tilde{j} = 0, \ldots, \tilde{n} - 1 \) [6].

Since \( W^*_{n,n} \) reproduces polynomials belonging to \( P^2 \), then we can prove the following.

**Lemma 1.** Let \( W^*_{n,n} \) be defined as in (6). Then for any polynomial \( g \in P^2, 1 \leq s < m \) and any function \( f \) such that \( D^v \) exists, where \( 0 \leq v + \tilde{v} < s < m \)

\[
E_{v,\tilde{v},s}(t, \tilde{t}) = \begin{cases} 
D^{v,\tilde{v}}(R(t, \tilde{t}) - W^*_{n,n}R(t, \tilde{t})) & \text{if } 0 \leq v + \tilde{v} < s, \\
D^{v,\tilde{v}}W^*_{n,n}R(t, \tilde{t}) & \text{if } s \leq v + \tilde{v} < m,
\end{cases}
\]  

with \( R(x, \tilde{x}) := f(x, \tilde{x}) - g(x, \tilde{x}) \).

**Proof.** We do not present the proof because it is similar to that of Lemma 8.2 in [11] for tensor product of quasi-interpolating splines. \( \square \)

Now we say that a collection of product partitions \( \{X_{m,n} \times \tilde{X}_{m,\tilde{n}}: n = n_1, n_2, \ldots; \tilde{n} = \tilde{n}_1, \tilde{n}_2, \ldots\} \) of \( D \) is quasi-uniform (q.u.) if there exists a positive constant \( \sigma \) such that

\[
\frac{\Delta}{\delta} \leq \sigma, \quad \frac{\Delta}{\delta} \leq \sigma, \quad \frac{\tilde{\Delta}}{\tilde{\delta}} \leq \sigma, \quad \frac{\tilde{\Delta}}{\tilde{\delta}} \leq \sigma,
\]

where

\[
\Delta := \max_{1 \leq i \leq n(m-1)} (x_i - x_{i-1}), \quad \delta := \min_{1 \leq i \leq n(m-1)} (x_i - x_{i-1})
\]

and

\[
\tilde{\Delta} := \max_{1 \leq i \leq \tilde{n}(m-1)} (\tilde{x}_i - \tilde{x}_{i-1}), \quad \tilde{\delta} := \min_{1 \leq i \leq \tilde{n}(m-1)} (\tilde{x}_i - \tilde{x}_{i-1}).
\]

We shall call a sequence of spline spaces \( \{S_{m,n} \times S_{m,\tilde{n}}\} \) quasi uniform if they are based on a sequence of q.u. partitions.

We denote by \( H \) the norm of the primary partition \( \Pi_{m,n} \)

\[
H := \max_{1 \leq i \leq n} (\tau_i - \tau_{i-1});
\]

likewise we define \( \tilde{H} \) for the partition \( \tilde{\Pi}_{m,\tilde{n}} \). We assume that \( H \to 0 \) as \( n \to \infty \) and \( \tilde{H} \to 0 \) as \( \tilde{n} \to \infty \).

Furthermore, for any \( t > 0 \), for any region \( \Theta \subset \mathbb{R}^2 \), and for any function \( \psi \in C(\Theta) \) we define

\[
\omega(D^{s-1}\psi; t; \Theta) := \max_{0 \leq \nu \leq s-1} \omega(D^{s-1}\psi; t; \Theta),
\]

where \( D^r \) is the \( r \)th total derivative operator and \( \omega(\psi; t; \Theta) \) is the modulus of continuity of the function \( \psi \), given by

\[
\omega(\psi; t; \Theta) := \sup_{\|\theta\| \leq t} |\psi(x + \theta, \tilde{x} + \tilde{\theta}) - \psi(x, \tilde{x})|.
\]
Now, we define the following parameters describing the spacing of the partitions above introduced:

\[
H_j := \max_{p_j \leq i \leq q_j-1} (\tau_{i+1} - \tau_i), \quad \Delta_j := \max_{(m-1)p_j \leq i \leq (m-1)q_j-1} (x_{i+1} - x_i),
\]

\[
\delta_{l, m-v} := \min_{l, r \leq m-v \leq r \leq l} (x_{r+m-v} - x_r), \quad v = 0, \ldots, m-1,
\]

\[
h_j := \min_{p_j \leq i \leq q_j-1} (\tau_{i+1} - \tau_i), \quad h^* := \min_{0 \leq j \leq n-1} h_j, \quad \delta^* := \min_{0 \leq j \leq n-1} \delta_{l, m-v},
\]

\[
R_{m,n} := \max_{1 \leq i, j \leq n, |r-j|=1} (x_{i+m-v} - x_i).
\]

and likewise \( \tilde{H}_j \), \( \tilde{A}_j \), \( \tilde{\delta}_{l, m-v} \), \( \tilde{h}_j \), \( \tilde{h}^* \), \( \tilde{\delta}^* \), \( R_{m,n} \) for the partitions \( \tilde{X}_{m,n}, \tilde{\Pi}_{m,n} \).

Finally we let:

\[
H_j := H_j + \tilde{H}_j, \quad \Delta_j := \Delta_j + \tilde{\Delta}_j,
\]

\[
H^* := \max_{0 \leq j \leq n-1} |H_j|, \quad \tilde{H}^* := \min_{0 \leq j \leq n-1} |H_j|, \quad \Delta^* := \max_{0 \leq j \leq n-1} |\Delta_j|.
\]

Let \( D_{p_j, \tilde{p}_j} \equiv [\tau_{p_j}, \tau_{q_j}] \times [\tilde{\tau}_{p_j}, \tilde{\tau}_{q_j}] \) with \( j = 0, \ldots, n-1 \) and \( j = 0, \ldots, \tilde{n}-1 \). We define for any \((x, \tilde{x}) \in D_{p_j, \tilde{p}_j} \) and \( f \in C^{s-1}(D_{p_j, \tilde{p}_j}) \) with \( 1 \leq s \leq m \), \((t, \tilde{t}) \in D_{p_j, \tilde{p}_j} \)

\[
R(x, \tilde{x}) := f(x, \tilde{x}) - \sum_{j=0}^{s-1} \sum_{j=0}^{s-1-j} D^{j,j} f(t, \tilde{t})(x-t)^j(\tilde{x}-\tilde{t})^j \frac{j!}{j!}.
\]

Then \( R \) and its derivatives of total order less or equal to \( s-1 \) are zero at \((t, \tilde{t})\). Hence, by (7), to give a bound for \( |E_{v, e}(t, \tilde{t})| \) it is only necessary to estimate \( |D^{s-1} W_{m,n} R(t, \tilde{t})| \).

In our discussion we need the following.

**Lemma 2.** Let \( f \in C^{s-1}(D_{p_j, \tilde{p}_j}) \), \( 1 \leq s \leq m \). Then, for \( i = p_j, \ldots, q_j \) and \( \tilde{i} = \tilde{p}_j, \ldots, \tilde{q}_j \)

\[
|R(\tau_i, \tilde{\tau}_i)| \leq \frac{m^s H_{j,i}^{s-1}}{(s-1)!} \omega(D^{s-1} f; H_{j,i}; D_{p_j, \tilde{p}_j}),
\]

with \( H_{j,i} \) defined by (14).

**Proof.** By the Taylor series for \( R \) we have

\[
R(\tau_i, \tilde{\tau}_i) \leq \frac{1}{(s-1)!} \sum_{k=0}^{s-1} \binom{s-1}{k} \frac{\partial^{s-1} R(\eta_i, \eta_{\tilde{i}})}{\partial x^{s-1-k} \partial \tilde{x}^k} \left| \tau_i - t \right|^{s-1-k} \left| \tilde{\tau}_i - \tilde{t} \right|^k,
\]

where \((\eta_i, \eta_{\tilde{i}})\) is a point on the straight line between \((t, \tilde{t})\) and \((\tau_i, \tilde{\tau}_i)\).

Since \( q_j - p_j \leq m \), it follows that

\[
|\eta_i - t| \leq |\tau_i - t| \leq |\tau_q - \tau_{p_i}| \leq m H_j
\]
and likewise on $\tilde{x}$-axis. Then, by the definition of $R$ we have

$$|R(\tau, \tilde{\tau})| \leq \frac{m_{s-1}}{(s-1)!} \sum_{k=0}^{s-1} \binom{s-1}{k} \left| \frac{\partial^{s-1} f(\eta_l, \tilde{\eta}_l)}{\partial x^{s-1-k} \partial \tilde{x}^k} - \frac{\partial^{s-1} f(t, \tilde{t})}{\partial x^{s-1-k} \partial \tilde{x}^k} \right| H_j^{s-1-k} \tilde{H}_j^k$$

$$\leq \frac{m_{s-1}}{(s-1)!} \sigma(D^{s-1} f; mH_j{\tilde{j}}; D_{p_{j, \tilde{j}}}) \sum_{k=0}^{s-1} \binom{s-1}{k} H_j^{s-1-k} \tilde{H}_j^k$$

and, from the subadditivity of modulus of continuity, we obtain the thesis. \[ \square \]

In the next theorem we give a local estimate for $|Ev, \tilde{v}, s(t, \tilde{t})|$.

**Theorem 3.** Let $t \in [x_l, x_{l+1}]$, $\tilde{t} \in [\tilde{x}_l, \tilde{x}_{l+1}]$. If $f \in C^{s-1}(D_{p_j, \tilde{p}_j})$, with $1 < s < m$, with $D_{p_j, \tilde{p}_j}$ defined as in Lemma 2, then for any $v$, $\tilde{v}$ such that $0 < v + \tilde{v} < m$, it results that

$$\max_{t \in [x_l, x_{l+1}]} |Ev, \tilde{v}, s(t, \tilde{t})| \leq K_j H_j^{s-v-\tilde{v}-1} \omega(D^{s-1} f; H_j; D_{p_j, \tilde{p}_j})$$

where $K_j$ is a constant depending on $s$, $j$, $v$, $\tilde{v}$, $m$ and proportional to $R_{m,n}, R_{m,\tilde{n}}, H_j; H_{\tilde{j}}, A_j, A_{\tilde{j}}$.

**Proof.** We assume $R$ as in (16). Since to give a bound for $|Ev, \tilde{v}, s(t, \tilde{t})|$, from Lemma 1, we only need to estimate $|D^{v, s} W_n R(t, \tilde{t})|$, then from (6) we can write

$$|Ev, \tilde{v}, s(t, \tilde{t})| \leq \sum_{i=p_{j, \tilde{j}}} a_i \sum_{i=p_{j, \tilde{j}}} |D^v \omega_{m,i}(t)||D^\tilde{v} \omega_{m,i}(\tilde{t})||R(\tau, \tilde{\tau})|$$

and the thesis follows immediately by applying Lemma 3.4 of [6] and Lemma 2 to this inequality.

An explicit expression for the constant $K_j$ can be found in [3]. \[ \square \]

Let $\|g\|_\infty := \max_{(x, \tilde{x}) \in D} |g(x, \tilde{x})|$; from the above local estimate a global estimate result can be deduced.

**Theorem 4.** Suppose $(t, \tilde{t}) \in D$, let $f \in C^{s-1}(D)$ with $1 < s < m$: then for $v$, $\tilde{v}$ such that $0 < v + \tilde{v} < s-1$

$$\|Ev, \tilde{v}, s\|_\infty \leq K H^{s-v-\tilde{v}-1} \omega(D^{s-1} f; H; D)$$

and for $v$, $\tilde{v}$ such that $s < v + \tilde{v} < m$

$$\|Ev, \tilde{v}, s\|_\infty \leq K \tilde{H}^{s-v-\tilde{v}-1} \omega(D^{s-1} f; H; D),$$

where $K$ is a constant depending on $s$, $v$, $\tilde{v}$, $m$ and proportional to $R_{m,n}, R_{m,\tilde{n}}, H^*/h^*, H^*/\tilde{h}^*, \Delta^*/\delta^*, \Delta^*/\tilde{\delta}^*$.

An explicit expression for the constant $K$ can be found in [3].

Now we can prove the following corollary:
Corollary 5. If the hypotheses of Theorem 3 hold, then for any sequence of q.u. nodal spline spaces \( \{W_{n}^{*}\} \), it results that

\[
\max_{t \in [x_{i}, x_{i+1}]} |E_{\nu, \tilde{\nu}, \nu}(t, \tilde{t})| \leq K^{*}H_{j}^{s-v-\tilde{v}-1} \omega(D^{s-1} f; H_{j}; D_{p, \tilde{p}}),
\]

where \( K^{*} \) is a constant depending only on \( m, s, \nu, \tilde{\nu}, \sigma \).

Proof. Since the sequence of splines is quasi-uniform, then

\[
\frac{A_{j}}{\delta_{j,m-\nu}} \leq 2\sigma, \quad \frac{\delta_{j,m-\nu}}{A_{j}} \leq 2\sigma.
\]

Moreover, the quasi-uniformity of the partitions \( X_{m,n}, \tilde{X}_{m,\tilde{n}} \) ensures also the quasi-uniformity of the other two \( \Pi_{m,n}, \tilde{\Pi}_{m,\tilde{n}} \). So we can write

\[
\frac{H_{j}}{h_{j}} \leq 2\sigma, \quad \frac{H_{j}}{h_{j}} \leq 2\sigma.
\]

Finally from quasi-uniformity hypothesis we can deduce that also \( R_{m,n} \) and \( R_{m,\tilde{n}} \) are uniformly bounded. Therefore, from Theorem 3 the thesis holds. \( \square \)

We remark that from Corollary 5 immediately follows that

\[
\|D^{s}(f - W_{n}^{*}f)\|_{\infty} \to 0, \quad n, \tilde{n} \to +\infty, \quad 0 \leq \nu + \tilde{\nu} \leq s - 1.
\]

Finally, we can prove:

Theorem 6. If \( f \in C^{p}(D) \) with \( 0 \leq p < m - 1 \), then for any sequence of q.u. nodal spline spaces \( \{W_{n}^{*}\} \)

\[
\omega(D^{p}W_{n}^{*}f, \Delta^{*}, D) = O(\omega(D^{p}f, \Delta^{*}, D)).
\]

Proof. As in [5, 11] by the definition of the operator \( W_{n}^{*} \) and from Corollary 5, since \( H_{j} \leq m\Delta_{j} \), we have, for any \( [t, \tilde{t}] \in [x_{i}, x_{i+1}] \times [\tilde{x}_{i}, \tilde{x}_{i+1}] \) and \( r, \tilde{r} \) such that \( 0 \leq r + \tilde{r} \leq p \), that:

(i) \( W_{n}^{*}f \in C^{p}(D) \);

(ii) \( |D^{r}(f - W_{n}^{*}f)(t, \tilde{t})| \leq c_{1}\omega(D^{p}f; \Delta_{j}; D_{p, \tilde{p}}) \); 

(iii) \( |D^{r+1}W_{n}^{*}f(t, \tilde{t})| \leq c_{2}A_{j}^{-1}\omega(D^{p}f; \Delta_{j}; D_{p, \tilde{p}}) \); 

(iv) \( |D^{r+1}W_{n}^{*}f(t, \tilde{t})| \leq c_{3}\omega(D^{p}f; \Delta_{j}; D_{p, \tilde{p}}) \),

with \( D_{p, \tilde{p}} \) defined as in Lemma 2 and \( c_{1}, c_{2}, c_{3} \) real constants.

Now, it suffices to show that for any \((u_{1}, v_{1}), (u_{2}, v_{2}) \in D, \) with \((u_{1}, v_{1}) \neq (u_{2}, v_{2}) \)

\[
|D^{r}W_{n}^{*}f(u_{1}, v_{1}) - D^{r}W_{n}^{*}f(u_{2}, v_{2})| \leq c_{4}\omega(D^{p}f; h_{x} + h_{y}; D), \quad (17)
\]

where \( c_{4} \) is a real constant, \( h_{x} = |u_{1} - u_{2}| \) and \( h_{y} = |v_{1} - v_{2}| \).
We distinguish three different cases according to the position of the points \((u_1, v_1), (u_2, v_2)\) in \(D\).

- If \((u_1, v_1), (u_2, v_2) \in [x_{i_1}, x_{i_1+1}] \times [\tilde{x}_{i_1}, \tilde{x}_{i_1+1}]\), with \(l_j = p_j(m-1), \ldots, q_j(m-1) - 1, l_j = \tilde{p}_j(m-1), \ldots, \tilde{q}_j(m-1) - 1, \) then by (iii) and (iv) we can deduce:

\[
\left| D^{r,f} W_{n \alpha}^* f(u_1, v_1) - D^{r,f} W_{n \alpha}^* f(u_2, v_2) \right| \\
\leq C \Delta_{ij}^{-1} \omega(D^\rho f; \Delta_{ij}; D_p, \rho) \left[ |u_1 - u_2| + |v_1 - v_2| \right],
\]

where \(C = \max(c_2, c_3)\). Now, from subadditivity of modulus of continuity, we obtain:

\[
\left| D^{r,f} W_{n \alpha}^* f(u_1, v_1) - D^{r,f} W_{n \alpha}^* f(u_2, v_2) \right| \leq c_4 \omega(D^\rho f; \gamma, \alpha; D_p, \rho),
\]

from which (17) holds and the thesis immediately follows.

- If \((u_1, v_1) \in [x_{i_1}, x_{i_1+1}] \times [x_{j_1}, x_{j_1+1}], (u_2, v_2) \in [x_{i_1}, x_{i_1+1}] \times [\tilde{x}_{j_1}, \tilde{x}_{j_1+1}], \) with \(l_j, l_j = p_j(m-1), \ldots, q_j(m-1) - 1, l_j, l_j = \tilde{p}_j(m-1), \ldots, \tilde{q}_j(m-1) - 1, \) then

\[
\left| D^{r,f} W_{n \alpha}^* f(u_1, v_1) - D^{r,f} W_{n \alpha}^* f(u_2, v_2) \right| \\
\leq \left| D^{r,f} f(u_1, v_1) - D^{r,f} f(u_2, v_2) \right| + \left| D^{r,f} W_{n \alpha}^* f(u_1, v_1) - D^{r,f} f(u_1, v_1) \right| \\
+ \left| D^{r,f} W_{n \alpha}^* f(u_2, v_2) - D^{r,f} f(u_2, v_2) \right|.
\]

Since

\[ \Delta_{ij} \leq \sigma(h_i + h_j), \quad \Delta_{ij} \leq \sigma(h_i + h_j), \]

then from (ii) (17) holds and the thesis follows.

- If \((u_1, v_1) \in [x_{i_1}, x_{i_1+1}] \times [\tilde{x}_{j_1}, \tilde{x}_{j_1+1}], (u_2, v_2) \in [x_{i_1}, x_{i_1+1}] \times [\tilde{x}_{j_1}, \tilde{x}_{j_1+1}], \) with \(l_j = l_1 + 1, l_j = l_1 + 1, \) since we can write

\[
\left| D^{r,f} W_{n \alpha}^* f(u_1, v_1) - D^{r,f} W_{n \alpha}^* f(u_2, v_2) \right| \leq \left| D^{r,f} W_{n \alpha}^* f(x_{i_1+1}, \tilde{x}_{j_1+1}) \right| \\
+ \left| D^{r,f} W_{n \alpha}^* f(x_{i_1+1}, \tilde{x}_{j_1+1}) - D^{r,f} W_{n \alpha}^* f(u_2, v_2) \right| \\
\leq c_4 \omega(D^\rho f; \gamma, \alpha; D_p, \rho),
\]

then the thesis follows. The cases \(l_j = l_j, l_j = l_j + 1\) and \(l_j = l_j + 1, l_j = l_j + 1\) are analogous. \(\square\)

4. Numerical evaluation of 2D Cauchy Principal Value integrals based on the spline operator \(W_{n \alpha}^*\)

In this section we consider the numerical evaluation of Cauchy Principal Value integrals of the form

\[
I(\xi; \theta; f) = \int_a^b \int_\tilde{a}^\tilde{b} w_1(x) w_2(\tilde{x}) \frac{f(x, \tilde{x})}{(x - \xi)(\tilde{x} - \theta)} \, dx \, d\tilde{x},
\]

with \(\xi \in (a, b), \theta \in (\tilde{a}, \tilde{b})\) by cubatures based on the spline operator \(W_{n \alpha}^*\). We assume:

- \(w_1(x) \in L_1([a, b]) \cap DT(N_\delta(\xi)), w_2(\tilde{x}) \in L_1([\tilde{a}, \tilde{b}]) \cap DT(N_\delta(\theta)),\) with DT the class of Dini-Type functions and \(N_\delta(\lambda) := [\lambda - \delta, \lambda + \delta], \delta > 0;\)

\[ \text{with} \quad w_1(x) \in L_1([a, b]) \cap DT(N_\delta(\xi)), w_2(\tilde{x}) \in L_1([\tilde{a}, \tilde{b}]) \cap DT(N_\delta(\theta)), \delta > 0; \]
- $f \in H_p(\mu, \mu)$, i.e. $f$ is a continuous function with all partial derivatives of $f$ of order $j = 0, \ldots, p$, $p \geq 0$, continuous, and each derivative of order $p$ satisfying a Hölder condition of order $0 < \mu \leq 1$ [12].

The integral (18) is approximated by

$$I(\xi; \theta; f) = I(\xi; \theta; W_{n\mu}^* f) + \mathcal{E}_{n\mu}(\xi; \theta; f).$$

(19)

We remark that, because of the polynomial reproduction property of the operator $W_{n\mu}^*$, we have $\mathcal{E}_{n\mu}(\xi; \theta; f) = 0$, for any $f \in P_m$.

Moreover, we can write

$$I(\xi; \theta; f) = \sum_{i=0}^{n} \sum_{j=0}^{\hat{n}} \lambda_i(\xi) \lambda_j(\theta) f(\tau_i, \tau_j) + \mathcal{E}_{n\mu}(\xi; \theta; f),$$

where

$$\lambda_i(\xi) = \int_{a}^{b} w_1(x) \omega_{m,i}(x) \frac{1}{x - \xi} \, dx \quad \text{and} \quad \lambda_j(\theta) = \int_{a}^{b} w_2(x) \omega_{m,j}(x) \frac{1}{x - \theta} \, dx$$

are the weights of the one-dimensional quadratures based on nodal splines [6, 15].

Now in order to study the convergence of the cubatures (19) we need the next lemma.

**Lemma 4.** Let $r_{n\mu}(x, \tilde{x}) = f(x, \tilde{x}) - W_{n\mu}^* f(x, \tilde{x})$ and $f \in H_p(\mu, \mu)$ with $p \geq 0$ and $0 < \mu \leq 1$. Then for any sequence of u. nodal spline spaces $\{W_{n\mu}^* f\}$, for any $0 < v < \mu$ and $x \neq \xi, \tilde{x} \neq \theta$

$$\frac{|r_{n\mu}(x, \tilde{x}) - r_{n\mu}(x, \theta)|}{|\tilde{x} - \theta|^v} \leq K^*_1 \Delta^{*(p+\mu)(1-\frac{\mu}{2})},$$

where $K^*_1$ is a real constant.

**Proof.** From Theorem 6, since $f \in H_p(\mu, \mu)$, we can deduce

$$|W_{n\mu}^* f(x, \tilde{x}) - W_{n\mu}^* f(x, \theta)| \leq \tilde{K}_1 |\tilde{x} - \theta|^\mu,$$

(20)

where $\tilde{K}_1$ is a real constant. Therefore, by means of (20) and from definition of $f \in H_p(\mu, \mu)$, we have

$$|r_{n\mu}(x, \tilde{x}) - r_{n\mu}(x, \theta)| \leq C |\tilde{x} - \theta|^\mu,$$

where $C$ is a real constant. Then from [5] and Theorem 4 we get

$$\frac{|r_{n\mu}(x, \tilde{x}) - r_{n\mu}(x, \theta)|}{|\tilde{x} - \theta|^v} = \frac{|r_{n\mu}(x, \tilde{x}) - r_{n\mu}(x, \theta)|^{1-\frac{\mu}{2}}}{|\tilde{x} - \theta|^v} |r_{n\mu}(x, \tilde{x}) - r_{n\mu}(x, \theta)|^{\frac{\mu}{2}} \leq C |r_{n\mu}(x, \tilde{x})| + |r_{n\mu}(x, \theta)|^{1-\frac{\mu}{2}} \leq K^*_1 \Delta^{*(p+\mu)(1-\frac{\mu}{2})},$$

which proves the lemma. Obviously the role of the variables $x$ and $\tilde{x}$ can be interchanged. □

Now, we can state the following convergence theorem:
Theorem 8. Let $f \in H_p(\mu, \mu)$ with $0 < \mu \leq 1$, $0 \leq p < m - 1$.

Let $H \rightarrow 0$ as $n \rightarrow \infty$ and $\bar{H} \rightarrow 0$ as $\bar{n} \rightarrow \infty$ and let \{\$W_{n, \bar{n}}^* f\} be a quasi-uniform sequence of nodal spline spaces. Then for the remainder term we have

$\varepsilon_{n, \bar{n}}(\xi; \theta; f) = O(A^* p^+ \mu^{-1})$,

where $\gamma$ is a real number with $0 < \gamma < \mu$, small as we like.

Proof. Since $\varepsilon_{n, \bar{n}}(\xi; \theta; f) = I(\xi; \theta; r_n)$, we can write

$\varepsilon_{n, \bar{n}}(\xi; \theta; f) = \int_a^b w_1(x) \frac{S_{n, \bar{n}}(x) - S_{n, \bar{n}}(\xi)}{x - \xi} \, dx$

$+ \left[ \int_a^b \frac{w_2(\xi)}{\xi - \theta} \, d\xi \right] \int_a^b w_1(x) \frac{r_{n, \bar{n}}(x, \theta) - r_{n, \bar{n}}(\xi, \theta)}{x - \xi} \, dx$

$+ \left[ S_{n, \bar{n}}(\xi) + r_{n, \bar{n}}(\xi, \theta) \int_a^b \frac{w_2(\xi)}{\xi - \theta} \, d\xi \right] \int_a^b w_1(x) \frac{r_{n, \bar{n}}(x, \theta)}{x - \xi} \, dx$,

where

$S_{n, \bar{n}}(x) := \int_a^b w_2(\xi) \frac{r_{n, \bar{n}}(x, \xi) - r_{n, \bar{n}}(x, \theta)}{\xi - \theta} \, d\xi$.

For $0 < \nu < \mu$ and $\bar{x} \neq \theta$, from Lemma 7 we obtain

$\frac{|r_{n, \bar{n}}(x, \bar{x}) - r_{n, \bar{n}}(x, \theta)|}{|\bar{x} - \theta|^\nu} \leq K^* A^* p^+ \mu^{-1}$,

with $\delta = \nu (p + \mu)/\mu$. Therefore, for the definition given for the function $S_{n, \bar{n}}$, we have

$|S_{n, \bar{n}}(x)| \leq L^* A^* p^+ \mu^{-1}$, with $L^*$ real constant.

Now, since

$|S_{n, \bar{n}}(x) - S_{n, \bar{n}}(\xi)| \leq K_0 |x - \xi|^\mu - \epsilon$,

where $\epsilon$ is a positive real number such that $\epsilon < \mu$ and $K_0$ is a constant depending only on $\epsilon$ [13], then we obtain

$|S_{n, \bar{n}}(x) - S_{n, \bar{n}}(\xi)| \leq C A^* (p + \mu - \delta)(1 - \frac{\mu}{\mu - \epsilon}) = C A^* (p + \mu - \delta_1)$,

where $\delta_1 = \delta + (p + \mu - \delta)(\nu/\mu - \epsilon)$ and $C$ is a real constant.

From (21), (22) and (24) by setting $\gamma = \max(\delta, \delta_1)$, for a sufficiently small $\nu$, the thesis immediately follows. \qed

5. Final remarks

In this paper we have considered a bivariate approximation operator defined as tensor product of two univariate nodal spline interpolation schemes and we have studied its smoothness.
Then we have proposed integration rules for CPV integrals (18), based on the above operator, we have obtained some convergence results and derived an error bound when $f \in H_p(\mu, \mu)$, $p \geq 0$, $0 < \mu \leq 1$. An other interesting question concerning the application of nodal splines to numerical integration of other kinds of CPV 2D integrals has been considered in [1].

References